is G-ergodic at 0, and this implies the ergodicity of F at 0. We have thus obtained a contradiction, as $0 \in \mathcal{N}$. Hence $\mathcal{N} = \emptyset$ and our corollary is proved.

Piecing together Theorem 2 and Corollary 2, we get our main result:

THEOREM 4. *Let $E \subset \hat{G}$ be closed and scattered and let m be a topological mean on $L^\infty(G)$. Then the map*

$$A\colon L_E^\infty(G) \to L_E^\infty(\tilde{G})$$

defined by $(Af)\hat{}\,(\chi) = m(f\bar{\chi})$ for $f \in L_E^\infty(G)$ and $\chi \in \hat{G}$ is an isometry of Banach spaces $L_E^\infty(G)$ and $L_E^\infty(\tilde{G})$ and it does not depend on the choice of m. Almost periodic functions are fixed points of A.

We end with simple corollaries to Theorem 4. Denote by $\boldsymbol{R}$ the additive group of real numbers.

EXAMPLE 1 (cf. [5]). Let $G = \boldsymbol{R} = \hat{G}$ and let $(p_n)_1^\infty$, $(q_n)_1^\infty$ be two sequences of integers, $p_n, q_n \geqslant 2$. Let

$$E_n = \{p_1 \cdot \ldots \cdot p_n \cdot k\colon\ k = 0, \pm 1, \ldots, \pm q_n\}$$

and let $E = \bigcup_{n=1}^{\infty} E_n$. If $K \subset (-\frac{1}{2}, \frac{1}{2})$ is compact and countable, then by [5], Example (I), $L_{E+K}^\infty(G) = AP_{E+K}(G)$. It is easy to see that $E+K$ is closed and scattered; hence by Theorem 4 we get $L_{E+K}^\infty(\tilde{G}) = AP_{E+K}(G)$.

COROLLARY 3. (cf. [2], Corollary of Theorem 1). *Let $E \subset \hat{G}$ be closed, scattered and independent. Then every $f \in L_E^\infty(G)$ is a Fourier transform of a discrete measure with a support in E.*

Proof. E is Sidon in $(\hat{G})_d$, so $L_E^\infty(\tilde{G}) = AP_E(G) = l^1(E)\hat{}$. By Theorem 4 $L_E^\infty(G) = AP_E(G) = l^1(E)\hat{}$.

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Some ergodic theorems for commuting L_1 contractions

by

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Abstract. Let $T_1, T_2, \ldots, T_k$ be commuting submarkovian operators on L_1 and suppose for some $1 < p \leqslant \infty$, $\|T_i\|_p \leqslant 1$, $1 \leqslant i \leqslant k$. Then for $f \in L_1$

$$(1/n)^k \sum_{i_1=0}^{n-1} \ldots \sum_{i_k=0}^{n-1} T_1^{i_1} \ldots T_k^{i_k} f(x)$$

converges pointwise as $n \to \infty$. Also, the local ergodic theorem is proved for k-parameter semigroups of L_1 isometries.

Introduction. Let (X, Σ, μ) be a σ-finite measure space and let $L_p = L_p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, be the usual Banach spaces of complex-valued functions. A linear operator T on L_1 is *submarkovian* if it is a positive contraction ($Tf \in L_1^+$ if $f \in L_1^+$ and $\|T\|_1 \leqslant 1$). Suppose T is submarkovian and $\|T\|_p \leqslant 1$ for some $p > 1$. Akcoglu and Chacon [2] showed that

$$\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} T^i f(x)$$

exists and is finite a.e. for every $f \in L_1$. In this paper we extend their result to the case of multiple ergodic averages of k commuting submarkovian operators. In obtaining this result we generalize Akcoglu's pointwise ergodic theorem [1] to the case of k noncommuting positive L_p contractions. The final section of the paper contains a proof of the local ergodic theorem for strongly continuous semigroups of (not necessarily positive) L_1 isometries. This result provides a partial answer to the question of whether the local ergodic theorem holds for k-parameter semigroups of nonpositive L_1 contractions.

