STUDIA MATHEMATICA, T. LXX. (1981)

# On functions with scattered spectra on lca groups

by

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**Abstract.** Let $G$ be a locally compact abelian group. For a closed subset $E$ of the dual group $\hat{G}$ we denote by $L_E^\infty(G)$ the space of all essentially bounded measurable functions with their Fourier transforms (which are pseudomeasures on $\hat{G}$) supported by $E$. As $\hat{G}$ with a discrete topology is a dual of $\tilde{G}$, the Bohr compactification of $G$, we may consider $L_E^\infty(\tilde{G})$ as well. Our main theorem is that if $E$ is closed and scattered, that is, if it contains no non-empty perfect subset, then the Banach spaces $L_E^\infty(G)$ and $L_E^\infty(\tilde{G})$ are isometric. This isometry is canonical in the following sense: if $m$ is a topological mean on $G$ and if $F \in L_E^\infty(\tilde{G})$ corresponds to $f \in L_E^\infty(G)$ by this map, then $\hat{F}(\chi) = m(f\overline{\chi})$ for every $\chi \in \hat{G}$.

**Introduction.** For a given $f \in L^\infty(G)$ ($G$ being locally compact abelian group) we consider its "Fourier series" with respect to some topological mean $m$ on $G$:

$$(*) \qquad \sum_{\chi \in \hat{G}} m(f\overline{\chi}) \cdot \chi.$$

It is not difficult to see that there are only countably many $\chi$ such that $m(f\overline{\chi}) \neq 0$ and, moreover, there exists an $F \in L^\infty(\tilde{G})$, $\tilde{G}$ being the Bohr compactification of $G$, such that $(*)$ is just the Fourier series of $F$. In general, of course $(*)$ depends on the choice of the mean $m$ and does not determine $f$ uniquely. But this turns out to be the case under some additional conditions on $f$, e.g. if the spectrum of $f$ is scattered, as we shall see later; so if $E \subset \hat{G}$ is closed and scattered, we obtain an injective map:

$$L_E^\infty(G) \ni f \to F \in L_E^\infty(\tilde{G}).$$

It has been proved by Mrs. Lust-Piquard that this map is an isometry of the Banach spaces $L_E^\infty(G)$ and $L_E^\infty(\tilde{G})$ for $E$ countable and discrete. We generalize this to an arbitrary scattered set $E$. Thus functions with scattered spectra appear to be in some sense similar to almost periodic functions, as they have a kind of extension to $\tilde{G}$. Our result is based on a theorem of Woodward ([6], Theorem 9 (ii)); we extend the notion of ergodicity introduced in his paper, but our extension is different from that given in [4], as we use topological means only. Nevertheless the present results are substantially related to these of [4] and [6].

**Preliminaries.** Let $G$ be a locally compact abelian group, $\hat{G}$ its dual and $\tilde{G}$ its Bohr compactification. The dual of $\tilde{G}$ is $\hat{G}$ with a discrete topology: we denote it by $(\hat{G})_d$. Let $L^\infty(G)$ be the usual Banach space of all Haar-measurable, essentially bounded functions on $G$ (with a "supess" norm $\|\cdot\|_\infty$). $C(G)$ and $AP(G)$ are its closed subspaces consisting of continuous functions and almost periodic functions, respectively. $C(\tilde{G})$ may be identified in a standard way with $AP(G)$. For $f \in L^\infty(G)$ we denote by $\sigma(f) \subset \hat{G}$ the spectrum of $f$ in $L^\infty(G)$, i.e. the support of the pseudo-measure $\hat{f}$ on $\hat{G}$. By $M(G)$ we denote the Banach algebra (with respect to convolution $*$) of finite regular Borel measures on $G$ and by $L^1(G)$ its subalgebra of Haar-integrable functions on $G$.

Let us now fix some Haar measure $dx$ on $G$. Put

$$\mathscr{P}(G) = \Big\{u \in L^1(G)\colon\ u \geqslant 0,\ \int_G u(x)\,dx = 1\Big\}.$$

Note that $\mathscr{P}(G)$ is a semigroup with respect to convolution.

