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A new group algebra and lacunary sets in discrete noncommutative groups (1)

by

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Abstract. We define for a discrete group G the $*$-Banach algebra $l^1[G]$, and the equality $l^1[G] = l^1(G)$ characterizes the amenability of a discrete group G. We consider the lacunary sets with respect to that algebra, which in the amenable case coincide with Sidon sets introduced by A. Figà-Talamanca and M. A. Picardello [11]. We give the construction of unconditional Sidon sets.

Introduction. For a discrete group G we introduce the new $*$-Banach algebra under convolution $l^1[G]$, which is equal to the closure of functions with finite support with respect to the norm $\|f\|_U = \||f|\|_{VN(G)}$.

Using the results of A. Derighetti and A. Hulanicki concerning the characterization of an amenable group, we see that $l^1[G] = l^1(G)$ *if and only if G is an amenable group*.

M. Leinert discovered a new class of lacunary infinite sets which could not be Sidon sets. We propose a new class of lacunary sets, U-Sidon sets, which contains Sidon sets and Leinert sets. We also introduce another class of "lacunary" sets (amenable sets) and *we show that that class is closed under finite union*. When the group G is amenable, every set is an amenable set and every U-Sidon set is a Sidon set.

M. A. Picardello has shown that weak Sidon sets are $\Lambda(p)$ sets for every $p < \infty$. Using the same argument (the Rademacher functions), we prove that *U-Sidon sets are $\Lambda(p)$ sets for every $p < \infty$*. We also show that *all infinite group G is not U-Sidon set or*, which is the same, $l^1[G] \neq C^*_\varrho(G)$.

The remainder of Section 3 shows that the *union of a Leinert and a U-Sidon set is again a U-Sidon set*. If we take the union of an infinite Sidon set and an infinite Leinert set, then we obtain a U-Sidon set which is not a Sidon set or a Leinert set.

The main result of Section 4 is Theorem 4.1, which gives *a sufficient condition for a set to be U-Sidon*.

(1) This work was done during the author's stay at the University of British Columbia, Canada.

0. Preliminaries. We establish some notations and definitions. For a discrete group G,

$$l^p(G) = \Big\{f\colon \Big\{\sum_{x\in G}|f(x)|^p\Big\}^{1/p} < \infty\Big\}.$$

For a bounded function $f\in l^\infty(G)$, we denote $|f|(x) = |f(x)|$; $c_0(G)$ will denote the space of all functions on G which tend to zero at infinity. We refer to P. Eymard [9] for the basic definition, properties and theorems of $A(G), B_\varrho(G), B(G), C_\varrho^*(G)$ and $\mathrm{VN}(G)$.

We recall only that $C_\varrho^*(G)$ and $\mathrm{VN}(G)$ are, respectively, the C^*-algebra and the von Neumann algebra generated by the operators $\varrho(f)$ on $l^2(G)$, where ϱ is the left regular representation and $f\in l^1(G)$.

Since every $T\in \mathrm{VN}(G)$ is of the form $\varrho(f)$ where $f = T(\delta_e)$, we shall identify T with $T(\delta_e)$, where, for $x\in G$, δ_x is the Dirac function at x. So for $f\in\mathrm{VN}(G)$ the operator norm is

$$\|f\|_{\mathrm{VN}(G)} = \sup\{\|f*g\|_2\colon \|g\|_2 = 1,\ g\in l^2(G)\}$$

where the convolution is defined as

$$(f*g)(x) = \sum_{y\in G} f(xy^{-1})g(y)$$

for every $f, g\in l^2(G)$.

For the function $f\in l^\infty(G)$ we define the involution by $f^*(x) = \overline{f(x^{-1})}$ and the left and right translation by $(xf)(y) = f(x^{-1}y)$, $(fx)(y) = f(yx)$. Note that

$$\delta_x*f = xf \quad\text{ and }\quad f*\delta_x = f_{x^{-1}}.$$

For $f, g\in l^\infty(G)$, $(f\cdot g)(x) = f(x)g(x)$ for every $x\in G$.

Let $L(G)$ denote the set of all functions on G with finite support.

