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Gleason measures and some inner products on the algebra of bounded operators on a Hilbert space

by

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Abstract. This paper represents a synthesis of the efforts of the authors to relate the study of Gleason measures to operator theory. We will begin by using results on Gleason measures to characterize certain inner products on $\mathscr{L}(\mathscr{H})$ — the algebra of bounded linear operators on the Hilbert space $\mathscr{H}$. Then we show that an algebraic condition on the inner product gives an equivalent characterization, which furthermore holds in the case where Gleason Theorem fails. The proof proceeds via results in operator theory, and it is our hope that these results may suggest a new direction of attack on the problem of Gleason measures [1], [2], [4].

§ 1. Let us begin with some preliminaries.

1.1. DEFINITION. Let $\mathscr{H}, \mathscr{K}$ be separable complex Hilbert spaces. Let $P(\mathscr{H})$ denote the lattice of projections in $\mathscr{H}$ and let $\mathscr{L}(\mathscr{K})^+$ stand for the cone of all positive bounded linear operators acting in $\mathscr{H}$. A mapping σ: $P(\mathscr{H}) \to \mathscr{L}(\mathscr{K})^+$ is said to be a *positive Gleason measure* when for any sequence $P_1, P_2 \ldots$ of mutually orthogonal projections in $P(\mathscr{H})$, $\sigma(\sum P_i) = \sum \sigma(P_i)$ in the weak operator topology and $\sigma(1) = 1$.

A positive Gleason measure σ is said to be *orthogonally scattered* if for $P, Q \in P(\mathscr{H})$, $P \perp Q$, we have $\sigma(P)\sigma(Q) = 0$.

It is easy to see that any positive Gleason measure σ has a unique extension to be a positive linear mapping from $\mathscr{L}(\mathscr{H})$ to $\mathscr{L}(\mathscr{K})$.

1.2. DEFINITION. A mapping σ: $P(\mathscr{H}) \to \mathscr{K}$ is said to be a *$\mathscr{K}$-valued orthogonally scattered Gleason measure* when for any sequence $\{P_k\}$ of pairwise orthogonal projections in $P(\mathscr{H})$, $\sigma(\sum_k P_i) = \sum_k \sigma(P_i)$ in the weak topology in $\mathscr{K}$, and $(\sigma(P), \sigma(Q)) = 0$ for $P \perp Q$.

1.3. DEFINITION. The *correlation function* of σ is the function $C_\sigma(P, Q) = \langle \sigma(P), \sigma(Q) \rangle$.

The main result on the structure of correlation functions is the following [2]

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1.4. THEOREM. *Suppose σ is a $\mathscr{H}$-valued, orthogonally scattered Gleason measure and* $\dim \mathscr{H} \geqslant 3$. *Then the correlation function of σ is given by* $C_\sigma(P,Q) = \operatorname{tr}(MPQ+NQP)$, *where* tr *denotes the usual faithful normal semifinite trace on $\mathscr{L}(\mathscr{H})$ and M, N are non-negative members of the trace-class. Furthermore, M and N are unique when* $\dim(\mathscr{H}) = +\infty$.

We now wish to introduce another definition.

1.5. DEFINITION. An *inner product* on $\mathscr{L}(\mathscr{H})$ is a non-negative sesquilinear form $\langle\cdot,\cdot\rangle\colon \mathscr{L}(\mathscr{H})\times\mathscr{L}(\mathscr{H})\to C$. That is to say, $\langle\cdot,\cdot\rangle$ has the following properties:

(i) $\langle A, A\rangle \geqslant 0$,

(ii) $\langle \alpha A+\beta B, C\rangle = \alpha\langle A, C\rangle+\beta\langle B, C\rangle$,

(iii) $\langle A, B\rangle = \overline{\langle B, A\rangle}$.

We then have the following elementary observation.

1.6. LEMMA. *Let σ be a $\mathscr{H}$-valued Gleason measure, and extend σ to be $\tilde\sigma\colon \mathscr{L}(\mathscr{H})\to\mathscr{H}$ in a standard way (taking first the linear combinations and then passing to the limit in the uniform operator topology). Then the correlation function of $\tilde\sigma$ defines an inner product on $\mathscr{L}(\mathscr{H})$, i.e.,*

$$\langle A, B\rangle = C_{\tilde\sigma}(A, B) = \langle\tilde\sigma(A), \tilde\sigma(B)\rangle$$

is an inner product on $\mathscr{L}(\mathscr{H})$. Notice that the correlation function has the property that P, Q orthogonal projections implies that $C_\sigma(P,Q) = 0 = \langle P, Q\rangle$.