Let $(k_a)$ be the Fejér averaging kernel on $G$ as constructed in [6], page 285. It has the following properties:

(1) $k_a \in \mathscr{P}(G)$.

(2) $\|k_a * u - k_a\|_{L^1(G)} \to 0$ for any $u \in \mathscr{P}(G)$.

There exists a natural homomorphism $\varrho\colon M(G) \to M(\tilde{G})$ given by:

$$\int_{\tilde{G}} \varphi\, d\varrho(\mu) = \int_G \varphi\, d\mu \quad \text{for} \quad \mu \in M(G),\ \varphi \in C(\tilde{G}) = AP(G).$$

As $\varrho(\mu)\hat{\ }(\chi) = \hat{\mu}(\chi)$ for $\chi \in \hat{G}$, we have $\varrho(\mu * \nu) = \varrho(\mu) * \varrho(\nu)$ for $\mu, \nu \in M(G)$.

$\mathscr{P}(G)$ acts by convolution on $L^\infty(G)$ as well as on $L^\infty(\tilde{G})$:

$$L^\infty(G) \ni f \to u * f \in L^\infty(G),$$

$$L^\infty(\tilde{G}) \ni F \to \varrho(u) * F \in L^\infty(\tilde{G})$$

for $u \in \mathscr{P}(G)$.

For $f \in L^\infty(G)$ $(F \in L^\infty(\tilde{G}))$ denote by $\bar{\mathscr{P}}(f)$ $(\bar{\mathscr{P}}(F))$ the norm closure of the orbit of $f$ $(F)$ under $\mathscr{P}(G)$. It is a closed convex subset of $L^\infty(G)$ $(L^\infty(\tilde{G}))$. In both cases, owing to (1) and (2), the assumption of Eberlein's Theorem ([1], th. 3.1) is fulfilled. Thus we have:

PROPOSITION 1. *Let $f \in L^\infty(G)$ $(F \in L^\infty(\tilde{G}))$ and let $c$ $(C)$ be a constant in $L^\infty(G)$ $(L^\infty(\tilde{G}))$. Then the following statements are equivalent:*

(i) $c \in \bar{\mathscr{P}}(f)$ $(C \in \bar{\mathscr{P}}(F))$.

(ii) $\|k_a * f - c\|_\infty \to 0$ $(\|\varrho(k_a) * F - C\|_\infty \to 0)$.



*Moreover, if $(k_a * f)$ $((\varrho(k_a) * F))$ is Cauchy in $L^\infty(G)$ $(L^\infty(\tilde{G}))$, then there exists a $c$ $(C)$ such that* (i) *and* (ii) *hold.*

Let $m$ be a topological mean on $L^\infty(G)$ that is a linear, positive, normed functional invariant under the action of $\mathscr{P}(G)$. Using $m$, we can identify the constant $c$ in Proposition 1. In fact, if $k_a * f \to c$ in $L^\infty(G)$, then $m(f) = m(k_a * f) \to m(c) = c$, so $c = m(f)$. Notice that $\int_{\tilde{G}} (\varrho(u) * F)(x)\,dx = (\varrho(u) * F)\hat{\ }(0) = \hat{u}(0)\cdot\hat{F}(0) = \int_{\tilde{G}} F(x)\,dx$ for $u \in \mathscr{P}(G)$ and $F \in L^\infty(\tilde{G})$. Thus in the bracket version of Proposition 1 we obtain $C = \hat{F}(0) = \int_{\tilde{G}} F(x)\,dx$.

**Topologically ergodic functions.** We begin with introducing the notion of (topological) ergodicity for $f \in L^\infty(G)$ and (topological) $G$-ergodicity for $F \in L^\infty(\tilde{G})$.