For a subset X in $l^\infty(G)$ and $E\subset G$, we denote

$$X_E = \{f\in X\colon \operatorname{supp} f\subset E\}$$

and

$$X|_E = X(E) = \{f|_E\colon f\in X\},$$

where $f|_E$ denote the restriction of f to the set E.

For an $f\in L(G)$ we define the L^{2n}-norm as

$$\|f\|_{L^{2n}} = \{(f*f^*)^n(e)\}^{1/2n},$$

where the power is a convolution power. Let $L^{2n} = L^{2n}(\hat{G})$ denote the completion of $L(G)$ with respect to this L^{2n}-norm [see [19]].

Let $M\big(A(G)\big) = \{f\in l^\infty(G)\colon f\cdot A(G)\subset A(G)\}$ with the operator norm. Note that if G is an amenable group, then $M\big(A(G)\big) = B(G)$. We always have $B(G)\subset M\big(A(G)\big)$.

It is not difficult to verify the following lemma:

LEMMA 0.1. *Let $f, g\in \mathrm{VN}(G)$ and let G be a discrete group; then*

(a) $|f*g|\leqslant |f|*|g|$;

(b) *If $f\in\mathrm{VN}(G)$, then* $\||f|\|_{\mathrm{VN}(G)}\geqslant \|f\|_{\mathrm{VN}(G)}$;

(c) *If $f\geqslant g\geqslant 0$, then* $\|f\|_{\mathrm{VN}}\geqslant\|g\|_{\mathrm{VN}}$.

1. Group algebras of a discrete group G. We start with the following proposition.

PROPOSITION 1.1. *Let G be a discrete group; then*

$$U_\varrho(G) = \{f\in\mathrm{VN}(G)\colon |f|\in\mathrm{VN}(G)\}$$

with the norm $\|f\|_U = \||f|\|_{\mathrm{VN}}$ is a $*$-Banach algebra under convolution.

Proof. Using Lemma 0.1, we have for $f, g\in U_\varrho(G)$:

(α_1) $$\|f*g\|_U = \||f*g|\|_{\mathrm{VN}}\leqslant \||f|*|g|\|_{\mathrm{VN}}\leqslant\|f\|_U\|g\|_U.$$

We also note of course that $\|\ \|_U$ is a norm. To prove that $U_\varrho(G)$ is a Banach space, we observe that if $\{f_n\}$ is a Cauchy sequence in $U_\varrho(G)$, then $\{|f_n|\}$ is Cauchy in $\mathrm{VN}(G)$. Hence there exists an $f\in\mathrm{VN}(G)$ such that

(α_2) $$\lim_n\|\,|f|-|f_n|\|_{\mathrm{VN}} = 0.$$

So $f\in U_\varrho(G)$ and we see that $\lim_n\|f-f_n\|_U = 0$. Since $|f^*| = |f|^*$, we infer that $U_\varrho(G)$ is a $*$-Banach algebra.

Remark. For $f\in U_\varrho(G)$ and $x\in G$

(α_3) $$\|xf\|_U = \|fx\|_U = \|f^*\|_U = \|f\|_U.$$

Now we consider another a $*$-Banach algebra, which is vital for us. Since every function $f\in\mathrm{VN}(G)$ has the "Fourier series"

(α_4) $$f\sim\sum_{x\in G} f(x)\delta_x,$$

we can ask when this series is unconditionally convergent in the operator norm $\|\ \|_{\mathrm{VN}}$, or, which is the same, when the set of Dirac functions $\{\delta_x\}_{x\in G}$ is an unconditional basic sequence.

Let $l^1[G] = \{f\in\mathrm{VN}(G)$: the "Fourier series" of $f\sim\sum_{x\in G}f(x)\delta_x$ is unconditionally convergent in $\|\ \|_{\mathrm{VN}}$ norm$\}$. For $f\in l^1[G]$ let

(α_5) $$\|f\|_S = \sup\{\|\chi_F f\|_{\mathrm{VN}}\colon F\in\mathfrak{F}\},$$

where χ_F denotes the characteristic function of the set F and $\mathfrak{F}$ is the class of all finite subsets of G.

