Method of orthogonal projections and approximation of the spectrum of a bounded operator II

by

ANDRZEJ POKRZYWA (Warsaw)

Abstract. A necessary and sufficient condition is given in order that a compact subset of the complex plane be a limit of spectra of operators $A_n = P_n A |P_nH$, where $A$ is a given bounded operator on a Hilbert space $H$ and $(P_n)$ is a sequence of orthogonal projections converging strongly to the identity operator on $H$. This paper is a continuation of the study of asymptotical behaviour of spectra of operators $A_n = P_n A |P_nH$, where $P_n$ is a sequence of orthogonal projections in Hilbert space $H$ converging strongly to the identity operator and $A$ is a bounded operator in $H$. We use the same notations as in the first authors paper [4] on this subject.

The main result is the following theorem, which completes Theorem 1 in paper [4].

**Theorem.** If $A$ is a bounded operator on a separable Hilbert space $H$ and its essential numerical range $W_e(A)$ contains an interior point, then for any sequence $(E_n)$ of finite nonvoid subsets of the interior of $W_e(A)$ there exists a sequence $(F_n)$ of orthogonal projections of finite rank such that $F_n \rightarrow 1$ strongly and

$$E_n \subseteq \Sigma(A_n) \subseteq E_n \cup (\Sigma(A) \setminus W_e(A)),$$

where

$$A_n = P_n A |P_nH \in L(P_nH).$$

Before proving this theorem, we need some auxiliary lemmas. The idea of conformal mapping in the proof of Lemma 1 is taken from Herrero's paper [3].

We start with the following simple example.

**Example.** Let $H$ stand for a separable Hilbert space, the operator $S$ satisfying the relations $R_n = \sigma_n$ (where $(\sigma_n)$ is a fixed orthonormal basis in $H$) is called the bilateral shift. It is known that $S$ is a normal operator and that its spectrum $\Sigma(S)$ is a unit circle $S(0, 1)$. Define a sequence
is a selfadjoint projection in \( H \times H \). This shows that \( N = QR(\mathbb{S}^{1} \times \mathbb{S}^{1}) \), where

\[
R = \bigoplus_{j=1}^{m} \left[ \begin{array}{cc}
\mu_{j} & -\mu_{j+1} \\
\mu_{j+1} & \mu_{j}
\end{array} \right] \in L(H \times H), \quad Q = \bigoplus_{j=1}^{m} \left[ \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right].
\]

Note that \( R \) is a normal operator with \( \Sigma(R) = \Sigma(N) \), and all normal operators which satisfy this relations are unitary equivalent, note also that the operators \( P_nR \) are orthogonal projections of the rank \( n \).

In this way we have proved the following lemma.

**Lemma 2.** Let \( x_{n} \) be an interior point of a convex polygon \( \Omega \) with extremal points \( \{\mu_{j}\}_{j=0}^{n} \). If \( R \in L(H) \) is a normal operator with \( \Sigma(R) = \Sigma(N) \), then there exists a sequence \( \{P_{n}\} \) of orthogonal projections in \( H \) such that

\[
\dim P_{n} = n, \quad \Sigma(P_{n}R_{1}P_{n}) = \{x_{n}\},
\]

\[
\text{dist}(\Sigma(P_{n}R_{1}P_{n}), \Omega) \to 0, \quad \text{with} \quad n \to \infty.
\]

**Lemma 3.** Let \( H \) stand for a Hilbert space with orthonormal basis \( \{e_{n}\}_{n=1}^{\infty} \). Let \( x_{n} \) be an interior point of the triangle \( \{\mu_{1}, \mu_{2}, \mu_{3}\} \). Define the operator \( A \in L(H) \) by the relations \( A_{e_{n}} = \mu_{n}e_{n+1} \), \( n = 0, 1, 2, \ldots \) Then for any \( \varepsilon > 0 \) there exists a projection \( P \in P_{1}(H) \) such that

\[
\Sigma(\lambda_{1}P_{1}P_{1}) = \{x_{n}\}, \quad \left\| e_{n} - P_{1}e_{n} \right\| < \varepsilon.
\]

**Proof.** By Lemma 2 there exists a \( Q \in P_{1}(H) \) such that \( \Sigma(QA_{1}P_{1}) = \{x_{n}\} \) and \( \text{dist}(\Sigma(QA_{1}P_{1}), \Sigma(A)) < \varepsilon/4 \), where \( \varepsilon = \delta(\mu_{1}, \mu_{2}, \mu_{3}) \) (>0), so there exists a unit vector \( x \in H \) such that \( \delta(x, x) < \varepsilon/4 \).