DEFINITION 1. We call $f \in L^\infty(G)$ $(F \in L^\infty(\tilde{G}))$ *ergodic* (*$G$-ergodic*) at $\chi \in \hat{G}$ if $f\bar{\chi}$ $(F\bar{\chi})$ fulfils one of the equivalent conditions of Proposition 1. If $f(F)$ is ergodic ($G$-ergodic) at every $\chi \in \hat{G}$, we call it *ergodic* (*$G$-ergodic*).

Remark 1. Let $f \in L^\infty(G)$ $(F \in L^\infty(\tilde{G}))$. If $\chi \in \hat{G}$ is not a cluster point of $\sigma(f)$ $(\sigma(F))$ in $\hat{G}$, then $f(F)$ is ergodic ($G$-ergodic) at $\chi$.

Proof. Let $V$ be a neighbourhood of 0 in $G$ such that $(\chi + V) \cap \sigma(f) \subset \{\chi\}$ $((\chi + V) \cap \sigma(F) \subset \{\chi\})$. Let $a_0$ be such that the support of $\hat{k}_a$ is contained in $V$ for $a > a_0$. Then $k_a * f\bar{\chi} = c$ $(\varrho(k_a) * F\bar{\chi} = C)$ for $a > a_0$ and some complex $c, C$. Since $(k_a * f\bar{\chi})_{a > a_0}$ $((\varrho(k_a) * F\bar{\chi})_{a > a_0})$ is constant, it is Cauchy.

Remark 2. If $\chi$ is an isolated point of $\sigma(f)$ and $m$ is a topological mean on $L^\infty(G)$, then $m(f\bar{\chi}) \neq 0$.

Following [4], we introduce maps $A_m$ and $B_\omega$:

DEFINITION 2. Let $m$ be a topological mean on $L^\infty(G)$. For $f \in L^\infty(G)$ we define $A_m f \in L^\infty(\tilde{G})$ as a functional on $L^1(\tilde{G})$. It is sufficient to define it on $C(\tilde{G}) = AP(G)$ and then to show that it is continuous in $L^1(\tilde{G})$-norm. So let

$$\langle A_m f, \varphi\rangle = m(f\varphi)$$

for $\varphi \in C(\tilde{G})$. Evidently $|\langle A_m f, \varphi\rangle| \leqslant \|f\|_\infty m(|\varphi|) = \|f\|_\infty \|\varphi\|_{L^1(\tilde{G})}$.

It is easy to check the following properties of $A_m\colon L^\infty(G) \to L^\infty(\tilde{G})$ (cf. [4], II.2. Lemma 3):

PROPOSITION 2.

(i) *$A_m$ is linear with norm equal to* 1.

(ii) *$(A_m f)\hat{\ }(\chi) = m(f\bar{\chi})$ for $f \in L^\infty(G)$, $\chi \in \hat{G}$.*

(iii) *$A_m \varphi = \varphi$ for $\varphi \in AP(G)$.*

(iv) *$A_m(\mu * f) = \varrho(\mu) * A_m f$ for $f \in L^\infty(G)$, $\mu \in M(G)$.*

(v) $A_m(f\chi) = A_m f\cdot\chi$ *for* $f \in L^\infty(G)$, $\chi \in \hat{G}$.

(vi) $\sigma(A_m f) \subset \sigma(f)$ *for* $f \in L^\infty(G)$.

DEFINITION 3. Let $\omega$ be an extension of $\delta_0 \in C(\tilde{G})^*$ ($\delta_0(\varphi) = \varphi(0)$ for $\varphi \in C(\tilde{G})$) to a normed functional on $L^\infty(\tilde{G})$. For $F \in L^\infty(\tilde{G})$ we define $B_\omega F \in L^\infty(G) = L^1(G)^*$ as follows:

$$\langle B_\omega F, u\rangle = \langle \omega, \varrho(\check{u}) * F\rangle$$

for $u \in L^1(G)$ (we put $\check{u}(x) = u(-x)$).

It is clear that $B_\omega F \in L^\infty(G)$ and $\|B_\omega F\|_\infty \leqslant \|F\|_\infty$.