Let $\mathfrak{a}$ denote the closure of $L(G)$ with respect to the $\|\ \|_U$ norm. Now we can state our main result:

THEOREM 1.2. *For every discrete group G, $l^1[G]$ is a $*$-Banach algebra and the norms $\| \|_S$ and $\| \|_U$ are equivalent on $l^1[G]$.*

Proof. Since $\mathfrak{a}$ is always a $*$-Banach algebra with the norm $\| \|_U$, it suffices to prove that $l^1[G] = \mathfrak{a}$. By the Orlicz–Pettis theorem $l^1[G] \subset \mathfrak{a}$. Let $f \in \mathfrak{a}$ and $\{E_n\}$ be any increasing sequence of finite sets contained in supp f. Since functions of finite support are dense in $\mathfrak{a}$, it is easy to see that $\{f\chi_{E_n}\}$ is a Cauchy sequence in $\mathrm{VN}(G)$; therefore $f \in l^1[G]$. Since the norm $\| \|_S$ is equivalent to the norm

$$(\alpha_6) \qquad \|f\|_{S^1} = \sup\{\|\varphi f\|_{\mathrm{VN}}\colon \|\varphi\|_\infty = 1\} = \||f|\|_{\mathrm{VN}},$$

the theorem follows.

Let us observe that we have the following inclusions:

$$(\alpha_7) \qquad l^1(G) \subseteq l^1[G] \subseteq C^*_\varrho(G)$$

and we have the following natural question: When do these algebras coincide? In Section 3 we shall see that for an infinite discrete group G we have $l^1[G] \neq C^*_\varrho(G)$.

The answer for the first inclusion is in the following.

PROPOSITION 1.3. *If G is a discrete group, then $l^1(G) = l^1[G]$ if and only if G is an amenable group.*

The proof results from the following characterization of an amenable groups due to A. Derighetti and A. Hulanicki [7], [14].

$(*)$ G is an amenable group if and only if $\|f\|_1 = \|f\|_{\mathrm{VN}}$ for every $f \in L(G)$ and $f \geqslant 0$.

Thus, since $L(G)$ is dense in $l^1(G)$ and $l^1[G]$, we conclude at once that if G is an amenable group, then $l^1(G) = l^1[G]$.

Conversely, let $l^1(G) = l^1[G]$; then by the Closed Graph Theorem we have

$$(\alpha_8) \qquad \|f\|_1 \leqslant C\|f\|_U$$

for every $f \in L(G)$ for some fixed constant $C > 0$.

But then

$$(\alpha_9) \qquad \|f^n\|_1 \leqslant C\|f^n\|_U,$$

where the power is the convolution power, for arbitrary integer $n > 0$.

Taking a positive f in (α_9), we obtain

$$(\alpha_{10}) \qquad \|f\|_1^n \leqslant C\|f\|_{\mathrm{VN}}^n.$$

Hence

$$(\alpha_{11}) \qquad \|f\|_1 \leqslant \|f\|_{\mathrm{VN}}$$

for every $f \in L(G)$ and $f \geqslant 0$. But we always have

$$(\alpha_{12}) \qquad \|f\|_1 \geqslant \|f\|_{\mathrm{VN}} \quad \text{for} \quad f \in l^1(G),$$

so the group G is an amenable by $(*)$.

Remark 1.4. By the same argument as above we infer that G is amenable if and only if $U_\varrho(G) = l^1(G)$.

It will be interesting to know when $l^1[G] = U_\varrho(G)$.

2. Properties of $l^1[G]$. Let $l^\infty[G]$ denote the dual Banach space of $l^1[G]$, and for $f \in l^\infty[G]$ let

$$(\beta_1) \qquad \|f\|_* = \sup\{|\langle f, g\rangle|\colon g \in L(G) \text{ and } \|g\|_U = 1\}.$$

Let $c_0[G]$ denote the closure of $L(G)$ in $l^\infty[G]$ with respect to the norm $\| \|_*$. Note that, for every $f \in L(G)$,

$$(\beta_2) \qquad \|f\|_* \leqslant \|f\|_{A(G)}.$$

Hence we obtain

$$(\beta_3) \qquad A(G) \subseteq c_0[G],$$

$$(\beta_4) \qquad B_\varrho(G) \subseteq l^\infty[G].$$

First we note the following

PROPOSITION 2.1. *For a discrete group G, we have*

(a) $l^\infty[G] \subseteq l^\infty(G)$;

(b) $l^\infty[G]$ *is an invariant under the translation ideal in* $l^\infty(G)$;

(c) $c_0[G]$ *is an invariant under the translation ideal in* $c_0(G)$ *and in* $l^\infty(G)$;

(d) $c_0[G]^* = U_\varrho(G)$.

