ment étant $MM'$, nous demandons que sa position finale $NN'$ soit située sur la bissectrice de $MM'$. Il faut alors que la valeur finale de l'angle $\vartheta$ soit

$$\vartheta' = \vartheta + \frac{\pi}{2}, \text{ donc } \frac{\vartheta'}{2} = \frac{1 + \tan \frac{\vartheta}{2}}{1 - \tan \frac{\vartheta}{2}}.$$

Cette valeur correspondant à $s = MN = \frac{1}{\cos \vartheta}$, on doit avoir

$$\frac{1 + \tan \frac{\vartheta}{2}}{1 - \tan \frac{\vartheta}{2}} = e^{\frac{1}{\cos \vartheta} \tan \frac{\vartheta}{2}} \frac{\vartheta}{2}.$$

ou, en posant $\frac{\vartheta}{2} = t$,

$$\frac{1 + t}{1 - t} = \frac{1 + t}{t(1 - t)}.$$

Il est aisé de voir que cette équation admet dans $(0, 1)$ une seule racine; on trouve $t = 0.540370$ ou $\vartheta = 56^\circ 46' 15''$.

L'angle $\vartheta_0$ étant ainsi choisi, le point $Q$ décrira un arc $MN'$. Cet arc et l'arc symétrique par rapport à la droite $NN'$ composent avec les segments $MN, NM'$ une courbe (2) non convexe en forme de coeur. L'aire de cette courbe est $2 \tan \vartheta_0 = 305293$, son périmètre $\frac{4}{\cos \vartheta_0} = 729942$.

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**The law of nought-or-one in the theory of probability**

by

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1. **Introduction.** In the calculus of probability a number of cases have been found in which it can be asserted *a priori* that a certain probability has either the value nought or the value one, or, to speak in terms of the theory of measure, that the measure of a certain set is either nought or one. The occurrence of this "law of nought-or-one" when one is dealing with probabilities concerning real numbers in the interval $0 \leq x \leq 1$ was explained by K. Knopp. The reason of it may be seen in the fact that the sets under consideration are homogeneous sets, i.e. sets which are equally distributed over the sub-intervals of $0 \leq x \leq 1$, and these homogeneous sets are, if they are measurable, necessarily sets of measure 0 or 1.

In more general cases in which a probability can only assume the extreme values 0 and 1, this probability is the measure of a set in an infinite product-space. In what follows I shall show that also here we can account for the occurrence of the "nought-or-one law" by means of the notion of a homogeneous set.

2. **The measure in infinite product-spaces.** Let $S_1, S_2, \ldots$ be an infinite sequence of spaces in each of which is defined a measure. This means that in every $S_i$ is given a complete field of sets (vollständiger Mengenkörper), the sets of which are referred

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to as the measurable sets in $S_i$, and that on this field is defined a non-negative completely additive set function $\mu_i$. We assume that $\mu_i(S_i) = 1$ for any $i = 1, 2, \ldots$.

The product-space $\mathcal{B}$ of the spaces $S_1, S_2, \ldots$ is the space whose points are the infinite sequences $(x_1, x_2, \ldots)$ of points $x_i \in S_i$. If the $E_i \subseteq S_i$ ($i = 1, 2, \ldots$) are sets in the space $S_i$, then we shall denote by

$$\{E_1, E_2, \ldots\}$$

the set in $\mathcal{B}$ formed by all sequences $(x_1, x_2, \ldots)$ with $x_i \in E_i$ ($i = 1, 2, \ldots$). We shall call (1) an elementary set if the $E_i$ are measurable and if only a finite number of them are proper parts of the corresponding $S_i$.

It was shown by Z. Łomnicki and S. Ulam that it is possible to define a measure $\mu$ in $\mathcal{B}$ with the property that any elementary set $E = \{E_1, E_2, \ldots\}$ is measurable and

$$\mu(E) = \mu_1(E_1) \mu_2(E_2) \ldots$$

This measure may be obtained in the following manner. We define $\mu(E)$ for an elementary set $E$ by (2). It can be proved that, if $E_1, E_2, \ldots$ are elementary sets and