1.7. DEFINITION. An inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{L}(\mathscr{H})$ is *orthogonal* if

(i) P, Q orthogonal projections implies $\langle P, Q\rangle = 0$;

and

(ii) for any decreasing sequence $\{P_n\}$ of orthogonal projections in $\mathscr{H}$, $P_n\to 0$ in the strong operator topology implies $\langle P_n, P_n\rangle\to 0$.

A non-negative, self-adjoint, trace class operator will be called an *s-operator*.

We shall now prove the following characterization of inner products on $\mathscr{L}(\mathscr{H})$.

1.8. THEOREM. *Let $\mathscr{H}$ be separable, complex Hilbert space* ($\dim\mathscr{H} > 2$) *with an orthogonal inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{L}(\mathscr{H})$. Then there exist two s-operators acting in $\mathscr{H}$, say M and N, such that the formula*

$$\langle A, B\rangle = \operatorname{tr} MAB^* + \operatorname{tr} NB^*A \tag{1}$$

holds for any $A, B\in\mathscr{L}(\mathscr{H})$.

The representation is unique when $\dim\mathscr{H} = \infty$. *Of course, formula* (1) *defines some inner product for any pair M, N of s-operators.*

Proof. Let $\mathscr{H}_0$ be the completion of $\mathscr{L}(\mathscr{H})/\{\langle A, A\rangle = 0\}$ (with respect to $\langle\cdot,\cdot\rangle$). Let us consider

$$\xi\colon P\to P,\quad P\in\operatorname{Proj}\mathscr{H}, \tag{2}$$

where P on the right-hand side is treated as an element of $\mathscr{H}_0$ (representant of an equivalence class). Thus by, (i) of Definition 1.7, ξ is an $\mathscr{H}_0$-valued, orthogonally scattered Gleason measure, and, by Theorem 1.4 there exist two s-operators M and N for which (2) holds. The orthogonally scattered measure ξ is bounded in the sense that $\|\xi P\|\leqslant\|\xi(I)\| < \infty$ for all $P\in\operatorname{Proj}\mathscr{H}$. Here I is the identity operator. Then, using Gleason's theorem [1], [4] we can extend ξ (in a unique, standard way) to a linear operator (which will be denoted by the same letter ξ). It is worth noting here that, in fact, ξ can be extended to the linear isometry on $\mathscr{H}$ onto $\xi(\mathscr{H})\subset\mathscr{H}_0$. By Theorem 1.4 we have

$$(\xi P, \xi Q) = \operatorname{tr} MPQ + \operatorname{tr} NQP. \tag{3}$$

Passing in (3) to the linear combinations of projectors and then to the limit (in uniform operator topology) we obtain

$$\langle\xi A, \xi B\rangle = \operatorname{tr} MAB^* + \operatorname{tr} NB^*A \tag{4}$$

for all $A, B\in\mathscr{L}(\mathscr{H})$. But, by (2), this entails (1).

To prove the uniqueness of representation (1) (in case $\dim = \infty$), let us assume that

$$\operatorname{tr} MAB^* + \operatorname{tr} NB^*A \equiv 0 \tag{5}$$

for some M and N. Let us fix an $x_0\in\mathscr{H}$, $\|x_0\| = 1$ and let V be an arbitrary unitary operator in $\mathscr{H}$. Then putting

$$[x_0]\colon y\to(y, x_0)x_0,\qquad A = V[x_0]\quad\text{and}\quad B = V, \tag{6}$$

we have

$$\operatorname{tr} MV[x_0]V^{-1} = -(Nx_0, x_0). \tag{7}$$

Since V is arbitrary, we have $(Mx, x) = \text{const}$ for all $x\in\mathscr{H}$; $\|x\| = 1$, which is possible only for $M = 0$, because M is nuclear and $\dim\mathscr{H} = \infty$. Similarly, $N = 0$. Hence the proof is completed.

1.9. Remarks. (a) If the Hilbert space $\mathscr{H}$ is real and so is the completion $\mathscr{L}(\mathscr{H})$ under real inner product $\langle\cdot,\cdot\rangle$, then we have the (unique) representation

$$\langle A, B\rangle = \operatorname{tr} MAB^* \tag{8}$$

for some s-operator M. This follows from the formula

$$(\xi P, \xi Q) = \operatorname{tr} MPQ,\qquad P, Q\in\operatorname{Proj}\mathscr{H} \tag{9}$$

for an orthogonally scattered measure in a real Hilbert space [2]. The proof of (9), in the real case, is very easy. Namely that (9) follows immediately from the formula

$$\|\xi(P+Q)\|^2 = \operatorname{tr} M(P+Q)^2,\qquad P, Q\in\operatorname{Proj}\mathscr{H}, \tag{10}$$

where M is given by Gleason's theorem:

(11) $$\|\xi P\|^2 = \operatorname{tr} MP, \qquad P \in \operatorname{Proj} \mathscr{H}.$$

In the complex case the formula

(12) $$\|\xi A\|^2 = \operatorname{tr} MAA^*$$

is valid only for normal operators and, in particular, does not hold for $A = P+iQ$ when P and Q do not commute.