Let $\{T(t_1, \ldots, t_k)\colon t_1 > 0, \ldots, t_k > 0\}$ be a strongly measurable semigroup of L_1 contractions. In considering the question of pointwise convergence of the ergodic averages

$$A(T, a)f = (1/a)^k \int_0^a \ldots \int_0^a T(t_1, \ldots, t_k) f\, dt_1 \ldots dt_k$$

it is necessary to define $A(T, a)f(x)$ in such a way that the question makes sense. Note that the existence of the Bochner integral

$$\int_0^a \ldots \int_0^a T(t_1, \ldots, t_k) f \, dt_1 \ldots dt_k$$

is guaranteed by the strong measurability of $\{T(t_1, \ldots, t_k)\}$ and the fact that $\|T(t_1, \ldots, t_k)\|_1 \leqslant 1$. Given $f \in L_1$, it is known [5], Lemma III.11.16, that there exists a scalar function $g(t_1, \ldots, t_k, x)$, measurable with respect to the usual product measure on $R_k^+ \times X$, which is uniquely determined up to a null set by the conditions

(i) for a.e. $(t_1, \ldots, t_k)$, $g(t_1, \ldots, t_k, \cdot) = T(t_1, \ldots, t_k) f$;

(ii) for a.e. x, the function $g(\cdot, x)$ is integrable over every finite k-dimensional rectangle in R_k^+ and the integral $\int_0^{a_1} \ldots \int_0^{a_k} g(t_1, \ldots, t_k, x) \, dt_1 \ldots dt_k$, as a function of x, equals $\int_0^{a_1} \ldots \int_0^{a_k} T(t_1, \ldots, t_k) f \, dt_1 \ldots dt_k$. We define

$$A(T, a) f(x) = (1/a)^k \int_0^a \ldots \int_0^a g(t_1, \ldots, t_k, x) \, dt_1 \ldots dt_k$$

for all $a > 0$. Then $A(T, a) f(x)$ is in the equivalence class of $(1/a)^k \int_0^a \ldots \int_0^a T(t_1, \ldots, t_k) f \, dt_1 \ldots dt_k$ for all $a > 0$. Note that $A(T, a) f(x)$ depends continuously on a outside a null set which is independent of $a > 0$.

Pointwise convergence for commuting submarkovian operators.

1. LEMMA. *Let $\{T(t): t > 0\}$ be a strongly measurable submarkovian semigroup satisfying $\|T(t)\|_p \leqslant 1$ for some $p > 1$. Then*

$$\sup_{a>0} |A(T, a) f(x)| < \infty \text{ a.e.}$$

for every $f \in L_1$.

Proof. By the Riesz convexity theorem we may assume $p < \infty$. By Lemmas III.11.16 and VIII.1.3 in [5] the semigroup $\{T(t): t > 0\}$, regarded as either an L_1 or L_p semigroup, is strongly continuous. If $0 < a \in Q^+$ ($=$ the set of nonnegative rationals) then

$$A(T, a) f = s - \lim_{n\to\infty} (1/an!) \sum_{m=0}^{an!-1} T(m/n!) f, \quad f \in L_p.$$

For any $\tau > 0$

$$\Big\| \sup_{n>0} \Big| (1/n) \sum_{j=0}^{n-1} T^j(\tau) f(x) \Big| \Big\|_p \leqslant (p/(p-1)) \|f\|_p$$

by Akcoglu's estimate [1]. Employing the Cantor diagonal process and Fatou's lemma one may show that

$$\|\sup_{a\in Q^+} |A(T, a) f(x)|\|_p \leqslant (p/(p-1)) \|f\|_p, \quad f \in L_p.$$

Since $A(T, a) f(x)$ depends continuously on a for a.e. x, it follows that

$$\|\sup_{a>0} |A(T, a) f(x)|\|_p \leqslant (p/(p-1)) \|f\|_p.$$

By the argument in [8], Lemma 1, we may define $T(0)$ so that $\{T(t): t \geqslant 0\}$ is a strongly continuous L_p semigroup. Since $A(T, a) f = A(T, a) T(0) f$, $f \in L_p$, the argument in [3], p. 551, shows that $\{T(t): t \geqslant 0\}$ is also strongly continuous as an L_1 semigroup. For $f \in L_1$ and $a > 0$, set

$$E(f, a) = \{x: \sup_{0<\tau<a} A(T, \tau) f(x) > 0\}$$

and

$$E(f) = \{x: \sup_{\tau>0} A(T, \tau) f(x) > 0\}.$$

Then

$$\int_{E(f,a)} T(0) f \, d\mu \geqslant 0$$

for all $a > 0$ by Lemma 1 in [7]. An application of the Lebesgue dominated convergence theorem yields $\int_{E(f)} T(0) f \, d\mu \geqslant 0$, $f \in L_1$. Now choose $0 < h \in L_1$. Then

$$\int_{E(f-\lambda h)} T(0)(f - \lambda h) \, d\mu \geqslant 0, \quad \text{for all } \lambda.$$