Proof. If $\varphi \in l^\infty[G]$, then $\varphi(x) = \langle\varphi, \delta_x\rangle$ so $|\varphi(x)| \leqslant \|\varphi\|_*$ for every $x \in G$.

The proof of (b) follows at once if we recall that for every $\varphi \in l^\infty(G)$ and $f \in l^1[G]$, $\varphi f, xf, fx \in l^1[G]$ for $x \in G$.

Now we show that $U_\varrho(G)$ is a dual space. The inclusion $c_0[G]^* \subseteq U_\varrho(G)$ follows immediately if we show that

$$(\beta_5) \qquad c_0[G]^* \subseteq \mathrm{VN}(G).$$

But since $A(G)$ is dense in $c_0[G]$ and (β_2) we get (β_5). Now let $f \in c_0[G]^*$ and $\varphi \in l^\infty(G)$; then $\varphi f \in c_0[G]^* \subset \mathrm{VN}(G)$. This implies that $f \in U_\varrho(G)$.

Conversely, let $f \in U_\varrho(G)$ and $u \in L(G)$ and $E = \mathrm{supp}(u)$, then

$$|\langle f, u\rangle| = |\langle f\chi_E, u\rangle| \leqslant \|f\chi_E\|_U\|u\|_* \leqslant \|f\|_U\|u\|_*.$$

Hence $f \in c_0[G]^*$ and the proposition follows.

Remark 2.2. Let E be a subset of G and let $l^1[G]|_E$ denote the quotient Banach space $l^1[G]/J_E$, where $J_E = \{f \in l^1[G]\colon f = 0 \text{ on } E\}$ and

$$l^1[G]_E = \{f \in l^1[G]\colon \mathrm{supp} f \subseteq E\},$$

then the mapping $l^1[G] \ni \varphi \to \varphi\chi_E \in l^1[G]_E$ gives the isomorphism of $l^1[G]|_E$ and $l^1[G]_E$ and this space will be denoted by $l^1[E]$.

PROPOSITION 2.3. *Let G be a discrete group. G is anamenable group if and only if there exists a non-zero continuous functional m on $l^\infty[G]$ invariant under translation of $x \in G$ (i.e. $m(xf) = m(f)$ for $f \in l^\infty[G]$ and $x \in G$).*

Proof. If G is an amenable, then by Proposition 1.3 $l^1(G) = l^1[G]$; hence $l^\infty(G) = l^\infty[G]$ and from the definition of amenability we have the existence of an invariant mean on $l^\infty[G] = l^\infty(G)$.

Conversely, let $m \in l^\infty[G]^*$, $m \neq 0$, m being invariant; then

$$(\beta_6)\qquad m\Big(\sum_{i=1}^{n} a_i x_i f\Big) = \Big(\sum_{i=1}^{n} a_i\Big) m(f).$$

But observe that

$$(\beta_7)\qquad \Big|m\Big(\sum_{i=1}^{n} a_i x_i f\Big)\Big| \leqslant \|m\| \Big\|\sum_{i=1}^{n} a_i \delta_{x_i}\Big\|_U \|f\|_*$$

So if $a_i \geqslant 0$ and if $m(f) \neq 0$, we get

$$(\beta_8)\qquad \Big|\sum_{i=1}^{n} a_i\Big| = \Big\|\sum_{i=1}^{n} a_i \delta_{x_i}\Big\|_1 \leqslant A \Big\|\sum_{i=1}^{n} a_i \delta_{x_i}\Big\|_U,$$

where $A = \|m\| \cdot \|f\|_* \cdot |m(f)|^{-1}$. The last inequality implies that $l^1(G) = l^1[G]$ and this, by Proposition 1.3, is equivalent to the amenability of G.

