The law of nought-or-one in the theory of probability

by

C. VISSER (Dordrecht).

1. *Introduction.* In the calculus of probability a number of cases have been found in which it can be asserted *a priori* that a certain probability has either the value nought or the value one, or, to speak in terms of the theory of measure, that the measure of a certain set is either nought or one[1]). The occurrence of this „law of nought-or-one" when one is dealing with probabilities concerning real numbers in the interval $0 \leqslant x \leqslant 1$ was explained by K. KNOPP[2]). The reason of it may be seen in the fact that the sets under consideration are homogeneous sets, i. e. sets which are equally distributed over the sub-intervals of $0 \leqslant x \leqslant 1$, and these homogeneous sets are, if they are measurable, necessarily sets of measure 0 or 1.

In more general cases in which a probability can only assume the extreme values 0 and 1, this probability is the measure of a set in an infinite product-space. In what follows I shall show that also here we can account for the occurrence of the „nought-or-one law" by means of the notion of a homogeneous set.

2. *The measure in infinite product-spaces.* Let $S_1, S_2, \ldots$ be an infinite sequence of spaces in each of which is defined a measure. This means that in every S_i is given a complete field of sets (vollständiger Mengenkörper), the sets of which are referred

---

[1]) See: A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung (Ergebnisse der Mathematik und ihrer Grenzgebiete, Zweiter Band, 3 Heft, Berlin 1933) p. 60—61.

[2]) K. Knopp, Mengentheoretische Behandlung einiger Probleme der diophantischen Approximationen und der transfiniten Wahrscheinlichkeiten, Math. Annalen 95 (1925) p. 409—426.

to as the measurable sets in S_i, and that on this field is defined a non-negative completely additive set function μ_i. We assume that $\mu_i(S_i)=1$ for any $i=1,2,\dots$.

The product-space $\mathfrak{P}$ of the spaces $S_1, S_2, \dots$ is the space whose points are the infinite sequences $(x_1, x_2, \dots)$ of points $x_i \in S_i$. If the $E_i \subset S_i$ $(i=1,2,\dots)$ are sets in the space S_i, then we shall denote by

$$(E_1, E_2, \dots) \tag{1}$$

the set in $\mathfrak{P}$ formed by all sequences $(x_1, x_2, \dots)$ with $x_i \in E_i$ $(i=1,2,\dots)$. We shall call (1) an *elementary set* if the E_i are measurable and if only a finite number of them are proper parts of the corresponding S_i.

It was shown by Z. Łomnicki and S. Ulam [3]) that it is possible to define a measure μ in $\mathfrak{P}$ with the property that any elementary set $\mathfrak{E}=(E_1, E_2, \dots)$ is measurable and

$$\mu(\mathfrak{E}) = \mu_1(E_1)\,\mu_2(E_2)\dots \tag{2}$$

This measure may be obtained in the following manner. We define $\mu(\mathfrak{E})$ for an elementary set $\mathfrak{E}$ by (2). It can be proved that, if $\mathfrak{E}_1, \mathfrak{E}_2, \dots$ are elementary sets and

$$\mathfrak{E} = \mathfrak{E}_1 + \mathfrak{E}_2 + \dots$$

while the $\mathfrak{E}_i$ are mutually disjoint, we have

$$\mu(\mathfrak{E}) = \mu(\mathfrak{E}_1) + \mu(\mathfrak{E}_2) + \dots$$

On putting for an arbitrary set $\mathfrak{A} \subset \mathfrak{P}$

$$\overline{\mu}(\mathfrak{A}) = \text{greatest lower bound} \sum_{i=1}^{\infty} \mu(\mathfrak{E}_i)$$

for all possible coverings $\mathfrak{E}_1 + \mathfrak{E}_2 + \dots \supset \mathfrak{A}$ with elementary sets, it is found that $\overline{\mu}$ is an exterior measure in the sense of Carathéodory [4]). Now Carathéodory's measurability theory ensures the existence of a complete field of sets on which $\overline{\mu}$ is completely additive and it can be proved that any elementary set belongs

[3]) Z. Łomnicki et S. Ulam, Sur la théorie de la mesure dans les espaces combinatoires et son application au calcul des probabilités I. Variables indépendantes, Fund. Math. 23 (1934) p. 237—278.