$$\mathcal{E} = E_1 + E_2 + \ldots$$

while the $E_i$ are mutually disjoint, we have

$$\mu(\mathcal{E}) = \mu(E_1) + \mu(E_2) + \ldots$$

On putting for an arbitrary set $\mathcal{A} \subseteq \mathcal{B}$

$$\bar{\mu}(\mathcal{A}) = \text{greatest lower bound} \sum_{i=1}^{\infty} \mu(E_i)$$

for all possible coverings $E_1 + E_2 + \ldots \supseteq \mathcal{A}$ with elementary sets, it is found that $\bar{\mu}$ is an exterior measure in the sense of Carathéodory. Now Carathéodory's measurability theory ensures the existence of a complete field of sets on which $\bar{\mu}$ is completely additive and it can be proved that any elementary set belongs

to it. If $\mathcal{A}$ belongs to this field we shall call $\mathcal{A}$ measurable and we shall write $\mu(\mathcal{A})$ instead of $\bar{\mu}(\mathcal{A})$ and call $\mu(\mathcal{A})$ the measure of $\mathcal{A}$.

3. A lemma. Let us take a fixed $n$ and consider the product space $\mathcal{Q}$ of the spaces $S_{n+1}, S_{n+2}, \ldots$. In $\mathcal{Q}$ we define a measure $\nu$ and a corresponding exterior measure $\overline{\nu}$ in the way it was done above for $\mathcal{B}$. Let $\mathcal{Y}$ be some set in $\mathcal{Q}$, let $E_1, E_2, \ldots, E_n$ be measurable sets in $S_1, S_2, \ldots, S_n$ respectively, and let $\mathcal{A}$ be the set of all sequences $(x_1, x_2, \ldots)$ with $x_i \in E_i$ ($i = 1, \ldots, n$), $(x_{n+1}, x_{n+2}, \ldots) \in \mathcal{B}$.

Then we have

$$\mu(\mathcal{A}) = \mu_1(E_1) \mu_2(E_2) \ldots \mu_n(E_n) \overline{\nu}(\mathcal{B})$$

A proof of this relation can be given by means of a reasoning similar to one used by Łomnicki and Ulam.

4. Homogeneous Sets. We shall say that a measurable set $\mathcal{A}$ in $\mathcal{B}$ is homogeneous, if for every elementary set $E$ with $\mu(E) > 0$ the quotient

$$\frac{\mu(\mathcal{A} \setminus E)}{\mu(E)}$$

has the same value. As is found on setting $\mathcal{E} = \mathcal{B}$ this value then necessarily is $\mu(\mathcal{A})$.

Theorem. The measure of a homogeneous set is either 0 or 1.

Proof. Let $\varepsilon$ be a positive number. We can find a covering

$$E_1 + E_2 + \ldots \supseteq \mathcal{A}$$

with elementary sets such that

$$\mu(E_1) + \mu(E_2) + \ldots \leq \mu(\mathcal{A}) + \varepsilon.$$

We have then

$$\mathcal{A} = \mathcal{A} E_1 + \mathcal{A} E_2 + \ldots$$

and hence

$$\mu(\mathcal{A}) = \sum_{i=1}^{\infty} \mu(\mathcal{A} E_i) \leq \mu(\mathcal{A}) \sum_{i=1}^{\infty} \mu(E_i) < \mu(\mathcal{A})^2 + \varepsilon \mu(\mathcal{A}).$$

\[\begin{array}{l}
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\end{array}\]
This holds for any \( \varepsilon > 0 \). Therefore

\[ \mu(\mathfrak{U}) \leq \mu(\mathfrak{V})^2. \]

Hence, since \( 0 \leq \mu(\mathfrak{U}) \leq 1 \), either \( \mu(\mathfrak{U}) = 0 \) or \( \mu(\mathfrak{U}) = 1 \).

5. We shall now consider an important class of homogeneous sets. Suppose that \( \mathfrak{U} \) is measurable and such that the relation

\[ (x_1, x_2, \ldots) \in \mathfrak{U} \]

depends only on the asymptotic behaviour of the sequence \( x_1, x_2, \ldots \). By this is meant that a relation \( (x_1, x_2, \ldots) \in \mathfrak{U} \) remains true when a finite number of the \( x_i \) are replaced by others. Then the set \( \mathfrak{U} \) is homogeneous and has consequently either the measure 0 or the measure 1.

