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Resolving Banach spaces
by
RICHARD EVANS (Berlin)

Abstract. In this paper we define a property of a projection algebra on a Banach space which we show to be necessary and sufficient for the existence of a resolution of the space taking values in a Banach lattice with order-continuous norm (proper L-space).

0. Introduction. If $K$ is a compact Hausdorff space and $f \mapsto T_f$ an isometric embedding of $C(K)$ into $B(X)$ for some Banach space $X$, the question arises whether it is possible to find a norm resolution for $X$ over $K$, i.e. a mapping $x \mapsto (x)$ of $X$ into some Banach function space over $K$ with the properties (a) $\|x\| = \|x\|$ for all $x$ in $X$, (b) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y$ in $X$, (c) $\|f(x) - f(y)\| = \|f\| \cdot \|x - y\|$ for all $x, y$ in $X$, (d) $\|x\| \leq (T_f(x), x)$ for all $f$ in $C(K)$, $x$ in $X$.

In [4] Cunningham showed the existence of such a resolution in the case where the operators $T_f$ have a lattice property similar to that in $M$-spaces. He also showed how $X$ can then be represented as a space of vector-valued functions over $K$, the function module representation, in such a way that the operators $T_f$ on $X$ correspond to multiplication by $f$ in the representation. In his doctoral thesis [5] the author showed how to construct a representation of an analogous type, the integral module representation, in the case where the embedded copy of $C(X)$ is a strongly closed algebra generated by $L^p$-projections, that is projections $E$ which have the norm decomposing property

$$\|x\|^p = \|Ex\|^p + \|x - Ex\|^p$$

for all $x$, some $p \in [1, \infty]$.

Attempts to extend the concept of $L^p$-projections by introducing projections with a more general decomposition property have failed, since it turns out that in all but trivial cases the only projections with the apparently more general property are the $L^p$-projections themselves (e.g. [3]). In this paper we define a decomposition property, not of individual projections, but of a complete projection algebra and show that this is general enough to completely describe the case where $X$ has a resolution taking values in a Banach lattice with order-continuous norm.

1. Projection algebras. A (linear) projection on a Banach space $X$ is a linear mapping $E : X \to X$ such that $E^2 = E$. It follows that $(I - E)^2$
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Theorem 1.1. Let \( \mathcal{A} \) be a projection algebra on a Banach space \( X \) consisting solely of bicontractive projections. Let \( f : T \to \mathcal{A} \) be the algebra isomorphism between the step functions in \( C(K) \) and the operators in \( \mathcal{L}(X) \) (the Stonean space of \( \mathcal{A} \)). Then \( \|f\| \leq \|f\|_{\mathcal{A}} \) for all step functions \( f \) in \( C(K) \), where \( M = \sup_{f \in \mathcal{A}} \|f\| \) is the norm of \( f \).

Note. Since all \( f \)’s are bicontractive, we have \( 1 \leq M \leq 2 \).

Proof. Let \( f \) be a positive step function. Then we can write \( f \) in the form \( f = \sum \alpha_i E_i \), where all \( \alpha_i \)’s are positive and \( 0 = D_0 \leq D_1 \leq D_2 \leq \ldots \leq D_n \). In this case we clearly have \( \|f\| = \sum \alpha_i \). The step function is then equal to \( \sum \alpha_i E_i \), whereby the projections \( E_i \) correspond to the clopen sets \( D_i \) in \( K \). Since all the \( E_i \)’s have norm 1, we have immediately \( \|T\| \leq \sum \alpha_i = \|f\| \).

Now suppose that \( f \) is an arbitrary step function. Then there is a clopen set \( D \) in \( K \) such that \( (2 \mathcal{X}_D - 1) \cdot f \) is positive. Let \( B \) be the corresponding projection in \( \mathcal{A} \). We then have

\[
\|T\| = \|2B - I - (2B - I)T\| \leq \|2B - I\| \cdot \|2B - I\| T \leq M \cdot \|2B - I\| T \leq M \cdot \|2B - I\| = M \cdot \|T\|_{\mathcal{A}} = \|T\| \leq \|f\|.
\]

For the other inequality, let \( f = \sum \alpha_i E_i \) be the representation of a continuous step function in which the \( D_i \)’s are disjoint and nonempty. In this case \( \|f\| \) is clearly the maximum of the \( \|\alpha_i\|’s \), say \( \|\alpha_i\| \). Let \( E_i \) be the projection in \( \mathcal{A} \) corresponding to \( D_i \). Then for \( x \in E_i X \), \( x = 0 \) we have \( \|x\| = \|\alpha_i E_i x\| = \|\alpha_i\| \|x\| \) so that \( \|T\| \geq \|\alpha_i\| = \|f\| \).

Note that the bounds in this lemma are the best possible since the functions corresponding to \( 2B - I \), \( B \in \mathcal{A} \), all have unit norm.

The result of Lemma 1.3 motivates the following definition:

1.4. Lemma. Let \( \mathcal{A} \) be a projection algebra on a Banach space \( X \) and \( K \) the Stonean space of \( \mathcal{A} \). The natural correspondence between the continuous step functions on \( K \) and the operators in \( \mathcal{L}(X) \) is an isomorphism. Since we have \( \|2B - I\| = \|2B - I\| \), \( T \) is equivalent to \( \|2B - I\| = 1 \).

A mirror-projection is clearly bi-contractive, since

\[ 2(Bx) = [2Bx - x - x + 2(Bx - I)x] = 2(Bx) \]

This shows that \( T \) is an isometry if and only if all projections in \( \mathcal{A} \) are mirror-projections.