[4]) C. Carathéodory, Vorlesungen über reelle Funktionen, 2 Auflage (Leipzig und Berlin 1927), Kapitel V.

to it. If $\mathfrak{A}$ belongs to this field we shall call $\mathfrak{A}$ measurable and we shall write $\mu(\mathfrak{A})$ instead of $\overline{\mu}(\mathfrak{A})$ and call $\mu(\mathfrak{A})$ the measure of $\mathfrak{A}$.

3. *A lemma.* Let us take a fixed n and consider the product space $\mathfrak{Q}$ of the spaces $S_{n+1}, S_{n+2}, \dots$. In $\mathfrak{Q}$ we define a measure ν and a corresponding exterior measure $\overline{\nu}$ in the way it was done above for $\mathfrak{P}$. Let $\mathfrak{B}$ be some set in $\mathfrak{Q}$, let $E_1, E_2, \dots, E_n$ be measurable sets in $S_1, S_2, \dots, S_n$ respectively, and let $\mathfrak{A}$ be the set of all sequences $(x_1, x_2, \dots)$ with

$$x_i \in E_i \ (i=1,\dots,n), \quad (x_{n+1}, x_{n+2}, \dots) \in \mathfrak{B}.$$

Then we have

$$\mu(\mathfrak{A}) = \mu_1(E_1)\,\mu_2(E_2)\dots\mu_n(E_n)\,\overline{\nu}(\mathfrak{B}).$$

A proof of this relation can be given by means of a reasoning similar to one used by Łomnicki and Ulam [5]).

4. *Homogeneous Sets.* We shall say that a measurable set $\mathfrak{A}$ in $\mathfrak{P}$ is *homogeneous*, if for every elementary set $\mathfrak{E}$ with $\mu(\mathfrak{E})>0$ the quotient

$$\frac{\mu(\mathfrak{A}\mathfrak{E})}{\mu(\mathfrak{E})}$$

has the same value. As is found on setting $\mathfrak{E}=\mathfrak{P}$ this value then necessarily is $\mu(\mathfrak{A})$.

Theorem. *The measure of a homogeneous set is either 0 or 1.*

Proof. Let ε be a positive number. We can find a covering

$$\mathfrak{E}_1 + \mathfrak{E}_2 + \dots \supset \mathfrak{A}$$

with elementary sets such that

$$\mu(\mathfrak{E}_1) + \mu(\mathfrak{E}_2) + \dots < \mu(\mathfrak{A}) + \varepsilon.$$

We have then

$$\mathfrak{A} = \mathfrak{A}\mathfrak{E}_1 + \mathfrak{A}\mathfrak{E}_2 + \dots$$

and hence

$$\mu(\mathfrak{A}) = \sum_{i=1}^{\infty} \mu(\mathfrak{A}\mathfrak{E}_i) = \mu(\mathfrak{A}) \sum_{i=1}^{\infty} \mu(\mathfrak{E}_i) < \mu(\mathfrak{A})^2 + \varepsilon\,\mu(\mathfrak{A}).$$

[5]) L. c. [1]), §§ 3 and 4.

This holds for any $\varepsilon > 0$. Therefore

$$\mu(\mathfrak{A}) \leqslant \mu(\mathfrak{A})^2.$$

Hence, since $0 \leqslant \mu(\mathfrak{A}) \leqslant 1$, either $\mu(\mathfrak{A}) = 0$ or $\mu(\mathfrak{A}) = 1$.

5. We shall now consider an important class of homogeneous sets. Suppose that $\mathfrak{A}$ is measurable and such that the relation

$$(x_1, x_2, \ldots) \in \mathfrak{A}$$

depends only on the asymptotic behaviour of the sequence $x_1, x_2, \ldots$. By this is meant that a relation $(x_1, x_2, \ldots) \in \mathfrak{A}$ remains true when a *finite* number of the x_i are replaced by others. *Then the set $\mathfrak{A}$ is homogeneous and has consequently either the measure* 0 *or the measure* 1.