3. Lacunary sets. Let E be a subset of a discrete group G; then we have the following increasing sequence of Banach spaces:

$$(\gamma_1)\qquad l^1(E) \subseteq l^1[E] \subseteq C^*_\varrho(G)_E \subseteq l^2(E).$$

We recall the definitions of special lacunary sets which were introduced by A. Figà-Talamanca and M. A. Picardello [11], [19].

E is called a *Sidon set* if $l^1(E) = C^*_\varrho(G)_E$,

and

E is called a *Leinert set* if $l^2(E) = C^*_\varrho(G)_E$.

Now we would like to propose the following two classes of sets:

E is called an *unconditional Sidon set* or a *U-Sidon set*, if $l^1[E] = C^*_\varrho(G)_E$

and

E is an *amenable set* if $l^1[E] = l^1(E)$.

By Proposition 1.3 we see that if the group G is an amenable, then every set in G is an amenable set and every unconditional Sidon set is a Sidon set.

Now we present the following characterization theorems:

THEOREM 3.1. *Let E be a subset of a discrete group G; then the following statements are equivalent:*

(a) *E is an unconditional Sidon set;*

(b) *Given $\varphi \in l^\infty(E)$, there is a $\psi \in M(A(G))$ such that $\varphi(x) = \psi(x)$ for $x \in E$ and $\|\psi\|_{M(A(G))} \leqslant \varkappa \|\varphi\|_\infty$ for some positive $\varkappa$;*

(c) $l^\infty[E] = B_\varrho(E)$;

(d) $c_0[E] = A(E)$;

(e) $l^1[E] = \mathrm{VN}(G)_E$;

(f) *There is a constant $\varkappa > 0$ such that for every $f \in L(G)_E$*

$$\|f\|_U \leqslant \varkappa \|f\|_{\mathrm{VN}};$$

(g) *For each function $d\colon E \to \{1, -1\}$, there exists a $u \in M(A(G))$ such that $\|u\|_{M(A(G))} \leqslant K$ and*

$$\sup_{x \in G} |u(x) - d(x)| \leqslant 1 - \delta \qquad (\delta > 0)$$

where K and δ depend only on E;

The proof follows by the standard considerations. For amenable groups the proof was given by J. Cygan [5] and M. A. Picardello [19].

THEOREM 3.2. *Let E be a subset of a discrete group G; then the following statements are equivalent:*

(a) *E is an amenable set;*

(b) *The restriction algebra $A(E)$ has a bounded approximate unit;*

(c) *The Leptin constant $\Omega(E)$ is finite, where*

$$\Omega(E) = \sup\{\omega(K)\colon K \subseteq E \text{ and } K \text{ is finite}\}$$

and $\omega(K) = \inf\{|KV|/|V|\colon V \text{ is finite and } V \neq \emptyset\}$.

(d) $B(E) = B_\varrho(E)$.

(e) *There is a constant $a > 0$ such that*

$$\|f\|_1 \leqslant a \|f\|_U \quad \textit{for} \quad f \in L(G)_E.$$

For the case where $E = G$ the above statements are well known (see [12], [17], [18]). The proof of our theorem follows by the adoption of the corresponding theorems in the amenable case, and so we omit it.

We recall that the set E in a discrete group G is a *weak Sidon set* if $l^\infty(E) = B(E)$. Now we can state

PROPOSITION 3.3. (a) *Every weak Sidon set is a U-Sidon set.*

(b) *Every Leinert set is a U-Sidon set.*

Proof. (a) This follows from Theorem 3.1 (b) and the fact that $B(G) \subseteq M(A(G))$.