In this case we can extend the isometry to the closure of \( \mathcal{L}(X) \). However, the product of commuting mirror-projections need not be one. It has been shown (2, Sect. 2) that in the classical Banach spaces \( L^p \)-spaces, \( C(K) \)-spaces, Lindestrauss spaces) all bicontractive projections are mirror-projections. As an immediate corollary of Lemma 1.3 we now obtain:

1.5. Corollary. Let \( \mathcal{A} \) be a projection algebra on a Banach space \( X \) and \( K \) the Stonean space of \( \mathcal{A} \). The natural correspondence between the continuous step functions on \( K \) and the operators in \( \mathcal{L}(X) \) is an isomorphism if and only if all projections in \( \mathcal{A} \) are mirror-projections.

1.6. Proposition. Let \( \mathcal{A} \) be a projection algebra on a Banach space \( X \) and \( K \) the Stonean space of \( \mathcal{A} \). Then \( \mathcal{L}(X) \) is isometrically a bicontractive algebra isomorphism to \( C(K) \) if and only if all projections in \( \mathcal{A} \) are mirror-projections.

Proof. Since an algebra-isomorphism maps idempotent elements into idempotent elements, it is the natural correspondence of Corollary 1.5 (except perhaps for a homeomorphism of \( K \)).
In view of the above results, when $\mathfrak{H}$ consists of mirror-projections, we shall identify $\lim \mathfrak{H}$ and $\mathcal{C}(\mathfrak{H})$ and write $\pi_{f}$ for the action of the operator corresponding to the function $f$ on the element $x$ in $\mathcal{X}$.

2. Decomposition properties. Having in the last section obtained an embedding of $\mathcal{C}(\mathfrak{H})$ in $B(\mathcal{X})$ for the case where $\mathcal{X}$ is the Stonean space of a Boolean algebra of mirror-projections, we now wish to obtain a norm resolution for $\mathfrak{H}$ over $\mathcal{X}$. In general, such a resolution need not exist since the projections in $\mathfrak{H}$ need not decompose the norms of elements in $\mathfrak{H}$ in a consistent manner. Thus we shall need to demand some further property of $\mathfrak{H}$ which will guarantee a consistent decomposition. A seemingly weak property of this nature is contained in the following definition:

2.1. Definition. A projection algebra $\mathfrak{H}$ on a Banach space $\mathcal{X}$ is said to be monotone, if the following condition holds:

If $E_{1}, E_{2}, \ldots, E_{n}$ are pairwise orthogonal elements of $\mathfrak{H}$ with $\bigvee_{i=1}^{n} E_{i} = I$ and for some $x, y$ in $\mathcal{X}$, we have $|E_{i}x| \geq |E_{j}y|$ for all $i, j$, then also $|x| \geq |y|$, i.e. if all parts of $x$ are larger (in norm) than the respective parts of $y$, then $x$ itself is larger than $y$.

Unfortunately this property is far too stringent for our purpose. Indeed, the author has shown {\[E_{i} = E_{j}\]} that in all but trivial cases a monotone algebra consists only of $F_{p}$-projections for some fixed $p$. We can weaken Definition 2.1 by requiring only that $|x| \geq |y|$ whenever $|E_{x}| \geq |E_{y}|$ for a larger number of projections than merely a single projection of the identity. Since it itself is in $\mathfrak{H}$, it is clearly vacuous to demand $|E_{x}| = |E_{y}|$ for all $E$ in $\mathfrak{H}$. The required modification is the subject of the following definition:

2.2. Definition. A subset $A$ of a Boolean algebra $\mathfrak{H}$ is said to be co-final in $\mathfrak{H}$ if for all $E \in \mathfrak{H}$, $E \neq 0$, there is an $E$ in $A$ with $0 < E \leq F$.

A projection algebra on a Banach space $\mathcal{X}$ is said to be uniformly decomposing if and only if $|x| \geq |y|$ whenever the set $\{E \in \mathfrak{H} : |E_{x}| \geq |E_{y}|\}$ is co-final in $\mathfrak{H}$, $x, y \in \mathcal{X}$.

Note. It then follows that $|E_{x}| \geq |E_{y}|$ for all $E$ in $\mathfrak{H}$.

In the case of complete projection algebras there is another formulation of the uniformly decomposing property which turns out to be more useful for the construction of a norm resolution.

2.3. Lemma. Let $\mathfrak{H}$ be a complete projection algebra on the Banach space $\mathcal{X}$. The following two statements are equivalent:

(a) $\mathfrak{H}$ is uniformly decomposing;

(b) For all $x, y$ in $\mathcal{X}$ there is an $E$ in $\mathfrak{H}$ such that $F \leq E \Rightarrow |E_{x}| \geq |E_{y}|$ and $F \perp E \Rightarrow |E_{x}| \leq |E_{y}|$ for $F$ in $\mathfrak{H}$.

Proof. (a) $\Rightarrow$ (b) Let $x, y$ be given. Set $E = \bigvee \{F : \text{if } F \in \mathfrak{H}, |F_{x}| \geq |F_{y}| \text{ for } F \leq E\}$. Since $\mathfrak{H}$ is uniformly decomposing, we clearly have $|F_{x}| \geq |F_{y}|$ for all $F \leq E$. Suppose $F \perp E$ and $\tilde{F} \in \mathfrak{H}$, $\tilde{F} \neq 0$. Then either there is an $\tilde{F}' \leq \tilde{F}$ with $|F_{x}| \leq |F_{y}|$ or we have $\tilde{F} \perp E$ which then means $\tilde{F} = 0$ since $F \perp E$. In the latter case set $\tilde{F} = \tilde{F}'$. Either way we now have $\tilde{F} \leq \tilde{F} \Rightarrow |F_{x}| \geq |F_{y}|$ with $|\tilde{F}_{x}| \leq |\tilde{F}_{y}|$. Thus the set of all $\tilde{F} \in \mathfrak{H}$ with $|\tilde{F}_{x}| \leq |\tilde{F}_{y}|$ is co-final in $\mathfrak{H}$. Since $\mathfrak{H}$ is uniformly decomposing, we have $|F_{x}| \leq |F_{y}|$ as required.