The proof of Theorem 1.4 that the authors know [2] is very long and complicated and it heavily depends on the classical result of Gleason [1]. A direct proof of representation (1) would give at the same time a new proof of Theorem 1.4.

(b) The condition $\dim \mathscr{H} > 2$ is closely connected with the fact that Gleason's theorem is not true for the case $\dim \mathscr{H} = 2$. One can easily construct a counterexample for Theorem 1.4 (and Theorem 1.8) in the case $\dim \mathscr{H} = 2$.

(c) The additional requirement $\langle A, A\rangle = 0$ if $A = 0$ is equivalent to the condition $\big((M+N)x, x\big) > 0$ for $x \neq 0$. In fact, for $x \in \mathscr{H}$, $\|x\| = 1$, we have $0 < \langle [x], [x]\rangle = \operatorname{tr} M[x] + \operatorname{tr} N[x] = \big((M+N)x, x\big)$.

(d) It follows immediately from (1) that every orthogonal inner product in $\mathscr{L}(\mathscr{H})$ is continuous with respect to both the variables in the strong operator topology and it is continuous with respect to any one variable in the weak operator topology.

§ 2.

2.1. In this section, we will present an operator-theoretic approach to the problem of inner products on $\mathscr{L}(\mathscr{H})$. It is our hope that these results may suggest a new direction of attack on the problem of Gleason measures, as mentioned in the Remarks above.

2.2. Definition. The inner product $\langle\cdot,\cdot\rangle$ on $\mathscr{L}(\mathscr{H})$ is said to be *algebraic* if (i) for fixed A, the functional $X \to \langle X, A\rangle$ is ultra-weakly continuous [0], [3], and

(ii) $$\langle AB, C\rangle + \langle 1, A^*CB^*\rangle = \langle A, CB^*\rangle + \langle B, A^*C\rangle \quad \text{for all} \quad A, B, C \in \mathscr{L}(\mathscr{H}).$$

We remark that an algebraic inner product also satisfies the condition that $X \to \langle A, X\rangle$ is an ultra-weakly continuous conjugate-linear functional.

Our main result is the following

2.3. Theorem. *Let $\mathscr{H}$ be a separable Hilbert space, let $\langle\cdot,\cdot\rangle$ stand for an algebraic inner product on $\mathscr{L}(\mathscr{H})$. Let tr be the usual faithful normal semifinite trace on $\mathscr{L}(\mathscr{H})$ associated with an orthogonal basis. Then there exist two non-negative elements M, N of the trace class such that $\langle A, B\rangle = \operatorname{tr}(MAB^* + NB^*A)$ for all $A, B \in \mathscr{L}(\mathscr{H})$. Furthermore, M and N are unique when $\dim(\mathscr{H}) = +\infty$.*

2.4. Remarks. Notice that for projections P and Q, Theorem 2.3 yields the same result as Theorem 1.4 and 1.8, and that Theorem 2.3 is valid when $\dim(\mathscr{H}) = 2$, whereas Theorems 1.4 and 1.8 are not.

Before beginning the proof of Theorem 2.3, we need some preliminary results of an operator-theoretic nature.

2.5. Lemma. *Let $\varphi\colon \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ be a linear map. Then $\varphi(A) = MA + AN$ for fixed M, N if and only if $\varphi(A) = XA + \delta(A)$, where X is fixed and δ is a derivation. Furthermore, if the range of φ is contained in the compact operators (trace-class), then M and N may be taken to be compact (trace-class).*

Proof. If $\varphi(A) = XA + \delta(A)$, then since $\delta(A) = \delta_Y(A) = YA - AY$ for some Y, we have $\varphi(A) = XA + YA - AY = (X+Y)A + A(-Y)$. Conversely, if $\varphi(A) = MA + AN$, let $X = M+N$. It suffices to put $\delta(A) = \varphi(A) - XA$.

Since $X = \varphi(1)$, we have that if the range of φ is contained in the compact operators (trace-class), so is the range of δ.