Note that \( x = \sum_{i=1}^{m} a_{i}y_{i} \), where \( \|y_{i}\| = 1 \), \( Ay_{i} = \mu_{i}y_{i} \), \( a_{i} \geq 0 \), \( i = 0, 1, 2 \).

Hence \( 1 = \|x\|^{2} = \sum_{i=1}^{m} a_{i}^{2} \), \( \langle A_{1}x, x \rangle = \sum_{i=1}^{m} a_{i}^{2} \mu_{i} \) because \( y_{i} \) are orthogonal.

Therefore

\[
\|x\|^{2} > \mu_{3} - \langle A_{1}x, x \rangle = (1 - a_{3}^{2}) \mu_{3} - a_{3}^{2} \mu_{2} + \frac{1}{2} \mu_{1}^{2} > (1 - a_{3}^{2}) \lambda;
\]

so \( 1 - a_{3}^{2} < \varepsilon/4 \) and

\[
\|x - y_{3}\|^{2} = \|(a_{0} - 1)y_{0} + a_{1}y_{1} + a_{2}y_{2}\|^{2} = a_{0}^{2} \leq \frac{\varepsilon^{2}}{1 + a_{3}} < \frac{\varepsilon^{2}}{2}.
\]

Let \( U \) stand for the unitary operator satisfying the relations \( Us - x \) for \( s \) orthogonal to \( e_{0} \) and \( y_{i} \), \( Ue_{0} = x_{1} \), \( Uy_{i} = e_{i} \). It is obvious that \( U^{*} = 1 \) and \( A = UAU \), let \( R = UQ_{1}U \). The operators \( Q_{1} \) and \( P \) are unitary equivalent, and \( \Sigma(P_{1}P_{1}) = \{x_{n}\} \). Now the lemma follows.
from the inequality
\[ \left\| e_0 - \frac{P_{\mathcal{P}_0}}{\|P_{\mathcal{P}_0}\|} \right\|^2 < 2 \|y_0 - Qy_0\|^2 \leq 2 \|y_0 - Qy_0\|^2 = 2 \inf_{\varphi \in \mathcal{P}} \|y_0 - \varphi\| \leq 2 \|y_0 - x\| < \varepsilon. \]

**Lemma 4.** If \( \Omega \) is a convex subset of \( C, \quad E(\mu, \Omega) < \varepsilon, \) then \( (1 - \varepsilon)\mu + \varepsilon \in \Omega. \)

**Proof.** There exists a \( \lambda \in \Omega \) such that \( |\mu - \lambda| < \varepsilon. \) Let
\[ \varepsilon = \frac{1}{1 - \varepsilon} |\mu - \lambda|; \]
then \( |\varepsilon| < \delta \) and \( s + \varepsilon \in \Omega, \) this implies \( (1 - \varepsilon)\mu + \varepsilon \in \Omega. \)

**Lemma 5.** If projections \( P, Q \) in \( H \) are defined by the formulas
\[ P = \sum_{i=1}^{n} \langle x_i, x_i \rangle x_i, \quad Q = \sum_{i=1}^{n} \langle y_i, y_i \rangle y_i, \]
where \( \{x_i\}_{i=1}^{n}, \{y_i\}_{i=1}^{n}, \) are two orthonormal sets such that \( \langle x_i, y_i \rangle = 0 \) for
\( i \neq j, \) \( |x_i - y_i| < \varepsilon, \) \( j = 1, 2, \ldots, n, \) then \( \|P - Q\| < \varepsilon. \)

**Proof.** Note that there exists an orthonormal set \( \{x_j\}_{j=1}^{n} \) such that
\[ y_j = a_jx_j + b_jx_{j+1}, j = 1, 2, \ldots, n, \]
where
\[ |a_j|^2 \leq \|x_j - y_j\|^2 \leq \varepsilon^2. \]

The following identity holds:
\[ (P - Q)x = \sum_{j=1}^{n} b_j (\langle x, x_j \rangle x_j - a_jx_{j+1} + \langle x, y_j \rangle y_{j+1}); \]
so
\[ \|P - Q\|^2 < \varepsilon^2 \sum_{j=1}^{n} (\langle x_j, x_j \rangle |a_j|^2 + \langle x, y_j \rangle |b_j|^2) \leq \varepsilon^2 \|x\|^2, \]
which follows from the Bessel inequality since the vectors \( \{b_jx_j - a_jx_{j+1}, y_j\}_{j=1}^{n}, \)
\( x_j, y_j \)} form an orthonormal set.