(b) $\Rightarrow$ (a) Suppose that $|x| < |y|$ for some pair $x, y$ in $\mathcal{X}$. Choose $\varepsilon > 0$ with $|x| < |y|$ and use (b) to find an $E$ in $\mathfrak{H}$ with $F \in \mathfrak{H}$, $|F_{x}| = |F_{y}| \geq |F(1-\varepsilon)y|$ and $F \perp E$, $|F_{x}| = |F(1-\varepsilon)y|$. Let $\tilde{F}$ be the carrier projection of $x$, that is, the smallest projection which maps $x$ onto itself. Then $\tilde{F} \perp \tilde{E}$ since $|x| < (1-\varepsilon)|y|$. Thus $\tilde{E} \perp (1-\varepsilon)E$. Let $\tilde{E} \leq \tilde{E} \perp (1-\varepsilon)E$, $\tilde{F} \neq 0$. Then $|\tilde{F}_{x}| = |\tilde{F}_{y}|$ and $F \perp E$ so that $|F_{x}| \leq |F(1-\varepsilon)y| < |F_{y}|$. Thus the set $\{F : F \in \mathfrak{H}, |F_{x}| \geq |F_{y}|\}$ is not co-final in $\mathfrak{H}$.

Thus a complete uniformly decomposing projection algebra contains, for each pair $x, y$, a projection which divides the space into the part where $x$ is larger (in norm) than $y$ and the part where $y$ is larger. This is clearly a necessary condition for the existence of a norm resolution whose values lie in a lattice with order-continuous norm. We shall see in the next section that it is also sufficient.

Since Proposition 1.6 refers to projection algebras consisting only of mirror-projections, it would seem that we must in future demand of $\mathfrak{H}$ that it is complete, uniformly decomposing and consists only of mirror-projections. The following simple lemma shows that the latter is redundant.

2.4. Lemma. Let $\mathfrak{H}$ be a (not necessarily complete) uniformly decomposing projection algebra on the Banach space $\mathcal{X}$. Then $\mathfrak{H}$ consists solely of mirror-projections.

Proof. Let $E$ be a projection in $\mathfrak{H}$ and $x$ an element in $\mathcal{X}$. For $x$ and $(2E-I)x$ we have:

for $F \leq E$, $|F_{x}| = |2F_{x} - F_{x}| = |2FE_{x} - F_{x}| = |(2E-I)x|$

for $F \perp E$, $|F_{x}| = |F_{x} - 2F_{x}| = |F_{x} - 2E(I-E)x| = |(2E-I)x|$

and since the set $\{F : F \leq E \text{ or } F \perp E\}$ is co-final in $\mathfrak{H}$, we have $|x| = |(2E-I)x|$. Since $x$ was arbitrary, $2E-I$ is an isometry.

Note that although a sub-algebra of an algebra consisting solely of mirror-projections naturally also consists solely of mirror-projections,
a sub-algebra of an uniformly decomposing algebra need not be uniformly decomposing. Indeed we have the following proposition.

2.5. Proposition. If \( \mathcal{A} \) is an uniformly decomposing projection algebra on a Banach space \( X \) such that every sub-algebra of \( \mathcal{A} \) is also uniformly decomposing, then \( \mathcal{A} \) is monotone.

Proof. Let \( E_1, E_2, \ldots, E_n \) be pairwise orthogonal projections in \( \mathcal{A} \) with \( \sum E_i = I \). Let \( \mathcal{A}_1 \) be the sub-algebra of \( \mathcal{A} \) generated by the \( E_i \)’s. \( \mathcal{A}_1 \) is atomic and its atoms are the \( E_i \)’s. Suppose that for some \( x, y \in X \)

\[ \langle E_i x, x \rangle > \langle E_f y, y \rangle \]

for all \( i \). Then \( \{ E_i \} \) in \( \mathcal{A}_1 \), \( \| E x \| > \| E y \| \) is co-finite in \( \mathcal{A}_1 \), which is supposed uniformly decomposing. Thus \( \| x \| > \| y \| \), which implies that \( \mathcal{A} \) is monotone.

As already noted, in all but trivial cases a monotone algebra consists solely of \( p \)-projections for some fixed \( p \) and for these algebra the problem of constructing a norm resolution has already been solved.

3. A norm resolution. We now turn to the construction of a norm resolution for the case where \( \mathcal{A} \) is a complete uniformly decomposing projection algebra. The first step in this direction is the following proposition, which relies heavily on Lemma 2.3.

3.1. Proposition. Let \( \mathcal{A} \) be a complete uniformly decomposing projection algebra on the Banach space \( X \). For each \( x, y \in X \), \( y 
eq 0 \), the quotient \( \| E x \| / \| E y \| \) converges (possibly to \( \infty \)) along each ultrafilter \( U \) containing the carrier projection of \( y \).

Proof. Let \( x, y \) be elements of \( X \) and \( U \) an ultrafilter in \( \mathfrak{A} \) containing the carrier projection of \( y \). Then \( \| E y \| > 0 \) for all \( E \) in \( U \) so that the quotient \( \| E x \| / \| E y \| \) is defined. Let us assume that \( \| E x \| / \| E y \| \) does not converge along \( U \) so that we can find two co-finite nets \( (E_\alpha)_{\alpha \in A_1}, (E_\alpha)_{\alpha \in A_2} \) in \( U \) with

\[ \| E_\alpha x \| / \| E_\alpha y \| > \lambda_1, \quad \| E_\beta x \| / \| E_\beta y \| > \lambda_2 \]

whereby \( \lambda_1 > \lambda_2 (\lambda_1, \lambda_2 \) possibly \( \infty \)).