(b) is a consequence of Theorem 3.1 (f) and the following argument: Let f have a finite support in E; then, since E is a Leinert set, we obtain for $f \in L(G)_E$

$$(\gamma_2)\qquad \|f\|_{\mathrm{VN}} \leqslant C \|f\|_2$$

for a fixed constant $C > 0$. But also $\operatorname{supp}|f| \subseteq E$, and so we get

$$(\gamma_3) \qquad \|f\|_U = \||f|\|_{VN} \leqslant C\||f|\|_2 = C\|f\|_2.$$

Since for every $f \in VN(G)$ we have

$$(\gamma_4) \qquad \|f\|_2 \leqslant \|f\|_{VN},$$

this implies that

$$(\gamma_5) \qquad \|f\|_U \leqslant C\|f\|_{VN(G)};$$

so E is a U-Sidon set.

Now we show some simple facts about the union:

PROPOSITION 3.4. *Let G be a discrete group and $E, F \subseteq G$. Then:*

(a) *If E is a Leinert set and F is a U-Sidon set, then $E \cup F$ is a U-Sidon set;*

(b) *If E and F are amenable sets, then $E \cup F$ is also an amenable set.*

Proof. (a) By [11], if E is a Leinert set, then $l^\infty(E) \subseteq M(A(G))$. Now take a U-Sidon set F; without loss of generality, we can assume that $E \cap F = \emptyset$. Let

$$(\gamma_6) \qquad \varphi \in l^\infty(E \cup F), \quad \text{and} \quad \varphi_1 = \chi_F\varphi,\ \varphi_2 = \chi_E\varphi.$$

Since F is a U-Sidon set, by Theorem 3.1 there exists a $\psi_1 \in M(A(G))$ such that $\psi_1 = \varphi_1$ on F. On the other hand, since E is a Leinert set, we can see that, by the strong property of a Leinert set, if we take $\psi_2 = \varphi_2 - \chi_E\psi_1$, then $\psi_2 \in M(A(G))$. Therefore $\psi = \psi_1 + \psi_2 \in M(A(G))$, but $\psi = \varphi$ on $E \cup F$.

(b) Now let E and F be amenable sets, and let $f \in l^1[E \cup F]$; then we can decompose $f = f_1 + f_2$, where $f_1 \in l^1[E]$, $f_2 \in l^1[F]$. The above decomposition follows from the well-known fact that, if the series is unconditionally convergent in a Banach space, then every subseries of it is also unconditionally convergent. Since $f_1 \in l^1(E)$ and $f_2 \in l^1(F)$, we have $f \in l^1(E \cup F)$ and this implies that $E \cup F$ is an amenable set.

M.A. Picardello has considered $\Lambda(2n)$ sets, i.e. $L^{2n}(\hat{G})_E = l^2(E)$ and he has proved that every weak Sidon set is a $\Lambda(2n)$ set (see [19]). We have slightly improved this theorem.

PROPOSITION 3.5. *If E is a U-Sidon set, then E is a $\Lambda(2n)$ set for every $n < \infty$.*

The proof is similar to the corresponding proof of an analogous theorem of M. A. Picardello [19], and so we omit it.

Now we present an application of the above proposition.

PROPOSITION 3.6. *Let G be a discrete group, then $l^1[G] = C_\varrho^*(G)$ if and only if G is a finite group.*

Proof. We use the following theorem of N.W. Rickert [21]:

If a group G is discrete and $A(G) \subseteq l^2(G)$, then G is a finite group.

In our case G is a U-Sidon set, so by Proposition 3.5 G is a $\Lambda(4)$ set and this implies that $A(G) = l^2(G)$, and so by the theorem of N. W. Rickert G is a finite group.

When the group G is finite, it is obvious that $l^1[G] = C_\varrho^*(G)$.

Remark 3.7. If G is a free group on two generators a and b, let $E = \{a^n b^n, a^{3^n} : n = 1, 2, 3, \ldots\}$, then by Proposition 3.4 (a) E is a U-Sidon set but it is not either Sidon set nor a Leinert set.

4. Construction of an unconditional Sidon set. In this section we give a sufficient condition for a set to be a U-Sidon set. We use the pseudonorms introduced by C. Akemann and Ph. Ostrand [1].