**Lemma 6.** If \( P, Q \in L(H), P = P^2, Q = Q^2, PQ = QP \) and \( \|P - Q\| < 1, \)
then \( P = Q. \)

**Proof.** Note that \( (P - Q)^{r} = (P - Q)^{2} \) and \( \|P - Q\|^2 < 1. \) Therefore
\( (P - Q)^{2} \) is a projection with a norm strictly less than 1; hence \( 0 = (P - Q)^{2} = P - Q - 2PQ. \) Multiplying this identity first on \( Q, \) then on \( P, \) we obtain \( P = PQ = Q. \)

Proof of the Theorem 1. It follows from Lemma 7 of [4] that it is enough to prove the theorem in the case where \( \mathcal{B} \) is one-point sets, therefore in the sequel we assume that \( \{x\} = \mathcal{B}. \) Then there exists a sequence \( \{b_n\}_{n=1}^{\infty} \) of positive numbers such that \( \mathcal{E}(\mu_n, \mathcal{B}) \cap \text{Int} W_+(\mathcal{A}), \quad 0 < \varepsilon, \)
\( 0 < \varepsilon < 1, \) and \( b_n \rightarrow \mathcal{E}(\mathcal{A}) = 0, \) where \( \mathcal{E} = W_+(\mathcal{A}) + (\mathcal{A}). \)

2. Let \( Q_n = E(C \setminus \mathcal{G}_n \setminus \mathcal{A}), \) and \( (\mathcal{P}_n)^{m_n} = P_f(H) \) be a sequence strongly

convergent to the identity operator. Let \( P_n \) stand for the orthogonal projection on the subspace \( \mathcal{P}_n = \mathcal{P}_n + \mathcal{Q}_n + \mathcal{Q}_n^*H. \) As \( \mathcal{P}_n \rightarrow 1 \) strongly with \( n \rightarrow \infty, \) it follows from Theorem 1 of [6] that there exists a sequence \( \{a_n\}_{n=1}^{\infty} \) such that
\[ |Q_n - E(C \setminus \mathcal{G}_n \setminus \mathcal{A})|_{P_{\mathcal{P}_n}, \mathcal{P}_n} < 1, \quad n \geq n_0. \]

For simplicity assume that \( \mathcal{P}_n = P_{\mathcal{P}_n}. \) Since \( Q_n \) commutes with \( A \) and \( \mathcal{P}_n, \)
so it commutes also with \( \mathcal{P}_nA \) and therefore \( \mathcal{Q}_n \) commutes also with \( E(C \setminus \mathcal{G}_n \setminus \mathcal{A})|_{\mathcal{P}_n}, \)
Lemma 6 shows that \( Q_n = E(C \setminus \mathcal{G}_n \setminus \mathcal{A})|_{\mathcal{P}_n}. \)

3. Now we fix \( n. \) Let \( \bar{Q}_n \) stand for the orthogonal projection with range \( Q_nH; \) then \( \bar{Q}_n Q_n = Q_n, \quad \bar{Q}_n \bar{Q}_n = \bar{Q}_n, \) and \( \bar{Q}_n < \mathcal{P}_n. \) The following identity holds:
\[ \mathcal{P}_n A \mathcal{P}_n = Q_n \mathcal{P}_n A \mathcal{P}_n + Q_n \mathcal{P}_n(\mathcal{P}_n - \mathcal{Q}_n)A(\mathcal{P}_n - \mathcal{Q}_n). \]
so the operator \( \mathcal{P}_n A|_{\mathcal{P}_n} \) may be represented by the operator matrix
\[ \begin{bmatrix} Q_n A Q_n & Q_n A(\mathcal{P}_n - \mathcal{Q}_n) \\ \mathcal{P}_n - \mathcal{Q}_n A(\mathcal{P}_n - \mathcal{Q}_n) & \mathcal{P}_n - \mathcal{Q}_n A(\mathcal{P}_n - \mathcal{Q}_n) \end{bmatrix}. \]