Let \( \lambda \) be a finite number with \( \lambda_1 > \lambda > \lambda_2 \). Then by Lemma 2.3 there is a projection \( E_\lambda \) in \( \mathfrak{A} \) such that \( \| E_\lambda x \| \leq \| E_\lambda y \| \) for \( F \leq E_\lambda \) and \( \| E_\lambda x \| \leq \| E_\lambda y \| \) for \( F \perp E_\lambda \). There are now two possibilities:

(a) \( E_\lambda \) lies in \( U \), but then \( E_\lambda \leq E_\lambda \) co-finitely for \( \gamma \in \Gamma \) so that

\[ \| E_\gamma x \| / \| E_\gamma y \| = 1 \lambda \| E_\gamma x \| / \| E_\gamma y \| \] is \( \lambda \) co-finitely.

(b) \( E_\lambda \) does not lie in \( U \), but then \( E_\lambda \perp E_\lambda \) co-finitely in \( \Gamma \) so that

\[ \| E_\gamma x \| / \| E_\gamma y \| = 1 \lambda \| E_\gamma x \| / \| E_\gamma y \| \] co-finitely.

In either case we have a contradiction, so that we may conclude that our assumption that the quotient does not converge was false.

Note. If \( K \) is the Stonean space of \( \mathfrak{A} \) the points of \( K \) are strictly speaking the ultrafilters in \( \mathfrak{A} \), nevertheless we shall write \( k \) for a point in \( K \) and \( U_k \) for the corresponding ultrafilter, with the interest of clarity.

For \( x \in X \), \( \text{sup} x \) will denote the clopen set in \( K \) corresponding to the carrier projection of \( x \), this is the same as \( \{ k \mid U_k \text{ contains the carrier projection of } x \} \).

The above proposition allows us to make the following definition:

3.2. Definition. Let \( \mathcal{A} \) be a complete uniformly decomposing projection algebra on the Banach space \( X \) and \( K \) the Stonean space of \( \mathfrak{A} \). For \( x, y \in X \), \( y \neq 0 \), we define

\[ \text{supp} x = \lim_{E \to X} \| E x \| / \| E y \| \]

for \( k \) in \( \text{supp} y \). Then \( x / y \) is continuous and finite almost everywhere for all \( k \in X \) and the mapping \( x / y \) from \( X \) into \( C(K) \) (resp. \( \text{supp} y \)) is sub-linear and absolutely homogeneous with respect to \( C(K) \) (resp. \( \text{supp} y \)).

Proof. Suppose \( x / y(k) = \lambda \), with \( 0 < \lambda < \infty \). Let \( \mathcal{A} \) be arbitrary between 0 and \( \lambda \). Then by 2.3 there is a projection \( E_\lambda \) in \( \mathfrak{A} \) such that

\[ \| E_\lambda x \| = \| E_\lambda y \| \| \lambda \| \| y \| \]

and a projection \( E_\lambda \) in \( \mathfrak{A} \) with

\[ \| E_\lambda x \| = \| E_\lambda y \| \| \lambda \| \| y \| \]

for all points \( k \) for which \( E_\lambda \) contains \( E_k \) in \( \mathfrak{A} \), then we have \( \lambda = \| y \| \) \( \leq \lambda + \epsilon \). These \( k \) form a clopen set containing \( k \). For \( \lambda = 0 \) or \( \infty \), an analogous argument with one projection suffices. In either case we have that \( x / y \) is continuous at \( k \).

Now let \( D \) be the clopen set \( \{ k \mid \text{supp} x \cap \text{supp} y \neq \emptyset \} \). Then for each natural number \( n \), \( \| E_k x \| > n \| E_k y \| \) is co-finite in \( \mathfrak{A} \) since \( \| E_k x \| / \| E_k y \| \to \infty \) along each ultrafilter \( U_k \) with \( k \in D \). Since \( \mathfrak{A} \) is uniformly decomposing, it follows that \( \{ x \} \geq \{ x \} \{ y \} \) for all \( x \) and thus that \( \text{supp} x \cap \text{supp} y \neq \emptyset \). Since \( D \leq \text{supp} y \) it follows that \( D = \text{supp} y \). Thus \( x / y \) is finite almost everywhere.

That the mapping \( x \to x / y \) is sub-linear follows immediately from the sub-linearity of the norm. To check absolute homogeneity let \( x \) be an element in \( X \) and \( f \) a function in \( C(K) \). We must show that for all \( k \in \text{supp} y \)

\[ (x / y(k))(k) = |k| (x / y(k))(k) \]

whenever \( x / y(k) \) is finite. Let \( k \) be a point in \( \text{supp} y \) and \( a = (x / y(k))(k) \).

\[ |E_k x| - |E_k a| = |E_k (x - a) u_a| \]

since the operator \( f \) lies in \( \text{lin} \mathfrak{A} \) and therefore commutes with \( E_k \). If \( D \) is the clopen set in \( K \) corresponding to \( E_k \) we have

\[ |E_k (x - a)| = |\bigcap_{k \in D} |E_k (x - a)| \]

by \( \text{supp} y \).
Since $a = f(b)$ and $f$ is continuous at $b$, we have $\|b(f - aI)\|_{\mathcal{C}_0} = 0$. It follows that
\[
\lim_{v_k \to 0} \frac{\|E(f - aI)\|_{\mathcal{C}_0}}{\|\alpha E\|_{\mathcal{C}_0}} = \lim_{v_k \to 0} \frac{\|E\|}{\|\alpha E\|_{\mathcal{C}_0}} = 0
\]
at each point $k$ where $x/y(k) = (\lim_{v_k \to 0} \|E\|)/\|\alpha E\|_{\mathcal{C}_0}$ is finite. Since
\[
\lim_{v_k \to 0} \frac{\|E\|}{\|\alpha E\|_{\mathcal{C}_0}} = (fz/y)(k) \quad \text{and} \quad \lim_{v_k \to 0} \frac{\|\alpha E\|_{\mathcal{C}_0}}{\|\alpha E\|} = (z/y)(k) = (z/y)(k)
\]
this is the required identity.