To prove this we consider two elementary sets,

\[ \mathfrak{C} = (E_1, E_2, \ldots) \quad \text{and} \quad \mathfrak{D} = (F_1, F_2, \ldots), \]

both having positive measure. There is an index \( n \) such that both \( E_i \) and \( F_i \) for \( i > n \) are identical with \( S_i \). Let \( \mathfrak{B} \) denote the set in the product-space \( \Omega \) (see §3) formed by all sequences \( (x_{n+1}, x_{n+2}, \ldots) \) taken from sequences \( (x_1, x_2, \ldots) \in \mathfrak{U} \). Then \( \mathfrak{A} \mathfrak{C} \) is the set of all sequences \( (x_1, x_2, \ldots) \) with \( x_i \in E_i \) \((i = 1, \ldots, n)\) and \( (x_{n+1}, x_{n+2}, \ldots) \in \mathfrak{B} \) and \( \mathfrak{A} \mathfrak{D} \) is the set of all sequences \( (x_1, x_2, \ldots) \) with \( x_i \in F_i \) \((i = 1, \ldots, n)\) and \( (x_{n+1}, x_{n+2}, \ldots) \in \mathfrak{B} \).

Here we have used the assumption made about \( \mathfrak{U} \). It follows from §3 that

\[ \mu(\mathfrak{A} \mathfrak{C}) = \mu(\mathfrak{A} \mathfrak{D}) = \mu_1(E_1) \ldots \mu_n(E_n) \mathbf{v}(\mathfrak{B}) = \mu(\mathfrak{C}) \mathbf{v}(\mathfrak{B}) \]

and

\[ \mu(\mathfrak{A} \mathfrak{D}) = \mu(\mathfrak{A} \mathfrak{C}) = \mu_1(F_1) \ldots \mu_n(F_n) \mathbf{v}(\mathfrak{B}) = \mu(\mathfrak{D}) \mathbf{v}(\mathfrak{B}). \]

Hence

\[ \frac{\mu(\mathfrak{A} \mathfrak{C})}{\mu(\mathfrak{D})} = \frac{\mu(\mathfrak{A} \mathfrak{D})}{\mu(\mathfrak{C})}, \]

and thus the homogeneity of \( \mathfrak{U} \) has been demonstrated.

6. Application to problems concerning probability of convergence. A real function \( f(x_1, x_2, \ldots) \) defined on \( \mathfrak{B} \) is said to be measurable if for any real number \( \alpha \) the set

\[ \{ f(x_1, x_2, \ldots) > \alpha \} \]

is measurable. It is not necessary to dwell any longer upon this definition, as everything is analogous to what may be said in the case of the Lebesgue measure.

Let be given a sequence

\[ f_n(x_1, x_2, \ldots) \quad (n = 1, 2, \ldots) \]

of measurable functions on \( \mathfrak{B} \). We shall denote by \( \mathfrak{C} \) its convergence set, i.e. the set of all points \( (x_1, x_2, \ldots) \in \mathfrak{B} \) for which

\[ \lim_{n \to \infty} f_n(x_1, x_2, \ldots) \]

exists and is a finite number. It is clear that we can write

\[ \mathfrak{C} = \bigcup_{i=1}^{\infty} \sum_{k=1}^{\infty} \bigcup_{l=1}^{\infty} \{ |f_{k+l} - f_k| < \frac{1}{l} \}. \]

Hence \( \mathfrak{C} \) is a measurable set. Its measure is the probability of convergence of the given sequence.

After what has been found in §5 the following theorem is evidently true.

**Theorem.** If the convergence of the given sequence depends only on the asymptotic behaviour of the sequence \( (x_1, x_2, \ldots) \), then its probability of convergence is either 0 or 1.

A particular case is obtained on assuming that

\[ f_n(x_1, x_2, \ldots) = \varphi_1(x_1) + \varphi_2(x_2) + \ldots + \varphi_n(x_n) \]

where the \( \varphi_i \) are measurable functions on the \( S_i \). In terms of the theory of probability, this means that we have a series of mutually independent chance variables and thus we find that such a series converges either with probability 0 or with probability 1. This result was obtained by Khintchine and Kolmogoroff.