2.6. Corollary. *$\varphi(A) = MA + AN$ if and only if*

$$\varphi(AB) = A\varphi(B) + \varphi(A)B - A\varphi(1)B.$$

Proof. By Lemma 2.5, $\varphi(A) = MA + AN$ if and only if $\varphi(A) = \varphi(1)A + \delta(A)$, where $\delta(A) = \varphi(A) - \varphi(1)A$ is a derivation. Then $\varphi(AB) - \varphi(1)AB = \delta(AB)$, and $\delta(AB) = A\delta(B) + \delta(A)B$ if and only if $\varphi(AB) - \varphi(1)AB = A[\varphi(B) - \varphi(1)B] + [\varphi(A) - \varphi(1)A]B$ if and only $\varphi(AB) = A\varphi(B) + \varphi(A)B - A\varphi(1)B$.

Proof of Theorem 2.3. Fix an orthonormal basis of $\mathscr{H}$, and let tr be the associated trace. Since $X \to \langle A, X\rangle$ is an ultra-weakly continuous conjugate-linear functional, there exists an element $\varphi(A)$ in the trace-class such that $\langle A, X\rangle = \operatorname{tr}\varphi(A)X^*$ and φ is a linear map from $\mathscr{L}(\mathscr{H})$ to the trace-class. We rewrite property (ii) of an algebraic inner product as $\langle AB, X\rangle = \langle A, XB^*\rangle + \langle B, A^*X\rangle - \langle 1, A^*XB^*\rangle$. Hence $\operatorname{tr}\varphi(AB)X^* = \operatorname{tr}\varphi(A)BX^* + \operatorname{tr}\varphi(B)X^*A - \operatorname{tr}\varphi(1)BX^*A$, and so $\operatorname{tr}\varphi(AB)X^* = \operatorname{tr}[\varphi(A)B + A\varphi(B) - A\varphi(1)B]X^*$. Thus $\varphi(AB) = \varphi(A)B + A\varphi(B) - A\varphi(1)B$.

By applying Corollary 2.6, we now have that $\langle A, X\rangle = \operatorname{tr}\varphi(A)X^* = \operatorname{tr}(MA + AN)X^* = \operatorname{tr}(MAX^* + NX^*A)$. We now wish to show that M and N are self-adjoint. Let $M = [m_{ij}]$, $N = [n_{ij}]$ be the matrix representations of M and N with respect to the fixed basis. Then for the matrix units E_{ii} and E_{ij} we have $\langle E_{ii}, E_{ij}\rangle = \operatorname{tr}(ME_{ii}E_{ij}^* + NE_{ij}^*E_{ii}) = \operatorname{tr}(NE_{ji}) = n_{ij}$; also $\langle E_{ij}, E_{ii}\rangle = \operatorname{tr}(ME_{ij}E_{ii}^* + NE_{ii}^*E_{ij}) = \operatorname{tr} NE_{ij} = n_{ji}$. From property (iii) of inner products, we have $n_{ji} = \overline{n_{ij}}$, and so N is self-adjoint. Similar use of the matrix units E_{ii} and E_{ji} shows that M is self-adjoint.

In order to study the question of uniqueness, suppose that $\operatorname{tr}(MAB^*+NB^*A) = \operatorname{tr}(M_1AB^*+N_1B^*A)$ for all $A, B \in \mathscr{L}(\mathscr{H})$. Then $\operatorname{tr}[(M-M_1)\times AB^*+(N-N_1)B^*A] = 0$ for all A and B, so it suffices to consider the case where $\operatorname{tr}(MAB^*+NB^*A) = 0$. Let $M = [m_{ij}]$, $N = [n_{ij}]$ be the matrix representations of M and N. Then for $A = E_{ii}$, $B = E_{ij}$ $(j \neq i)$ we have $AB^* = 0$, $B^*A = E_{ji}$, and so $0 = \operatorname{tr}(MAB^*+NB^*A) = \operatorname{tr} NE_{ji} = n_{ij}$. Since $n_{ij} = 0$ for $i \neq j$, N is diagonal. The choice of $A = E_{ii}$, $B = E_{ij}$ yields the fact that M is also diagonal. Then for $A = E_{ij}$, $B = E_{ij}$, we have $AB^* = E_{ii}$, $B^*A = E_{jj}$, and so $0 = \operatorname{tr}(MAB^*+NB^*A) = \operatorname{tr}(ME_{ii}+NE_{jj}) = m_{ii}+n_{jj}$. Fixing i, we have $n_{jj} = -m_{ii}$ for all j, and since N is diagonal, N is scalar. Similarly, M is scalar, and so $M = mI = -N$.