The mappings $x/y$ for each $y$ thus have the properties of a norm resolution except for the norm preserving property since the range space $C\mathcal{B}(supp\Gamma)$ is not normal and in general not even normable (as a lattice). We can however construct a Banach function lattice in $C\mathcal{B}(supp\Gamma)$ which is large enough to contain the functions $x/y$. This is the purpose of the following definitions and results.

3.4. Definitions. Let $X$ be a Banach space and $\mathcal{M}$ an arbitrary Boolean algebra of projections on $X$. We define the ordering $\geqslant$ (or $\supseteq$) on $\mathcal{M}$ if $\mathcal{M}$ is clear) by $x \geqslant y$ if there is an $E$ in $\mathcal{M}$ with $Ex = y$, i.e. $x$ is larger than $y$ if and only if it is an extension of it. This is clearly a partial order on $X$.

Let $X$ be a Banach space and $\mathcal{M}$ a complete uniformly decomposing Boolean algebra of projections on $X$ with Stonean space $K$. Suppose $\Gamma \subseteq X$ is a subset which is directed by $\geqslant$. The support of $\Gamma$ is the set $supp\Gamma := \bigcup_{C \subseteq supp\Gamma} C -$ a clopen subset of $K$. We define the mapping $m_j$ from $C(supp\Gamma)$ into $[0, \infty]$ by virtue of
\[
m_j(f) := \sup_{C \subseteq supp\Gamma} \|f\|_{C}
\]
and $C \subseteq C(supp\Gamma)$ by
\[
C := \{ f \in C(supp\Gamma), (f)_{\geqslant} \text{ is Cauchy} \}.
\]
Finally we define $M \subseteq C(supp\Gamma)$ as the set
\[
\{ f \in C(supp\Gamma), \text{the increasing net of positive } C \text{-functions which are majorized by } (f) \text{ is } M \text{-Cauchy} \}
\]
and extend $m_j$ to $M$ by means of
\[
m_j(f) := \sup_{C \subseteq supp\Gamma} \|m_j|_{C} = 0 \leqslant \|f\|, g \leqslant C_j,
\]
If $\Gamma$ consists solely of one element $y$, we shall write $M_j$ and $m_j$ instead of $M$ and $m_j$.

3.5. Proposition. $M_j$ is an order ideal in $C\mathcal{B}(supp\Gamma)$, $m_j$ is an order-continuous lattice norm for $M_j$ and $\Gamma$ is complete in $m_j$. Thus $M$ with $m_j$ is a Banach lattice with order-continuous norm.

Proof. It is clear from the definitions that $M_j$ is an order ideal in $C\mathcal{B}(supp\Gamma)$ and that $m_j$ is a lattice norm. It remains to show that $m_j$ is order-continuous and that $\Gamma$ is complete. Let $\{f_n\}$ be a downwards directed net in $M_j$, whose infimum is $0$. We may suppose without loss of generality that $|f_n|$ has a largest element, say $f$. Since $f$ lies in $M_j$, there is an $f_0$ in $C_j$ with $0 \leqslant f_n \leqslant f$ and $m_j(f - f_0) \leqslant \varepsilon/5$ for a given $\varepsilon > 0$. For each $\alpha$ we define $D_{\alpha} := \{ f \in \mathcal{M}, (f)_{\alpha} \leqslant (f_0)_{\alpha} \}$ and $D := \{ f \in \mathcal{M}, (f) \leqslant (f_0) \}$.

We then have $\int (f_0 - f)_{\alpha} \leqslant m_j(f_0)$ for all $\alpha$. Since the net converges in order to 0 and $f_0 - f_0 \alpha f$ is positive, we have $f_0 - f_0 \leqslant 0$. As $f_0$ lies in $C_j$, there is an $f_0$ in $C_j$ such that $\|f_0(y - y)\| \leqslant \varepsilon/5$ for $y \in C_j$. \(1 - \chi_0 f_0 y\) is 0, therefore by the completeness of $M$ there is an $a_0$ such that
\[
a \geqslant a_0 = \|1 - \chi_0 f_0 y\| \leqslant \varepsilon/5
\]
and then for $y \geqslant y_0$ we have
\[
\|1 - \chi_0 f_0 y\| \leqslant \|1 - \chi_0 f_0 (y - y_0)\| + \|1 - \chi_0 f_0 y_0\| \leqslant \|f_0(y - y_0)\| + \|1 - \chi_0 f_0 y_0\| \leqslant \varepsilon/5 + \varepsilon/5 = 2\varepsilon/5.
\]
We thus have $m_j((1 - \chi_0 f_0)_{\alpha}) \leqslant 2\varepsilon/5$. But then
\[
m_j((f_0 - f_0)_{\alpha}) \leqslant m_j(1 - \chi_0 f_0)_{\alpha} + m_j(\chi_0 f_0)_{\alpha} \leqslant m_j(1 - \chi_0 f_0)_{\alpha} + m_j(f_0)_{\alpha} \leqslant m_j(f_0)_{\alpha} + 2\varepsilon/5 + \varepsilon/5 \leqslant \varepsilon + \varepsilon < \varepsilon
\]
for $\alpha \geqslant a_0$.

Thus $(m_j(f_0))_{\alpha} \to 0$. This shows that $m_j$ is order-continuous.

In order to show that $M_j$ is complete, let $(f_n)$ be a monotone increasing Cauchy sequence of positive functions in $M_j$. Let $f$ be the supremum of $(f_n)$ in $C\mathcal{B}(supp\Gamma)$; if we show that $f$ is in $M_j$, we shall be finished since the order-continuity of $m_j$ implies that $(f_n)$ converges to $f$. Let $\varepsilon > 0$ be given. Since $(f_n)$ is Cauchy, there is an $n$ such that $m_j(f_n - f_0) < \varepsilon/2$ for all $n > n$. Since $f_n$ is in $M_j$, there is a $g_n$ in $C_j$ with $m_j(f_n - g_n) < \varepsilon/2$.