DEFINITION. Let $\operatorname{supp} f = E$ be a finite set, for $k = 1, 2, \ldots$,

$$(\delta_1) \qquad N_k(f) = \sup\{\|f * g\|_2 : \operatorname{supp} g \subseteq (E^{-1}E)^k\}.$$

For a set $E \subseteq G$, let $E^{(n)} = E(E^{-1}E)^{2n}$. Note that

$$(\delta_2) \qquad N_k(f) \leqslant N_{k+1}(f) \leqslant \|f\|_{VN(G)}.$$

Let us first prove

LEMMA 4.1. *Let $E = \{x_1, x_2, \ldots\}$ be a fixed countable set in a discrete group G and let $E_n = \{x_1, x_2, \ldots, x_n\}$ and take $\varepsilon_n \geqslant 0$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then there exists a sequence of integers $\{k_n\}_{n=1}^\infty$ such that*

$$(\delta_3) \qquad N_{k_n}(f) \geqslant (1 - \varepsilon_n)\|f\|_{VN}$$

for every function f with support in E_n.

Proof. It was observed by C. Akemann and Ph. Ostrand [1] that for every $g \in L(G)$ we have

$$(\delta_4) \qquad \|g\|_{VN} = \lim_k N_k(g).$$

But if we consider the set $P_n = \{f \in L(G)_{E_n} : \|f\|_{VN} = 1\}$, then P_n is a compact set, hence by the classical Dini theorem the limit in formula (δ_4) is uniform on P_n. But this is our lemma.

Now we prove the following simple lemma which gives the construction of a U-Sidon set.

LEMMA 4.2. *Let F be a finite set and let $x \notin F(F^{-1}F)^{2k}$; then, for every function f such that $\operatorname{supp} f \subseteq F$ and an arbitrary complex a, we have*

$$(\delta_5) \qquad N_k(f + a\delta_x) \geqslant N_k(f).$$

Proof. First we observe that if $\operatorname{supp} f \subseteq F$, and $\operatorname{supp} g \subseteq (F^{-1}F)^k$ and $x \notin F(F^{-1}F)^{2k}$, then by the Pythagorean theorem we have

$$(\delta_6) \qquad \|(f+a\delta_x)*g\|_2^2 = \|f*g\|_2^2 + \|ag\|_2^2 \geqslant \|f*g\|_2^2.$$

But

$$(\delta_7) \qquad N_k^2(f+a\delta_x) \geqslant \sup\{\|f*g+a\delta_x*g\|_2^2 \colon \|g\|_2^2 = 1,\ g \in L(G),\ \operatorname{supp} g \subseteq (F^{-1}F)^k\}.$$

Hence by the inequality (δ_6) we get our lemma.

DEFINITION. Let G be a discrete group; a countable set $E = \{x_1, x_2, \ldots\}$ in G is said to be *strong dissociate* if

$$(\delta_8) \qquad x_m \notin (E_n \setminus \{x_m\})^{(k_n)}$$

for every $0 < m \leqslant n$ and every $n = 1, 2, \ldots$ ($E_n = \{x_1, x_2, \ldots, x_n\}$) and $\{k_n\}$ is a sequence from Lemma 4.1 for $\varepsilon_n = 1/n^2$.

THEOREM 4.3. *Every strong dissociate set in a discrete group G is an U-Sidon set.*

Proof. Since the norm $\|\cdot\|_U$ is equivalent to the norm $\|\cdot\|_S$ on $L(G)$ by Theorem 1.2, we would like to show that for a strong dissociate set E

$$(\delta_9) \qquad \|f\|_S \leqslant C\|f\|_{VN}$$

for every $f \in L(G)_E$.

Let $\operatorname{supp} f = E_n = \{x_1, x_2, \ldots, x_n\} \subseteq E$ and let $F = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ be an arbitrary subset of E_n. Then by Lemmas 4.1 and 4.2 we have

$$(\delta_{10}) \qquad \|f\|_{VN} \geqslant N_{k_n}(f) \geqslant (1-\varepsilon_{i_1})(1-\varepsilon_{i_2})\cdot\ldots\cdot(1-\varepsilon_{i_r})\|\chi_F f\|_{VN}.$$

Since $\varepsilon_n = 1/n^2$ and $\prod_{n=1}^{\infty} 1/(1-\varepsilon_n) = 2$, we get $\|f\|_S \leqslant 2\|f\|_{VN(G)}$ for every $f \in L(G)_E$.

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