To prove this we consider two elementary sets,

$$\mathfrak{E} = (E_1, E_2, \ldots) \text{ and } \mathfrak{F} = (F_1, F_2, \ldots),$$

both having positive measure. There is an index n such that both E_i and F_i for $i > n$ are identical with S_i. Let $\mathfrak{B}$ denote the set in the product-space $\mathfrak{Q}$ (see § 3) formed by all sequences $(x_{n+1}, x_{n+2}, \ldots)$ taken from sequences $(x_1, x_2, \ldots) \in \mathfrak{A}$. Then $\mathfrak{A}\mathfrak{E}$ is the set of all sequences $(x_1, x_2, \ldots)$ with $x_i \in E_i$ $(i = 1, \ldots, n)$ and $(x_{n+1}, x_{n+2}, \ldots) \in \mathfrak{B}$ and $\mathfrak{A}\mathfrak{F}$ is the set of all sequences $(x_1, x_2, \ldots)$ with $x_i \in F_i$ $(i = 1, \ldots, n)$ and $(x_{n+1}, x_{n+2}, \ldots) \in \mathfrak{B}$. Here we have used the assumption made about $\mathfrak{A}$. It follows from § 3 that

$$\mu(\mathfrak{A}\mathfrak{E}) = \bar{\mu}(\mathfrak{A}\mathfrak{E}) = \mu_1(E_1) \ldots \mu_n(E_n)\,\bar{\nu}(\mathfrak{B}) = \mu(\mathfrak{E})\,\bar{\nu}(\mathfrak{B})$$

and

$$\mu(\mathfrak{A}\mathfrak{F}) = \bar{\mu}(\mathfrak{A}\mathfrak{F}) = \mu_1(F_1) \ldots \mu_n(F_n)\,\bar{\nu}(\mathfrak{B}) = \mu(\mathfrak{F})\,\bar{\nu}(\mathfrak{B}).$$

Hence

$$\frac{\mu(\mathfrak{A}\mathfrak{E})}{\mu(\mathfrak{E})} = \frac{\mu(\mathfrak{A}\mathfrak{F})}{\mu(\mathfrak{F})}$$

and thus the homogeneity of $\mathfrak{A}$ has been demonstrated.

6. *Application to problems concerning probability of convergence.* A real function $f(x_1, x_2, \ldots)$ defined on $\mathfrak{P}$ is said to be measurable if for any real number α the set

$$\{f(x_1, x_2, \ldots) > \alpha\}$$

is measurable. It is not necessary to dwell any longer upon this definition, as everything is analogous to what may be said in the case of the Lebesgue measure.

Let be given a sequence

$$f_n(x_1, x_2, \ldots) \qquad (n = 1, 2, \ldots)$$

of measurable functions on $\mathfrak{P}$. We shall denote by $\mathfrak{C}$ its convergence set, i. e. the set of all points $(x_1, x_2, \ldots) \in \mathfrak{P}$ for which $\lim_{n \to \infty} f_n(x_1, x_2, \ldots)$ exists and is a finite number. It is clear that we can write

$$\mathfrak{C} = \prod_{i=1}^{\infty} \sum_{k=1}^{\infty} \prod_{l=1}^{\infty} \{|f_{k+l} - f_k| < \frac{1}{i}\}.$$

Hence $\mathfrak{C}$ is a measurable set. Its measure is *the probability of convergence* of the given sequence.

After what has been found in § 5 the following theorem is evidently true.

Theorem. *If the convergence of the given sequence depends only on the asymptotic behaviour of the sequence $(x_1, x_2, \ldots)$, then its probability of convergence is either* 0 *or* 1.

A particular case is obtained on assuming that

$$f_n(x_1, x_2, \ldots) = \varphi_1(x_1) + \varphi_2(x_2) + \ldots + \varphi_n(x_n)$$

where the φ_i are measurable functions on the S_i. In terms of the theory of probability, this means that we have a series of mutually independent chance variables and thus we find that such a series converges either with probability 0 or with probability 1. This result was obtained by Khintchine and Kolmogoroff [6]).

7. We consider again the sequence f_n and make the assumption that for any index p and arbitrary sequences

$$(x_1, \ldots, x_p, x_{p+1}, x_{p+2}, \ldots) \text{ and } (a_1, \ldots, a_p, x_{p+1}, x_{p+2}, \ldots)$$

$$\lim_{n \to \infty} |f_n(x_1, \ldots, x_p, x_{p+1}, x_{p+2}, \ldots) - f_n(a_1, \ldots, a_p, x_{p+1}, x_{p+2}, \ldots)| = 0.$$

[6]) A. Khintchine und A. Kolmogoroff, Über Konvergenz von Reihen, deren Glieder durch den Zufall bestimmt werden, Recueil Math. de Moscou 32 (1925) p. 668—677. See also: P. Lévy, Sur les séries dont les termes sont des variables éventuelles indépendantes, Studia Math. 3 (1931) p. 119—155.