But then for $g$ in $C_j$, $g \leqslant f_n \leqslant f$ we have
\[
m_j(g - g_n) \leqslant m_j(g - g_n) + m_j(g_n - g_n) \leqslant m_j(g - g_n) + m_j(f_n - g_n) + \varepsilon
\]
\[
\leqslant m_j(g) + m_j(f_n - g_n) + \varepsilon < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\]
As $3\varepsilon < \varepsilon$, we have $(g - g_n)_{\alpha} \to 0$ in order and thus also in norm. So $m_j(g - g_n) \leqslant \varepsilon$ which implies (since $\varepsilon$ was arbitrary) that $f$ is in $M_j$.
of a norm resolution for \( X \). However, we must first check that the \( M_r \)'s are large enough to contain the functions \( x/y \).

3.6. **Definition.** If \( \Gamma \subseteq X \) is directed and \( \text{supp} \Gamma \neq \emptyset \), then for non-zero \( y, \bar{y} \in \Gamma \), \( y \geq \bar{y} \) the functions \( x/y, x/\bar{y} \) for \( x \in X \) are clearly equal on their common domain of definition. Thus we can define \( x/_{\Gamma} = \sup \text{supp} \Gamma \to [0, \infty) \) as the unique continuous extension to \( \text{supp} \Gamma \) of the functions \( x/y \) on the supports \( y, \bar{y} \in \Gamma \).

**Note.** If \( \Gamma \) contains a maximal element \( y \), then \( x/_{\Gamma} = x/y \); this is in particular the case when \( \Gamma \) only contains one \( y \).

3.7. **Lemma.** With \( X, \mathcal{A}, K, \Gamma \) as above we have:

(i) \( x + z/_{\Gamma} = \bar{x} + \bar{z}/_{\Gamma} \) for all \( x, \bar{x} \in X \);

(ii) \( f(s/\Gamma) = f(s/\bar{\Gamma}) \) for all \( s \in X, f \in C(K) \).

Furthermore, for every \( x \in X \), \( x/_{\Gamma} \in M_{\Gamma} \) and \( m_{\Gamma}(x/_{\Gamma}) = \|Ex\| \), whereby \( E \) is the projection in \( K \) corresponding to \( \Gamma \).

**Proof.** (i) and (ii) follow obviously from the corresponding relations for \( x/y \).

Let \( x \) be an element in \( X \) for which \( x/_{\Gamma} \) is finite. We write \( f = x/\Gamma \in C_{(\text{supp} \Gamma)} \). Then \( f(y) = f(\text{supp} \Gamma) = x/y \) for \( y \in \Gamma \). If \( E_x \) denotes the carrier projection of an element \( y \), then we have \( y_x = E_x y \) for \( y, y_x \in \Gamma \); \( y_x \geq y \). Thus
\[
\|f(y) - f(y_x)\| = \|E_x f - E_{x/\Gamma} f_x\| = \|E_x f - E_{x/y} f_x\|.
\]

Since \( E_x f \leq E_{x/\Gamma} \), we have \( \|E_x f - E_{x/y} f_x\| = \|E_x f - E_{x/\Gamma} f_x\| \leq \|f(y) - f(y_x)\| \) as a consequence of the completeness of \( \mathcal{A} \). Thus \( f(y) = f(y_x) \) if \( y_x \) is Cauchy and \( f \) therefore lies in \( C_{\Gamma} \).

3.8. **Lemma.** With \( X, \mathcal{A}, K, \Gamma \) as above let \( \Gamma_b \) be any finite subset of \( X \) with \( \text{supp} \Gamma_b \cap \text{supp} \Gamma = \emptyset \). Then there is a directed set \( \Gamma \subseteq X \) containing \( \Gamma_b \) with \( \text{supp} \Gamma = K \).

**In particular,** there are directed sets \( \Gamma \) with \( \text{supp} \Gamma = K \) (simply set \( \Gamma_b = \{x\} \) for some non-zero \( x \)).

**Proof.** By Zorn's Lemma there is a maximal set \( \Gamma \subseteq X \) containing \( \Gamma_b \) such that the supports of distinct elements of \( \Gamma \) are disjoint. Let \( D = \{f(y) \mid y \in \Gamma, f \in C(K)\} \). If \( D \neq \emptyset \), then there is a non-zero element \( x \in X \) with \( \text{supp} x \subseteq K \). This would contradict the maximality of \( \Gamma \). Thus \( \bigcup \text{supp} \Gamma = K \). Let \( \Gamma \) be the set of all finite sums of elements in \( \Gamma, Y \).

Since the sum of two elements with disjoint support majorises both elements in the order \( \geq \), \( \Gamma \) is a directed set containing \( \Gamma_b \) for which \( \text{supp} \Gamma = K \).

3.9. **Theorem.** Let \( X \) be a Banach space and \( \mathcal{A} \) a complete uniformly decomposing projection algebra on \( X \) with Stonean space \( K \). Then there is a norm resolution for \( X \) with respect to \( \lim \mathcal{A} \equiv C(K) \) taking values in a Banach lattice of continuous numerical functions on \( K \) with order-continuous norm.

**Proof.** Let \( \Gamma \) be a directed set in \( X \) with \( \text{supp} \Gamma = K \). Then by Lemma 3.7, the mapping \( x \mapsto x/\Gamma \) is a norm resolution (since \( E \equiv I \) in this case) taking values in the Banach lattice \( M_{\Gamma} \), which is a lattice of continuous numerical functions on \( K \) with the order-continuous norm \( m_{\Gamma} \).

4. **Cycles and ideals.** With the help of the norm resolutions defined in the last section we can show that the \( \mathcal{A} \)-cycles and ideals have several nice properties. The reader is reminded of the following definitions.

4.1. **Definition.** Let \( X \) be a Banach space and \( A \) a commutative subset of \( B(X) \). A closed subspace \( J \) of \( X \) is called an \( A \)-cycle if \( J \) is an invariant subspace for every operator in \( A \) and an \( A \)-ideal if it is invariant for every operator in \( A_{\text{comm}} \).