By a theorem on a triangular matrix form ([1], p. 107) there exists an
orthonormal set \( \{x_1, \ldots, x_1, \ldots, x_k\} \) such that
\[ \bar{Q}_n = \sum_{j=1}^{k} \langle x_j, x_j \rangle x_j, \quad \mathcal{P}_n - \mathcal{Q}_n = \sum_{j=1}^{k} \langle x_j, x_j \rangle x_j \]
and
\[ \langle \bar{x}, x_j \rangle = 0 \quad \text{for} \quad i < j. \]

Let \( \tilde{b}_j = \langle \bar{x}, x_j \rangle; \) so \( \tilde{b}_j \) is a spectral measure. Since \( \bar{Q}_n A \bar{Q}_n \mathcal{P}_n = A|_{\mathcal{P}_n}, \)
we have
\[ \tilde{b}_j \mathcal{P}_n = \mathcal{P}_n A|_{\mathcal{P}_n}, \]
Since \( Q_n = E(C \setminus \mathcal{G}_n \setminus \mathcal{A})|_{\mathcal{P}_n}, \) we have
\[ \tilde{b}_j \mathcal{P}_n = \mathcal{P}_n A|_{\mathcal{P}_n} \cap \mathcal{G}_n, \]
therefore
\[ (1) \quad d(\mu_1, W_+(\mathcal{A})) < \varepsilon^* \quad (1 < i < r). \]
Now let
\[ \mu_j = \begin{cases} 
\nu_k, & -s \leq j \leq 0, \\
(1 - \delta_n)\nu_k + \delta_n\nu_n, & 1 \leq j \leq r.
\end{cases} \]
It follows from Lemma 4 and (1) that \( \mu_j \in \text{Int} W_r(A) \), whence \( 1 \leq j \leq r \).

Lemma 7 of [4] implies that there exist vectors \( \tilde{\delta}_j \) (for \( j = r+1, r+2, \ldots, 2r \)) such that \( \langle \tilde{\delta}_j, \tilde{\delta}_i \rangle = 0 \), and
\[ \langle A^*\tilde{\delta}_j, \tilde{\delta}_i \rangle = 0 \quad \text{for} \quad j > r, j \neq i, \]
\[ \langle A\tilde{\delta}_j, \tilde{\delta}_i \rangle = 0 \quad \text{for} \quad j > r, j < 2r. \]
Now put
\[ s_j = \frac{\tilde{\delta}_j}{\sqrt{1 - \delta_n^2 + \delta_n^2 + \delta_n^2 s_j}}, \]
\[ 1 \leq j \leq r \]
and note that:
\[ \langle s_j, s_i \rangle = \delta_n, \quad \langle A^*s_j, s_i \rangle = \mu_j, \quad \langle As_j, s_i \rangle = 0 \quad \text{for} \quad i < j. \]

Let an orthogonal projection \( P_\lambda \) be defined by the formula \( P_\lambda = \sum_{j=0}^s \langle s_j, s_j \rangle s_j s_j^* \).

It follows from (2) that
\[ \Sigma(P_\lambda A)[\lambda] = \{ \mu_j \}_{j=0}^r, \quad \Sigma(A) \cap G_\lambda, \]
\[ \{ \mu_j \}_{j=0}^s = W_r(A). \]

From the definition of \( \delta_n \) we see that \( \|s_j - \tilde{\delta}_j\| \leq \delta_n \) hence by Lemma 5
\[ \|P_\lambda - P_\mu\| \leq \sqrt{\delta_n}. \]

4. There exist complex numbers \( \mu_j \) (for \( j < k < 3r \)) such that: \( \mu_j \in \text{Int} W_r(A) \) and \( s_n \) is an interior point of the triangle \( \{ \mu_j, \mu_{j+r}, \mu_{j+2r} \} \)
\[ 1 \leq j \leq r \] (In the case where \( \mu_j = s_n \) we put \( \mu_{j+r} = \mu_{j+2r} = s_n \)).

Let \( \mu_{j+r} = \mu_j, k = 1, 2, \ldots \) By Lemma 7 of [4] there exist unit vectors \( s_j \) (for \( j < k < 3r \)) such that:
\[ \{ s_j \}_{j=0}^k \] is an orthonormal set in \( H \),
\[ \langle A^*s_j, s_i \rangle = \mu_j \quad (-s \leq j \leq 0), \]
\[ \langle As_j, s_i \rangle = 0 \quad \text{for} \quad j > r, j \neq i. \]

Now define orthogonal projections \( R_j \) (for \( 0 \leq j \leq r \)) by the formulas:
\[ R_0 = 0, \quad R_j = \sum_{i=0}^j \langle s_j, s_i \rangle s_i s_i^* \quad (1 \leq j \leq r). \]