PROPOSITION 3 (cf. [4], II.1). $B_\omega$: $L^\infty(\tilde{G}) \to L^\infty(G)$ *has the following properties*:

(i) $B_\omega$ *is linear with norm equal to* 1.

(ii) $B_\omega \varphi = \varphi$ *for* $\varphi \in C(\tilde{G})$.

(iii) $B_\omega(\varrho(\mu) * F) = \mu * B\omega F$ *for* $F \in L^\infty(\tilde{G})$, $\mu \in M(G)$.

(iv) $\sigma(B_\omega F) \subset \overline{\sigma(F)}$ *for* $F \in L^\infty(\tilde{G})$.

We omit an easy verification of (i)–(iv).

**The main result.** Let $m_0$ be a *weak cluster point of $(k_a)_a$ in $L^\infty(G)^*$. Taking a subnet, we may assume that $m_0 = \lim_a k_a$ in the *weak sense. Put $A_{m_0} = A$ and fix some $\omega$ as in Definition 3.

THEOREM 1. *Let* $F \in L^\infty(\tilde{G})$ *be* $G$*-ergodic at* $\chi \in \hat{G}$. *Then*

$$\big(A(B_\omega F)\big)\hat{}\,(\chi) = \hat{F}(\chi).$$

Proof. By (iii) of Prop. 2, the definition of $m_0$ and the $G$-ergodicity of $F$ at $\chi$ (Prop. 1 (ii)) we have:

$$\begin{aligned}A(B_\omega F)\hat{}\,(\chi) &= m_0(B_\omega F\overline{\chi}) = \lim_a \langle B_\omega F\overline{\chi}, k_a\rangle \\ &= \lim_a \langle B_\omega F, k_a\overline{\chi}\rangle = \lim_a \langle \omega, \varrho(k_a)\chi * F\rangle \\ &= \lim_a \big\langle \omega, \big(\varrho(k_a) * F\overline{\chi}\big)\chi\big\rangle = \langle \omega, \hat{F}(\chi)\chi\rangle \\ &= \hat{F}(\chi)\langle\omega, \chi\rangle = \hat{F}(\chi)\chi(0) = \hat{F}(\chi).\end{aligned}$$

COROLLARY 1. *If* $F \in L^\infty(\tilde{G})$ *is* $G$*-ergodic, then* $A(B_\omega F) = F$.

Proof. In fact, we have $A(B_\omega F)\hat{}\,(\chi) = \hat{F}(\chi)$ for every $\chi \in \hat{G}$ so $A(B_\omega F) = F$.

We call a closed subset of $\hat{G}$ *scattered* if it does not contain any non-empty perfect subset. We have the following

PROPOSITION 4. *Let* $f \in L^\infty(G)$. *If* $\sigma(f)$ *is contained in a closed, scattered subset* $E$ *of* $\hat{G}$, *then* $f$ *is ergodic* (*at every* $\chi \in \hat{G}$).



Proof. As $\sigma(f\overline{\chi}) = \sigma(f) + \chi$ and since translations are homeomorphisms, it is sufficient to show that $f$ is ergodic at 0. To prove our proposition take $u \in \mathscr{P}(G)$ such that the support of $\hat{u}$ is compact. Then the support of $(u*f)\hat{}$ is compact and contained in $E$ and thus by the Loomis theorem ([3], Th. 4) $u*f$ is almost periodic and, a fortiori, ergodic at 0 (Prop. 1). This means that $(k_a * u * f)_a$ is Cauchy. But by (1) we have

$$\|k_a * f - k_a * u * f\|_\infty \to 0$$

and so $(k_a * f)_a$ is Cauchy.

Now let us restrict $A$ to $L_E^\infty(G)$ (the space of all $f \in L^\infty(G)$ with $\sigma(f) \subset E$) where $E$ is closed and scattered.

THEOREM 2. *Let* $A$ *be as defined at the beginning of this section and let* $E \subset G$ *be closed and scattered. Then* $A$ *is an isometry from* $L_E^\infty(G)$ *onto* $G$*-ergodic elements of* $L_E^\infty(\tilde{G})$.