Setting

$$R^*(f, h)(x) = \sup_{a>0} |A(T, a) f(x) / A(T, a) h(x)|$$

and $\mu_h(A) = \int_A T(0) h \, d\mu$, $A \in \Sigma$, and noting that

$$\{x: R^*(f, h)(x) > \lambda\} \subseteq E(|f| - \lambda h),$$

one has $\mu_h(\{R^*(f, h) > \lambda\}) \leqslant (1/\lambda) \int T(0) |f| \, d\mu$, $\lambda > 0$. Thus $R^*(f, h) < \infty$ a.e. on $\operatorname{supp} T(0) h$. If we choose $0 < h \in L_1 \cap L_p$, then $h^*(x) < \infty$ a.e. where $h^*(x) = \sup_{a>0} |A(T, a) h(x)|$. Since $f^*(x)/h^*(x) \leqslant R^*(f, h)$ on $\operatorname{supp} T(0) h$, it follows that $f^*(x) < \infty$ a.e. on $\operatorname{supp} T(0) h$. Because $T(0)$ is a positive operator and $h > 0$ a.e., $T(0) g$ vanishes a.e. outside $\operatorname{supp} T(0) h$ for any $g \in L_1$. Thus $T(t) f = T(0) T(t) f$ vanishes a.e. outside $\operatorname{supp} T(0) h$ for every $t > 0$. Consequently, $f^*(x) = 0$ a.e. on $X - \operatorname{supp} T(0) h$ and we have

$$\sup_{a>0} |A(T, a) f(x)| < \infty \text{ a.e.}, \quad f \in L_1.$$

2. Lemma. *Let $\{T(t_1, \ldots, t_k)\colon t_1 > 0, \ldots, t_k > 0\}$ be a strongly measurable submarkovian semigroup satisfying*

$$\|T(t_1, \ldots, t_k)\|_p \leqslant 1, \quad t_1 > 0, \ldots, t_k > 0$$

for some $p > 1$. Then

$$\sup_{a>0} |A(T, a)f(x)| < \infty \text{ a.e.}, \quad f \in L_1. \blacksquare$$

Proof. The proof is by induction on k. The result holds for $k = 1 = 2^0$. Suppose it holds for $k = 2^m$. Let $\{T(t_1, \ldots, t_{2k})\colon t_1 > 0, \ldots, t_{2k} > 0\}$ be a submarkovian semigroup satisfying $\|T(t_1, \ldots, t_{2k})\|_p \leqslant 1$. For $u, x > 0$, set $\varphi_u(x) = (u/2\sqrt{\pi})x^{-3/2}e^{-u^2/4x}$ and

$$S(x_1, \ldots, x_k)f = \int_0^\infty \ldots \int_0^\infty \varphi_{x_1}(t_1)\varphi_{x_2}(t_2) \ldots \varphi_{x_k}(t_{2k})T(t_1, \ldots, t_{2k})f\,dt_1 \ldots dt_{2k}.$$

It is shown in [9] that $\{S(x_1, \ldots, x_k)\colon x_1 > 0, \ldots, x_k > 0\}$ is a strongly continuous submarkovian semigroup. Since $\int_0^\infty \varphi_u(x)\,dx = 1$, it is clear that $\|S(x_1, \ldots, x_k)\|_p \leqslant 1$, $x_1 > 0, \ldots, x_k > 0$. By Lemma 2.3 in [9] (see also [5], Lemma VIII.7.13), there exists a constant $C_k > 0$, depending only on k, such that for $f \in L_1^+$

$$A(S, \sqrt{a})f(x) \geqslant C_k A(T, a)f(x), \quad a > 0.$$

Since $\sup_{a>0} A(S, a)f(x) < \infty$ a.e. by assumption, we have

$$\sup_{a>0} |A(T, a)f(x)| < \infty \text{ a.e.}, \quad f \in L_1.$$

This establishes the result for $k = 2^m$, $m = 0, 1, 2, \ldots$ For $k \neq 2^m$, choose m so that $2^m > k$ and set

$$S(t_1, \ldots, t_{2^m}) = T(t_1, \ldots, t_k).$$

Then $A(S, a)f(x) = A(T, a)f(x)$ a.e. for all $a > 0$ and thus the lemma holds for all k. $\blacksquare$

The next theorem generalizes Akcoglu's pointwise ergodic theorem [1] to the case of multiple ergodic averages of noncommuting L_p contractions.