We have now shown that $M-M_1 = mI$, $N-N_1 = -mI$. Further, since M, M_1, N, and N_1 are members of the trace-class, so is mI. In case $\dim \mathscr{H} = +\infty$, this forces $m = 0$, proving the uniqueness. Notice that in case $\dim \mathscr{H} < \infty$, we have $\operatorname{tr}[(M-mI)AB^*+(N+mI)B^*A] = \operatorname{tr}(MAB^*+NB^*A)+m\operatorname{tr}(B^*A-AB^*) = \operatorname{tr}(MAB^*+NB^*A)$ for all A and B.

In order to show that M and N are non-negative, let x and y be unit vectors in $\mathscr{H}$, and let the operator A be given by $A(z) = \langle z, x\rangle y$ for $z \in \mathscr{H}$; thus $A^*(w) = \langle w, y\rangle x$. Then we have

$$\begin{aligned} 0 \leqslant \langle A, A\rangle &= \operatorname{tr}(MAA^*+NA^*A) = \sum_i \langle MAA^*e_i, e_i\rangle + \sum_i \langle NA^*Ae_i, e_i\rangle \\ &= \sum_i \langle e_i, y\rangle \|x\|^2 \langle My, e_i\rangle + \sum_i \langle e_i, x\rangle \|y\|^2 \langle Nx, e_i\rangle \\ &= \|x\|^2 \langle My, \sum_i \langle y, e_i\rangle e_i\rangle + \|y\|^2 \langle Nx, \sum_i \langle x, e_i\rangle e_i\rangle \\ &= \|x\|^2 \langle My, y\rangle + \|y\|^2 \langle Nx, x\rangle. \end{aligned}$$

If $\dim \mathscr{H} = +\infty$, then the fact that M and N are self-adjoint elements in the trace-class implies that the spectra $\sigma(M)$ and $\sigma(N)$ consist at most of sequences converging to 0. Let x be an arbitrary unit vector, let $\varepsilon > 0$ be arbitrary, and let y be a unit vector with $|\langle My, y\rangle| < \varepsilon$. Then $\langle Nx, x\rangle \geqslant -\langle My, y\rangle \geqslant -\varepsilon$, but ε arbitrary implies that $\langle Nx, x\rangle \geqslant 0$, and hence that $N \geqslant 0$. Similarly, reversal of the roles of x and y implies that $M \geqslant 0$.

If $\dim \mathscr{H} < \infty$, then for any real number λ, $\langle A, B\rangle = \operatorname{tr}[(M+\lambda)AB^* + (N-\lambda)B^*A]$. Thus for $A(z) = \langle z, x\rangle y$ with $\|x\| = \|y\| = 1$, we have $0 \leqslant \langle A, A\rangle = \langle (M+\lambda)y, y\rangle + \langle (N-\lambda)x, x\rangle$. We will choose λ so that $M+\lambda \geqslant 0$ and $N-\lambda \geqslant 0$. Let $\mu = \inf_{\|y\|=1} \langle My, y\rangle$, and let $\lambda = -\mu$; then for unit vectors y, $\langle (M+\lambda)y, y\rangle = \langle My, y\rangle - \mu = \langle My, y\rangle - \inf_{\|y\|=1} \langle My, y\rangle \geqslant 0$, and so $M+\lambda \geqslant 0$. Since $\dim \mathscr{H} < \infty$, there is a unit vector y_0 such that $\langle My_0, y_0\rangle = \mu$, and so $\langle (M+\lambda)y_0, y_0\rangle = 0$. Then for any unit vector x, we have $0 \leqslant \langle (M+\lambda)y_0, y_0\rangle + \langle (N-\lambda)x, x\rangle = \langle (N-\lambda)x, x\rangle$, and thus $N-\lambda \geqslant 0$. The proof is completed.

2.7. Remarks. (a) It is somewhat surprising that there are more difficulties involved in the finite-dimensional case than in the infinite-dimensional case. The problem, of course, is that the trace of the identity is finite.

(b) The condition that the inner product be algebraic may perhaps be unnatural. What is essential is that the map $\varphi\colon \mathscr{L}(\mathscr{H}) \to$ trace-class be of the form $\varphi(A) = MA+AN$ for some operators M and N in the trace-class.

(c) Suppose that for Hilbert–Schmidt operators A, B in $\mathscr{L}(\mathscr{H})$, the inner product $\langle A, B\rangle$ is continuous with respect to the Hilbert–Schmidt norm. Can one use the Riesz theorem to obtain a proof of Theorem 1.4, Theorem 1.8, or Theorem 2.3? This problem seems to be open. It follows from Theorem 1.4, Theorem 1.8 and Theorem 2.3 that these inner products are indeed continuous on the Hilbert–Schmidt operators with respect to the Hilbert–Schmidt norm.

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