Let $\mathfrak{A}_\lambda$ be the set in $\mathfrak{P}$ on which

$$\lim_{n\to\infty} \sup f_n(x_1, x_2, \ldots) > \lambda.$$

Since

$$\mathfrak{A}_\lambda = \sum_{i=1}^{\infty} \prod_{k=1}^{\infty} \sum_{l=k}^{\infty} \{f_l > \lambda + \frac{1}{i}\},$$

$\mathfrak{A}_\lambda$ is measurable. Clearly the relation $(x_1, x_2, \ldots) \in \mathfrak{A}_\lambda$ implies $(a_1, \ldots, a_p, x_{p+1}, x_{p+2}, \ldots) \in \mathfrak{A}_\lambda$ for any $a_1, \ldots, a_p$. Hence the set $\mathfrak{A}_\lambda$ is homogeneous. Consequently $\mu(\mathfrak{A}_\lambda) = 0$ or 1. It is plain, further, that $\mathfrak{A}_{\lambda_1} \supset \mathfrak{A}_{\lambda_2}$ for $\lambda_1 < \lambda_2$. Suppose that at least for one λ $\mu(\mathfrak{A}_\lambda) = 1$ and let β denote the least upper bound of those values. We put $\lambda_k = \beta - \frac{1}{k}$, $\lambda'_k = \beta + \frac{1}{k}$ $(k = 1, 2, \ldots)$ and

$$\mathfrak{A} = \mathfrak{A}_{\lambda_1}\mathfrak{A}_{\lambda_2}\ldots, \quad \mathfrak{B} = \mathfrak{A}_{\lambda'_1} + \mathfrak{A}_{\lambda'_2} + \ldots.$$

Then

$$\mu(\mathfrak{A}) = 1, \quad \mu(\mathfrak{B}) = 0.$$

A sequence $(x_1, x_2, \ldots)$ for which $\lim_{n\to\infty} \sup f_n \geqslant \beta$, belongs to $\mathfrak{A}$, a sequence $(x_1, x_2, \ldots)$ for which $\lim_{n\to\infty} \sup f_n > \beta$, to $\mathfrak{B}$. Hence on $\mathfrak{A} - \mathfrak{B}$ we have $\lim_{n\to\infty} \sup f_n = \beta$. Thus we find that the sequence f_n has for almost all points in $\mathfrak{P}$ the same limes superior. A similar statement is true for the limes inferior and we obtain the following

Theorem. *Let f_n be a sequence of measurable functions on $\mathfrak{P}$, such that*

$$\lim_{n\to\infty} |f_n(x_1, \ldots, x_p, x_{p+1}, x_{p+2}, \ldots) - f_n(a_1, \ldots, a_p, x_{p+1}, x_{p+2}, \ldots)| = 0$$

for any $a_1, \ldots, a_p$. Then there are two numbers α and β (the values $-\infty$ and $+\infty$ being admitted) such that for almost all sequences

$$\lim_{n\to\infty} \inf f_n(x_1, x_2, \ldots) = \alpha, \quad \lim_{n\to\infty} \sup f_n(x_1, x_2, \ldots) = \beta.$$

We conclude with the following application of this theorem. Let

$$\varphi_1(x_1), \quad \varphi_2(x_2), \ldots$$

be a sequence of measurable functions on $S_1, S_2, \ldots$ respectively,

or, to use the terminology of the theory of probability, a sequence of mutually independent chance variables. We put

$$s_n(x_1, x_2, \ldots) = \frac{\varphi_1(x_1) + \ldots + \varphi_n(x_n)}{n}.$$

The sequence of the functions s_n satisfies the hypotheses of the above theorem. Hence there exist two numbers α and β, $\alpha \leqslant \beta$, such that with probability 1

$$\lim_{n\to\infty} \inf s_n = \alpha, \quad \lim_{n\to\infty} \sup s_n = \beta.$$

So that, if the law of great numbers $(\alpha = \beta)$ fails to hold, this failure is in a certain sense the same for almost all sequences.

(Reçu par la Rédaction le 7. 11. 1937).