Proof. By Prop. 2 (iv)–(vi) $A$ maps $L_E^\infty(G)$ into $G$-ergodic elements of $L_E^\infty(\tilde{G})$. Take $F \in L_E^\infty(\tilde{G})$ which is $G$-ergodic. Cor. 1 shows that $A(B_\omega F) = F$ and so by Prop. 3 (iv) $A$ maps $L_E^\infty(G)$ onto $G$-ergodic members of $L_E^\infty(\tilde{G})$. Suppose now that $Af = 0$ for some $f \in L_E^\infty(G)$. This implies $(Af)\hat{}\,(\chi) = m_0(f\overline{\chi}) = 0$ for every $\chi \in \hat{G}$. By Remark 2 $\sigma(f) = \emptyset$ and so $f = 0$. Thus $A$ is injective on $L_E^\infty(G)$. It is isometric by the above and Prop. 2 (i), Prop. 3 (i).

Our last step is to show that every $F \in L_E^\infty(\tilde{G})$ is $G$-ergodic for $E$ closed and scattered. Theorem 3 is just an adaptation of [6], Th. 9 (ii).

THEOREM 3. *Let* $F \in L^\infty(\tilde{G})$ *be* $G$*-ergodic at every* $\chi \neq 0$. *Then* $F$ *is* $G$*-ergodic also at* 0.

Proof. Take $u \in \mathscr{P}(G)$. Of course, $\varrho(u) * F$ is $G$-ergodic at $\chi \neq 0$ and it is $G$-ergodic at 0 exactly when $F$ is. Using Theorem 1 and Prop. 3 (iii), we have

$$\begin{aligned}A(u * B_\omega F)\hat{}\,(\chi) &= A\big(B_\omega(\varrho(u) * F)\big)\hat{}\,(\chi) \\ &= \big(\varrho(u) * F\big)\hat{}\,(\chi) = \hat{u}(\chi)\hat{F}(\chi) \quad \text{for } \chi \neq 0.\end{aligned}$$

Thus $\varrho(u) * F = A(u * B_\omega F) + \text{const}$. Being an image of $\varrho(u) * F$ by $B_\omega$ $u * B_\omega F$ is ergodic at every $\chi \neq 0$ by Prop. 3(iii). As it is uniformly continuous, we may apply [6] Theorem 9(ii) to obtain the $G$-ergodicity of $A(u * B_\omega F)$ at 0. Of course the same is true for $\varrho(u) * F$ and hence for $F$.

COROLLARY 2. *Let* $F \in L^\infty(\tilde{G})$. *If* $\sigma(F)$ *is contained in a closed and scattered set* $E \subset \hat{G}$, *then* $F$ *is* $G$*-ergodic*.

Proof. In fact, let $\mathscr{N} = \{\chi \in \hat{G}\colon F \text{ is not } G\text{-ergodic at } \chi\}$. By Remark 1, $\mathscr{N} \subset E$, and so it either is empty or contains an isolated point. Suppose that there is an isolated point $\chi_0$ in $\mathscr{N}$ (we may assume that $\chi_0 = 0$). Let $u \in \mathscr{P}(G)$ be such that $\operatorname{supp}\hat{u} \cap \mathscr{N} = \{0\}$. (By $\operatorname{supp}\hat{u}$ we denote the support of $\hat{u}$.) Then $\varrho(u) * F$ is $G$-ergodic at $\chi \neq 0$, and so by Theorem 3 $\varrho(u) * F$

is $G$-ergodic at 0, and this implies the ergodicity of $F$ at 0. We have thus obtained a contradiction, as $0 \in \mathscr{N}$. Hence $\mathscr{N} = \emptyset$ and our corollary is proved.