Since \( \mathcal{A} \) itself is commutative, an \( A \)-ideal is an \( A \)-cycle. Also if \( J_0 \) is a family of \( A \)-cycles (resp. \( A \)-ideals), then \( \bigcap J_0 \) and \( \bigcup J_0 \) are also \( A \)-cycles (resp. \( A \)-ideals). In particular, we can define the \( A \)-cycle (resp. \( A \)-ideal) generated by a subset of \( X \) as the intersection of all \( A \)-cycles (\( A \)-ideals) containing it. In our context we are naturally interested in the \( \mathcal{A} \)-cycles and ideals where \( \mathcal{A} \) is a complete uniformly decomposing projection algebra on \( X \). The cycles generated by directed sets turn out to have a very simple form.

4.2. **Proposition.** Let \( \mathcal{A} \) be a complete uniformly decomposing projection algebra on a Banach space \( X \) and \( \Gamma \) a directed subset of \( X \). \( S(\mathcal{A}, \Gamma) \), the \( \mathcal{A} \)-cycle generated by \( \Gamma \), is isometrically isomorphic to the Banach space \( M_{\Gamma} \).

**Proof.** A simple calculation shows that \( S(\mathcal{A}, \Gamma) = \bigcup f(y) \in C(K) \), \( y \in \Gamma \). Consider the mapping \( f(y) = f(y) \in C(K) \), \( y \in \Gamma \). This is well-defined and since for \( y \geq \bar{y} \) in \( \Gamma \) we have \( f(y) \leq f(\bar{y}) \), \( f(y) \in C_{\Gamma} \) has norm \( \|f(y)\| \). Furthermore, since \( f(y) + f(\bar{y}) = (f(y) + f(\bar{y})) + f(y) \), the mapping is linear. This mapping then
extends to an isometry between $S(\mathcal{M}; J)$ and the closure in $\mathcal{M}_\Gamma$ of the functions in the form $f \circ \psi$, $f \in C(K)$, $y \in \Gamma$. Inspection of the definitions of $G_I$ and $M_\Gamma$ shows that this closure is all of $M_\Gamma$. Observe that this mapping maps an element $x$ onto a function whose absolute value is $x/\Gamma$.

One of the most interesting problems in the general theory of cycles is the question of the existence of a projection in $[A]_\text{closed}$ projecting onto an $A$-cycle $J$. The positive answer for cycles of the form $S(\mathcal{M}; J)$ is a simple corollary of the preceding proposition.

4.3. COROLLARY. Let $\mathcal{A}$ be a complete uniformly decomposing projection algebra on a Banach space $X$ and $J$ a directed subset of $X$. There is a contractive projection from $X$ onto $S(\mathcal{M}; J)$ which commutes with $\mathcal{A}$.

Proof. Let $J$ be the isometry between $S(\mathcal{M}; J)$ and $M_\Gamma$, which was constructed in the preceding proposition. We have $j(z) \leq |j(z)| = x/\Gamma$ for all $x$ in $S(\mathcal{M}; J)$. Since $M_\Gamma$ is order-complete, we can apply the Hahn-Banach theorem to obtain a linear mapping $T : X \to M_\Gamma$ which extends $j$ and for which $T < x/\Gamma$ for all $x$ in $X$. Since also $T < F_j + T(-x) \leq (-x)/\Gamma = x/\Gamma$, we have $[T\mathcal{A}] \leq x/\Gamma$. Considering the mapping $j = T$. This clearly maps $X$ into $S(\mathcal{M}; J)$ and for $x$ in $S(\mathcal{M}; J)$ we have $T(x) = j^*(x) = x$. This is thus a projection onto $S(\mathcal{M}; J)$. Since

$$[j^*(x)] = m_j(Tx) = m_j([T\mathcal{A}] < \mathcal{M}[x/\Gamma] = [E\mathcal{A}],$$

it is also contractive ($E$ as in 3.7). Let $F$ be a projection in $\mathcal{A}$. Then for all $x$ in $X$ we have

$$E_j^*T(I - E)x = \chi_D[j^*(I - E)x/\Gamma] \leq \chi_D((I - E)x/\Gamma) = \chi_D[(1 - \chi_D)(x/\Gamma) = 0,$$

where $D$ is the clopen set in $X$ corresponding to $F$. Since $E_j^*T(I - E)x \in S(\mathcal{M}; J)$, this implies that $E_j^*T(I - E)x = 0$. Since this holds for all projections in $\mathcal{A}$ and

$$E_j^*Tx = e_j^*TFx = E_j^*T(I - E)x = (I - E)j^*TFx,$$

the projection $j = T$ commutes with $\mathcal{A}$.

In our concrete case the other subspaces, the $\mathcal{A}$-ideals, have an even simpler form.

4.4. PROPOSITION. Let $\mathcal{A}$ be a complete uniformly decomposing projection algebra on a Banach space $X$. A closed subspace $J$ of $X$ is an $\mathcal{A}$-ideal if and only if it is the range of a projection in $\mathcal{A}$.