3. Theorem. *For some $1 < p < \infty$, let $T_1, \ldots, T_k$ be positive L_p contractions. If $f \in L_p$, then*

$$\frac{1}{n_1 \ldots n_k} \sum_{i_1=0}^{n_1-1} \ldots \sum_{i_k=0}^{n_k-1} T_1^{i_1} \ldots T_k^{i_k} f(x)$$

converges a.e. *to a finite limit as $n_1 \to \infty, \ldots, n_k \to \infty$ independently.*

Proof. The proof is by induction on k. The case $k = 1$ is Akcoglu's theorem. Assume the theorem has been established for $k-1$ operators. For $f \in L_p$, set

$$A_i(n)f = (1/n)\sum_{j=0}^{n-1} T_i^j f$$

and

$$M_i f(x) = \sup_{n>0} |A_i(n)f(x)|,$$

$1 \leqslant i \leqslant k$. By repeated application of Akcoglu's dominated estimate, one obtains

$$\|M_1 \ldots M_k f\|_p \leqslant (p/(p-1))^k \|f\|_p, \quad f \in L_p.$$

Thus

$$\sup_{n_1>0, \ldots, n_k>0} |A_1(n_1) \ldots A_k(n_k)f(x)| < \infty \text{ a.e.}, \quad f \in L_p.$$

Hence Banach's convergence principle may be applied to establish the result for k operators provided we can show that the limit exists for f in a dense subclass of L_p. Functions of the form

$$f + (I - T_k)g,$$

where $f, g \in L_p$ and $T_k f = f$, are dense in L_p by a corollary to the mean ergodic theorem [5], Corollary VIII. 5.2. We show that

$$\lim A_1(n_1) \ldots A_k(n_k)[f(x) + (I - T_k)g(x)] = \lim A_1(n_1) \ldots A_{k-1}(n_{k-1})f(x) \text{ a.e.}$$

Let

$$g_m = \sup_{i \geqslant m} |T_k^i g/i|, \quad m = 1, 2, \ldots$$

Since

$$\int [g_m(x)]^p\,d\mu \leqslant \int \Big[\sum_{i \geqslant m} (|T_k^i g|/i)^p\Big]\,d\mu = \sum_{i \geqslant m} \int (|T_k^i g|/i)^p\,d\mu \leqslant \|g\|_p^p \cdot \sum_{i \geqslant m} (1/i)^p$$

for every $m > 0$, it is clear that $\|g_m\|_p \to 0$ and consequently that $g_m(x) \searrow 0$ a.e. as $m \to \infty$. Setting

$$g_m^*(x) = M_1 \ldots M_{k-1} g_m(x),$$

we have

$$\|g_m^*(x)\|_p \leqslant (p/(p-1))^{k-1} \|g_m\|_p.$$

Again we have $\|g_m^*\|_p \to 0$ which implies $g_m^*(x) \searrow 0$ a.e. as $m \to \infty$. Thus

$$A_1(n_1) \ldots A_{k-1}(n_{k-1})[T_k^{n_k} g(x)/n_k] \to 0 \text{ a.e.}$$

as $n_1 \to \infty, \ldots, n_k \to \infty$ independently since

$$|A_1(n_1) \ldots A_{k-1}(n_{k-1})[T_k^{n_k} g(x)/n_k]| \leqslant g^*_{n_k}(x)$$

for all $n_1, \ldots, n_k$.