7. We consider again the sequence \( f_n \) and make the assumption that for any index \( p \) and arbitrary sequences

\[ (x_1, \ldots, x_p, x_{p+1}, x_{p+2}, \ldots) \quad \text{and} \quad (a_1, \ldots, a_p, x_{p+1}, x_{p+2}, \ldots) \]

\[ \lim_{n \to \infty} \left| f_n(x_1, x_2, x_{p+1}, x_{p+2}, \ldots) - f_n(a_1, a_2, x_{p+1}, x_{p+2}, \ldots) \right| = 0. \]
Let \( \mathcal{A}_i \) be the set in \( \mathcal{B} \) on which
\[
\lim_{n \to \infty} \sup_{n} f_n(x_1, x_2, ...) > \lambda.
\]
Since
\[
\mathcal{A}_i = \sum_{i=1}^{\infty} \prod_{k=1}^{\infty} \sum_{l=1}^{\infty} \{f_l > \frac{\lambda + 1}{l}\},
\]
\( \mathcal{A}_i \) is measurable. Clearly the relation \((x_1, x_2, ...) \in \mathcal{A}_2 \) implies \((a_1, a_2, x_{p+1}, x_{p+2}, ...) \in \mathcal{A}_2 \) for any \( a_1, a_2, x_{p+1}, x_{p+2}, ... \). Hence the set \( \mathcal{A}_2 \) is homogeneous. Consequently \( \mu(\mathcal{A}_2) = 0 \) or 1. It is plain, further, that \( \mathcal{A}_1 \subseteq \mathcal{A}_2 \) for \( \lambda_1 < \lambda_2 \). Suppose that at least for one \( \lambda \), \( \mu(\mathcal{A}_2) = 1 \) and let \( \beta \) denote the least upper bound of those values. We put \( \lambda_k = \beta - \frac{1}{k}, \lambda'_k = \beta + \frac{1}{k} \) \((k = 1, 2, ...) \) and
\[
\mathcal{A} = \mathcal{A}_{11} \mathcal{A}_{12} \ldots, \mathcal{B} = \mathcal{A}_{21} + \mathcal{A}_{22} + \ldots.
\]
Then
\[
\mu(\mathcal{A}) = 1, \quad \mu(\mathcal{B}) = 0.
\]
A sequence \((x_1, x_2, ...)\) for which \( \lim_{n \to \infty} \sup_{n} f_n \geq \beta \), belongs to \( \mathcal{A} \), a sequence \((x_1, x_2, ...)\) for which \( \lim_{n \to \infty} \sup_{n} f_n > \beta \), to \( \mathcal{B} \). Hence on \( \mathcal{A} - \mathcal{B} \) we have \( \lim_{n \to \infty} f_n = \beta \). Thus we find that the sequence \( f_n \) has for almost all points in \( \mathcal{B} \) the same limes superior. A similar statement is true for the limes inferior and we obtain the following

**Theorem.** Let \( f_n \) be a sequence of measurable functions on \( \mathcal{B} \), such that
\[
\lim_{n \to \infty} |f_n(x_1, x_2, ..., x_{p+1}, x_{p+2}, ...) - f_n(a_1, a_2, x_{p+1}, x_{p+2}, ...)\| = 0
\]
for any \( a_1, a_2 \). Then there are two numbers \( \alpha \) and \( \beta \) (the values \( -\infty \) and \( +\infty \) being admitted) such that for almost all sequences
\[
\lim_{n \to \infty} \inf_{n} f_n(x_1, x_2, ...) = \alpha, \quad \lim_{n \to \infty} \sup_{n} f_n(x_1, x_2, ...) = \beta.
\]
We conclude with the following application of this theorem. Let
\[
\varphi_1(x), \quad \varphi_2(x), \ldots
\]
be a sequence of measurable functions on \( S_1, S_2, \ldots \) respectively,

or, to use the terminology of the theory of probability, a sequence of mutually independent chance variables. We put
\[
s_n(x_1, x_2, ...) = \frac{\varphi_1(x_1) + \ldots + \varphi_n(x_n)}{n}.
\]
The sequence of the functions \( s_n \) satisfies the hypotheses of the above theorem. Hence there exist two numbers \( \alpha \) and \( \beta \), \( \alpha \leq \beta \), such that with probability 1
\[
\lim_{n \to \infty} \inf_{n} s_n = \alpha, \quad \lim_{n \to \infty} \sup_{n} s_n = \beta.
\]
So that, if the law of great numbers \( (\alpha = \beta) \) fails to hold, this failure is in a certain sense the same for almost all sequences.

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