Note that:
(i) \( R_i R_j = 0 \) for \( i \neq j \);
(ii) \( R_j A_{j+r} = \sum_{i=0}^j \langle s_{j+r}, s_i \rangle s_i s_i^* \mu_{j+r}^2 \) for \( j \geq 1, k = 0, 1, \ldots, \)
therefore the operator \( R_j A_{j+r} \) is normal for \( j \geq 1 \) and
\[ \Sigma(A_{j+r}) = \Sigma(A) \cap G_{s_n}, \]
\[ (1 \leq j \leq r); \]

(iii) if \( i > j \geq 1 \), then
\[ R_i A_{i+r} = \sum_{k=0}^r \langle s_{i+r}, s_k \rangle s_k s_k^* \mu_{i+r}^2 \mu_{i+r} \mu_{i+r} \]
\[ = \langle A_{i+r}, s_i \rangle s_i = 0; \]
hence \( R_i A_{i+r} = 0 \) also \( R_i A_{i+r} = R_i R_i R_i A_{i+r} = 0 \), and so
\[ R_i A_{i+r} = 0 \quad \text{if} \quad 0 \leq i < r \]

5. By Lemma 3 there exists a projection \( S_j \in P_j(H) \) (for \( 1 \leq j \leq r \)) such that:
\[ \|s_j - \tilde{\delta}_j\| = \delta_n, \quad \|s_j - \tilde{\delta}_j\| = \delta_n. \]
Put also \( S_j = R_j \) (note that:
(i) \( S_j S_j = 0 \), \( S_j = 0 \) for \( i \neq j \); hence if we define \( P_\lambda = \sum_{j=0}^r S_j \), then \( P_\lambda \in P_j(H) \);
(ii) if \( 0 \leq j < r \), then \( S_j A S_j = S_j R_j A R_j S_j = 0 \).

This shows that the operator \( A_{n} = A_{n} A_{n}^{*} A_{n}^{*} A_{n} \) may be represented in an upper triangular matrix form
\[ A_{n} = \begin{bmatrix} S_{n} & S_{n} & \cdots & S_{n} \\ 0 & S_{n} & \cdots & S_{n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{n} \end{bmatrix}, \]

Therefore
\[ \Sigma(A_{n}) = \bigcup_{j=0}^r \Sigma(S_j A_{j+r}) = \Sigma(A) \cap G_{s_n}. \]

It remains to prove that \( P_\lambda \to 1 \) strongly. Let
\[ P_\lambda = \sum_{j=0}^r \langle s_j, s_j \rangle s_j s_j^* \]
\[ + \sum_{j=0}^r \langle s_j, s_j \rangle s_j s_j^* \]
\[ \|P_{\lambda} - P_{\mu}\| \leq \delta_n \] this, with (3), gives \( \|P_{\lambda} - P_{\mu}\| \to 0 \). Since
\( P_{\lambda} \to 1 \) strongly, \( P_{\mu} \to 1 \) strongly; since \( P_{\lambda} > P_{\mu} \), also \( P_{\lambda} \to 1 \) strongly.

This ends the proof.

Remark 1. If \( A \in L(H) \) is a self-adjoint operator, then the operators \( A_{n} = P_{\lambda} A_{n}^{*} P_{\lambda} \) are also self-adjoint. Using the identity
\[ d(\lambda, E(A)) = \inf \{ \|E(\lambda - A)\| : \|E\| = 1 \}, \]
which is valid for any normal operator, we see that if \( P_{\lambda} \to 1 \) strongly, \( P_{\lambda} \in P_j(H) \), then for any \( s > 0 \) and \( \varepsilon > 0 \), large enough \( \Sigma(A_{n}) = \Sigma(A_{n}) + \varepsilon \).
In such a case the assumption of our Theorem that \( W_r(A) \) contains an interior point is not satisfied; \( W_r(A) \) is an interval.
However, the asymptotical behaviour of spectra of the operators $A_n$ is not clear in the case where $A$ differs from a selfadjoint operator by a compact one.

**Remark 2.** If $W_n(A)$ is a one-point set, then (6) $A$ is of the form $\lambda + K$ where $K$ is compact. Then for any sequence $(P_n)$ of projections in $H$ (not necessarily orthogonal) converging strongly to identity $\text{dist}(\sigma(A), \Sigma(P_n, A|\mathcal{P}_n)) \to 0$.