Applying the above argument to the special case $T_k = I$, we get

$$A_1(n_1) \ldots A_{k-1}(n_{k-1})[g(x)/n_k] \to 0 \text{ a.e.}$$

as $n_1 \to \infty, \ldots, n_k \to \infty$ independently. Hence

$$\begin{aligned}\lim A_1(n_1) \ldots A_k(n_k)[f(x)+(I-T_k)g(x)] &= \lim A_1(n_1) \ldots A_{k-1}(n_{k-1})[f(x)+g(x)/n_k - T_k^{n_k} g(x)/n_k] \\ &= \lim A_1(n_1) \ldots A_{k-1}(n_{k-1}) f(x) \text{ a.e.}\end{aligned}$$

By Banach's principle [5], Theorem IV. 11. 3,

$$A_1(n_1) \ldots A_k(n_k) f(x)$$

converges a.e. for every $f \in L_p$. ■

4. THEOREM. *Let $T_1, \ldots, T_k$ be commuting submarkovian operators satisfying $\|T_i\|_p \leqslant 1$, $1 \leqslant i \leqslant k$, for some $p > 1$. Then*

$$\lim_{n\to\infty} (1/n)^k \sum_{i_1=0}^{n-1} \ldots \sum_{i_k=0}^{n-1} T_1^{i_1} \ldots T_k^{i_k} f(x)$$

exists a.e. for $f \in L_1$.

Proof. For $x_1 \geqslant 0, \ldots, x_k \geqslant 0$, set

$$S(x_1, \ldots, x_k) = e^{-(x_1+\ldots+x_k)} \sum_{i_1=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} \frac{x_1^{i_1} \ldots x_k^{i_k}}{i_1! \ldots i_k!} T_1^{i_1} \ldots T_k^{i_k}.$$

Then $\{S(x_1, \ldots, x_k): x_1 \geqslant 0, \ldots, x_k \geqslant 0\}$ is a strongly continuous submarkovian semigroup satisfying

$$\|S(x_1, \ldots, x_k)\|_p \leqslant 1, \quad x_1 \geqslant 0, \ldots, x_k \geqslant 0.$$

Given $f \in L_1^+$ and $n > 0$ there exists $C_k > 0$, depending only on k, and $\alpha = \alpha(n, f)$, such that

$$A(S, \alpha)f(x) \geqslant C_k A(T, n) f(x) \text{ a.e.,}$$

where

$$A(T, n)f(x) = (1/n)^k \sum_{i_1=0}^{n-1} \ldots \sum_{i_k=0}^{n-1} T_1^{i_1} \ldots T_k^{i_k} f(x).$$

Since $\sup A(S, \alpha)f(x) < \infty$ a.e. by Lemma 2, we have

$$\sup_{n>0} A(T, n)f(x) < \infty \text{ a.e.,} \quad f \in L_1^+.$$

The existence of C_k is established in [5], p. 703. By Theorem 3, $\lim_{n\to\infty} A(T, n)f(x)$ exists a.e. for $f \in L_1 \cap L_p$ which is dense in L_1. By Banach's principle

$$\lim_{n\to\infty} A(T, n)f(x)$$

exists a.e. for $f \in L_1$. ■

It should be noted that the continuous parameter analogues of Theorems 3 and 4 are true and may be established by similar arguments.

A local ergodic theorem for semigroups of L_1 isometrics. In this section the local ergodic theorem for k-parameter semigroups of L_1 isometries is proved. A linear operator T is an L_1 isometry if $\|Tf\|_1 = \|f\|_1$ for every $f \in L_1$.

5. THEOREM. *Let $\{T(t_1, \ldots, t_k): t_1 \geqslant 0, \ldots, t_k \geqslant 0\}$ be a strongly continuous k-parameter semigroup of L_1 isometries. Then*

$$\lim_{\alpha \searrow 0} A(T, \alpha)f(x) = f(x) \text{ a.e.,} \quad f \in L_1.$$