Piecing together Theorem 2 and Corollary 2, we get our main result:

THEOREM 4. *Let $E \subset \hat{G}$ be closed and scattered and let $m$ be a topological mean on $L^\infty(G)$. Then the map*

$$A\colon L_E^\infty(G) \to L_E^\infty(\tilde{G})$$

*defined by $(Af)^\wedge(\chi) = m(f\bar{\chi})$ for $f \in L_E^\infty(G)$ and $\chi \in \hat{G}$ is an isometry of Banach spaces $L_E^\infty(G)$ and $L_E^\infty(\tilde{G})$ and it does not depend on the choice of $m$. Almost periodic functions are fixed points of $A$.*

We end with simple corollaries to Theorem 4. Denote by $\boldsymbol{R}$ the additive group of real numbers.

EXAMPLE 1 (cf. [5]). Let $G = \boldsymbol{R} = \hat{G}$ and let $(p_n)_1^\infty$, $(q_n)_1^\infty$ be two sequences of integers, $p_n, q_n \geqslant 2$. Let

$$E_n = \{p_1 \cdot \ldots \cdot p_n \cdot k\colon k = 0, \pm 1, \ldots, \pm q_n\}$$

and let $E = \bigcup_{n=1}^\infty E_n$. If $K \subset (-\frac{1}{2}, \frac{1}{2})$ is compact and countable, then by [5], Example (I), $L_{E+K}^\infty(G) = AP_{E+K}(G)$. It is easy to see that $E+K$ is closed and scattered; hence by Theorem 4 we get $L_{E+K}^\infty(\tilde{G}) = AP_{E+K}(G)$.

COROLLARY 3. (cf. [2], Corollary of Theorem 1). *Let $E \subset \hat{G}$ be closed, scattered and independent. Then every $f \in L_E^\infty(G)$ is a Fourier transform of a discrete measure with a support in $E$.*

Proof. $E$ is Sidon in $(\hat{G})_d$, so $L_E^\infty(\tilde{G}) = AP_E(G) = l^1(E)^\wedge$. By Theorem 4 $L_E^\infty(G) = AP_E(G) = l^1(E)^\wedge$.

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# Some ergodic theorems for commuting $L_1$ contractions

by

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**Abstract.** Let $T_1, T_2, \ldots, T_k$ be commuting submarkovian operators on $L_1$ and suppose for some $1 < p \leqslant \infty$, $\|T_i\|_p \leqslant 1$, $1 \leqslant i \leqslant k$. Then for $f \in L_1$

$$(1/n)^k \sum_{i_1=0}^{n-1} \ldots \sum_{i_k=0}^{n-1} T_1^{i_1} \ldots T_k^{i_k} f(x)$$

converges pointwise as $n \to \infty$. Also, the local ergodic theorem is proved for $k$-parameter semigroups of $L_1$ isometries.

**Introduction.** Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $L_p = L_p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, be the usual Banach spaces of complex-valued functions. A linear operator $T$ on $L_1$ is *submarkovian* if it is a positive contraction ($Tf \in L_1^+$ if $f \in L_1^+$ and $\|T\|_1 \leqslant 1$). Suppose $T$ is submarkovian and $\|T\|_p \leqslant 1$ for some $p > 1$. Akcoglu and Chacon [2] showed that

$$\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} T^i f(x)$$

exists and is finite a.e. for every $f \in L_1$. In this paper we extend their result to the case of multiple ergodic averages of $k$ commuting submarkovian operators. In obtaining this result we generalize Akcoglu's pointwise ergodic theorem [1] to the case of $k$ noncommuting positive $L_p$ contractions. The final section of the paper contains a proof of the local ergodic theorem for strongly continuous semigroups of (not necessarily positive) $L_1$ isometries. This result provides a partial answer to the question of whether the local ergodic theorem holds for $k$-parameter semigroups of nonpositive $L_1$ contractions.

Let $\{T(t_1, \ldots, t_k)\colon t_1 > 0, \ldots, t_k > 0\}$ be a strongly measurable semigroup of $L_1$ contractions. In considering the question of pointwise convergence of the ergodic averages

$$A(T, a)f = (1/a)^k \int_0^a \ldots \int_0^a T(t_1, \ldots, t_k) f\, dt_1 \ldots dt_k$$

it is necessary to define $A(T, a)f(x)$ in such a way that the question makes