The case of not necessarily orthogonal projections is easy to settle. Remark 3 and the corollary of Lemma 7 give a characterization of the asymptotical behaviour of spectra of the operators $P_n A|\mathcal{P}_n$.

**Lemma 7.** If $A \in \mathcal{L}(H)$ and $W_n(A)$ is not a one-point set, then for any bounded set $\Omega \subset C$ there exists an operator $C \in \mathcal{L}(H)$ with $C^{-1} \in \mathcal{L}(H)$ such that $\Omega \subset W_n(C^{-1}A)$.

**Proof.** If $a, b \in W_n(A)$, $a \neq b$, then there exists a number $r > 0$ such that

$$\Omega \subset K \left( \frac{a+b}{2}, r \left| \frac{a-b}{2} \right| \right).$$

It follows from Lemma 7 of [4] that there exists an orthonormal set $(y_k, \lambda_k)_{k=1}^\infty$ such that: $\langle Ax_k, y_k \rangle \to A$, $\langle Ay_k, y_k \rangle \to b$, $\langle Ax_k, y_k \rangle = \langle Ay_k, y_k \rangle = 0$.

Let $Q$ stand for the orthogonal projection on the subspace spanned by $(y_k, \lambda_k)_{k=1}^\infty$. Define operator $C \in \mathcal{L}(H)$ by the formula

$$Qz = (1 - Q)z + \sum_{n=1}^\infty \left( \langle x_n, y_n \rangle x_n + \langle x, y_n \rangle y_n \right);$$

then

$$(CQ)z = (1 - Q)z + \sum_{n=1}^\infty \left( \langle x_n, y_n \rangle x_n + \langle x, y_n \rangle y_n \right).$$

Let $v_n = \frac{1}{\sqrt{2}} (x_n + e^{i\theta}y_n)$ where $\theta \in R$. $(v_n(t))_{t=1}^\infty$ is an orthonormal sequence in $H$ and

$$\langle CQv_n, v_n \rangle = \langle CQv_n, (C^{-1})v_n \rangle =$$

$$= \frac{1}{2} \left( (1 + e^{i\theta}) \langle Ax_n, x_n \rangle + (1 - e^{i\theta}) \langle Ay_n, y_n \rangle \right) \to \frac{a+b}{2} + r e^{i\theta} \left( \frac{a-b}{2} \right) = \lambda(t).$$

This shows that $\lambda(t) \in W_n(C^{-1}A)$. Since $W_n(C^{-1}A)$ is a convex set and

$$\bigcup \{ \lambda(t) \} = S \left( \frac{a+b}{2}, r \left| \frac{a-b}{2} \right| \right),$$

we see that

$$\Omega \subset K \left( \frac{a+b}{2}, r \left| \frac{a-b}{2} \right| \right) \subset W_n(C^{-1}A).$$

**Corollary.** Suppose $A \in \mathcal{L}(H)$ is not of the form $\lambda + K$ where $K$ is a compact operator on $H$, then for each bounded set $\Omega \subset C$ there exists a sequence $(P_n)$ of projections in $H$ of finite rank (we do not assume $P_n = P_n^*$) such that:

$$P_n \to 1 \text{ strongly, } P_n^* \to 1 \text{ strongly}$$

and

$$\text{dist}(\Sigma(A_n), \Omega) \to 0, \text{ where } A_n = P_n A|\mathcal{P}_n.$$

**Proof.** By the previous lemma there exists a $C \in \mathcal{L}(H)$ with $C^{-1} \in \mathcal{L}(H)$ such that $\Omega \cup \Sigma(A) \subset W_n(CAC^{-1})$. Theorem implies that there exists a sequence $(Q_n)_{n=1}^\infty \subset P_f(H)$ such that $Q_n \to 1$ strongly and

$$\text{dist}(\Sigma(Q_nC_{AC^{-1}}|Q_nH), \Omega) \to 0.$$ Let $P_n = C^{-1}Q_nC$, so $P_n = P_n^*$ and it is easy to verify that $P_n \to 1$ strongly, $P_n^* \to 1$ strongly.

Moreover, $A_n = P_n A|\mathcal{P}_n = C^{-1}(Q_n C AC^{-1}|Q_nH) (P_n H)$, i.e., $A_n$ is similar to $Q_n C AC^{-1}|Q_nH$. Therefore $\Sigma(A_n) = \Sigma(Q_n C AC^{-1}|Q_nH)$, this and (4) end the proof.

**References**


*Received April 8, 1973*