Proof. Let $\tau(t_1, \ldots, t_k)$ denote the linear modulus [4] of $T(t_1, \ldots, t_k)$. Since each $\tau(t_1, \ldots, t_k)$ is an isometry, $\{\tau(t_1, \ldots, t_k): t_1 \geqslant 0, \ldots, t_k \geqslant 0\}$ is strongly continuous by the corollary in [6], p. 371. Clearly

$$\tau(t_1, \ldots, t_k) \cdot \tau(s_1, \ldots, s_k) f(x) \geqslant \tau(t_1+s_1, \ldots, t_k+s_k) f(x) \text{ a.e.}$$

for every $f \in L_1^+$. Therefore we must have

$$\tau(t_1, \ldots, t_k) \cdot \tau(s_1, \ldots, s_k) f(x) = \tau(t_1+s_1, \ldots, t_k+s_k) f(x) \text{ a.e.}$$

for every $f \in L_1^+$ since $\tau(t_1+s_1, \ldots, t_k+s_k)$ is a positive isometry. Consequently $\{\tau(t_1, \ldots, t_k): t_1 \geqslant 0, \ldots, t_k \geqslant 0\}$ is a strongly continuous k-parameter semigroup of positive L_1 isometries. Since the local ergodic theorem holds for $\{\tau(t_1, \ldots, t_k)\}$ [9], it follows that

$$\sup_{0<\alpha<1} A(\tau, \alpha)f(x) < \infty \text{ a.e.,} \quad f \in L_1^+.$$

Noting that

$$A(\tau, \alpha)|f(x)| \geqslant |A(T, \alpha)f(x)|, \quad \alpha > 0, \quad f \in L_1,$$

we see that

$$\sup_{0<\alpha<1} |A(T, \alpha)f(x)| < \infty \text{ a.e.,} \quad f \in L_1.$$

Because $T(t_1, \ldots, t_k)$ is strongly continuous at $t_1 = t_2 = \ldots = t_k = 0$ (for simplicity we assume $T(0, \ldots, 0) = I$), the class $\mathfrak{M} = \{A(T, \alpha)f: \alpha > 0, f \in L_1\}$ is dense in L_1. The argument in [9], p. 266 shows that

$$\lim_{\alpha \searrow 0} A(t, \alpha)f(x) = f(x) \text{ a.e.} \quad f \in \mathfrak{M}.$$

By Banach's principle [5], Theorem IV. 11. 3

$$A(T, a)f(x) \to f(x) \text{ a.e.}, \quad f \in L_1,$$

as $a \searrow 0$ through Q^+. Since $A(T, a)f(x)$ depends continuously on a a.e. it follows that

$$\lim_{a \searrow 0} A(T, a)f(x) = f(x) \text{ a.e.}, \quad f \in L_1. \blacksquare$$

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Addendum to the paper "Weak-strong convolution operators on certain disconnected groups"

by

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Abstract. In [1] G.I. Gaudry and the author obtained several results concerning L^p convolution operators and multipliers on a totally disconnected group where the indices of sucessive subgroups remain bounded. More specifically, estimates were obtained for kernels (resp. multipliers) having a strong singularity at the origin (resp. at infinity). In this note we show how to extend the results of [1] to the case where the indices are unbounded, and in doing so answer a question implicit in the work of Peyrière and Spector [2].

1. Introduction. Let G denote a compact abelian group having the following properties:

(i) there exists a strictly decreasing sequence $\{G_n\}_{n=0}^{\infty}$ of open compact subgroups of G such that the index $G_{n+1}: G_n$ of G_{n+1} in G_n is finite;

(ii) $\bigcup G_n = G$ and $\bigcap G_n = \{0\}$;

(iii) $|G_0| = 1$ where $|S|$ denotes the Haar measure of a (measurable) set S;

(iv) $|G_n| \cdot |G_{n-1}|^{-1} \downarrow 0$.

Let Γ denote the dual group of G and Γ_n the annihilator of G_n in Γ. Then $\{\Gamma_n\}$ is an increasing sequence of open compact subgroups of Γ and $\Gamma_n : \Gamma_{n+1} = G_{n+1} : G_n$. Such groups divide naturally into two classes: (a) where $G_{n+1} : G_n \leqslant b$ for some positive integer $b \geqslant 2$, and (b) where $G_{n+1}: G_n \to \infty$. Groups satisfying (a) were treated in [1] and from now on we shall suppose that (b) holds.

We refer the reader to [1] for all the required definitions and notation.

2. Convolution estimates. The following result takes the place of Theorems 2.1 and 2.2 of [1]. (There is no real need to consider the case $\theta > 0$ of [1].)

THEOREM 1. *Suppose $k \in L^1$. If*

$$|\hat{k}(\gamma)| \leqslant B\{|G_{n+1}|/|G_n|\}^{1/2}, \quad \gamma \in \Gamma_{n+1} \setminus \Gamma_n \tag{1}$$