Proof. Clearly the range of a projection in $\mathcal{A}$ is an $\mathcal{A}$-ideal. Now suppose $J$ is an $\mathcal{A}$-ideal and let $E$ be the supremum in $\mathcal{A}$ of the carrier projections $\mathcal{E}_y$ of elements $y$ in $J$. Clearly $J = E\mathcal{A}$; we shall show the reverse inclusion. Take $x$ in $E\mathcal{A}$ and $r \geq 0$. Since $E \subseteq \mathcal{E}_y$, there is a $y$ in $J$ with $[x - E_y] \leq r$. For each $z$ in $X$ let $F_z = [z - E_y]$, $x \in z = [x]$. Clopen $D$ in $X$, the Stone-

Ian space of $\mathcal{A}$, and $E_y$ the corresponding projection in $\mathcal{A}$. Since $E_y \subseteq \mathcal{E}_y$, there is an $E_y$ with $[x - E_y] \leq r$. Set $g = E_yx/y$. Consider the mapping $f \mapsto fE_yx$ for $f \in C(K)$. Since $E_y \leq E_y$ it is well-defined, is clearly linear and continuous with $C(K)$ and thus with $\mathcal{A}$. Also $fE_yx/y = fE_yx/y = E_y$ for all $y$. Hence $[fE_yx] = [fE_yx/y] \leq [y]$. The mapping therefore extends to a continuous linear mapping from $S(\mathcal{A}; y)$ into $S(\mathcal{A}; E_y)$. Let $F$ be a projection in $[\mathcal{A}]_\text{closed}$ mapping $X$ onto $S(\mathcal{A}; y)$, then $TF$ lies in $[\mathcal{A}]_\text{closed}$ and $rF = T = E_yx/y$. Since $y$ is in $J$ and $J$ is an $\mathcal{A}$-ideal, $E_yx/y$ is also in $J$. But $[x - E_y] \leq r$ and $x$ was arbitrary, so $x$ itself lies in $J$.

The results of this section generalize 2.10-2.12 of 5, which also form 4.2, 4.4 and 4.5 of 1.

5. A characterization and a representation theorem. This section is devoted to the proof of two theorems. The first is an application of the results of the last section to obtain a Banach space characterization of Banach lattices with order-continuous norm. The second is a representation theorem analogous to the function module representation of Cunningham [4] and our own integral module representation [5].

5.1. THEOREM. Let $X$ be a Banach space. Then the following properties of a complete uniformly decomposing projection algebra $\mathcal{A}$ on $X$ are equivalent:

(i) $[\mathcal{A}]_\text{closed} = \mathbb{N}$;

(ii) $\mathcal{A}$ is a maximal Boolean algebra of bounded projections;

(iii) Every $\mathcal{A}$-cycle is an $\mathcal{A}$-ideal.

Moreover, there is such an algebra on $X$ if and only if $X$ is isometrically isomorphic to a Banach lattice with order-continuous norm.

Proof. Clearly (i) is (ii) and by 4.3, (ii) is (iii).

Assume (iii) and let $I$ be a directed set in $\mathcal{A}$ with sup $\mathcal{A} = K$, the Stone-

Ian space of $\mathcal{A}$. By 4.4, $S(\mathcal{A}; I)$ is isometrically isomorphic to $M_\Gamma$, a Banach lattice with order-continuous norm. However, $S(\mathcal{A}; I)$ is an $\mathcal{A}$-cycle and therefore also an $\mathcal{A}$-ideal. By 4.4 $S(\mathcal{A}; I)$ is the range of a projection in $\mathcal{A}$. Since sup $\mathcal{A} = K$, this must be the identity, i.e. $S(\mathcal{A}; I) = X$. Thus $X$ is isometrically isomorphic to a Banach lattice with order-continuous norm. Note that the projections in $\mathcal{A}$ correspond to the projections $f \mapsto fE_y$ for clopen $D$ and these are exactly the band projections in $M_\Gamma$.

Now suppose that $X$ is isometrically isomorphic to a Banach lattice $M$ with order-continuous norm. Then $X$ is itself a Banach lattice with
order-continuous norm in the induced ordering. Let \( \mathfrak{W} \) be the Boolean algebra of band projections on \( X \). Since \( X \) has order-continuous norm, \( \mathfrak{W} \) is a complete projection algebra. Also a simple calculation shows that \( \mathfrak{W} \) is the centre of \( X \) (i.e. all operators \( T \) for which \( -aT \leq T \leq aT \) for some \( a \in \mathbb{N} \)). It follows from Lemma 2.3 that \( \mathfrak{W} \) is a uniformly decomposing since the band projection onto the band generated by \( \langle \epsilon \rangle_{\mathfrak{W}} \) clearly satisfies (b) of the lemma. Let \( T \) be an operator in \( \mathfrak{W}_{\text{non}} \) and \( x \) an element of \( X \). Set \( z = (|T|x - (|T|x))^+ \) and let \( E \) be the band projection onto the principal band generated by \( x \). Then

\[ |TEx| = |E| |Tx| \geq |E(\{T\} + 1)x| = (\{T\} + 1)|Ex|. \]

It follows that \( Ex = 0 \) and so also \( z = 0 \). Thus \( |Tx| \leq (\{T\} + 1)|x| \) for all \( x \) and \( T \) belonging to the centre of \( X \) which is \( \mathfrak{W} \). Thus \( \mathfrak{W} \) satisfies (i) and (ii) and completes the proof of the theorem.

The representation theorem is a direct generalisation of the author's integral module representation and shows that a Banach space \( X \) can be considered as a space of vector-valued functions on the Stonean space of any complete uniformly decomposing projection algebra on \( X \). The following definition defines the appropriate type of vector-valued function space.

5.3. Definition. Let \( M \) be a Banach lattice with order-continuous norm consisting of continuous numerical functions on the extremally disconnected compact Hausdorff space \( K \). Let \( \{X_k\} \) be a family of Banach spaces indexed by the points of \( K \).

By \( M(K; \{X_k\}) \) we denote the set of all functions \( x : K \to \bigoplus X_k \) such that:

(i) \( x(k) \in X_k \cup \{\infty\} \) for each \( k \) and

(ii) \( \|x\|_M = \sup_{k \in K} \|x(k)\|_{X_k} \) (with \( \|\cdot\|_{X_k} = +\infty \)).

A lattice module in \( M(K; \{X_k\}) \) is a subset \( \mathfrak{Y} \) for which:

(i) \( x, y \in \mathfrak{Y} \) in \( \mathfrak{Y} \) in \( Y \) with \( \|x\|_\mathfrak{Y} = \|y\|_\mathfrak{Y} \) \( \|x(k)\|_{X_k} \leq \|y(k)\|_{X_k} \) for each \( k \);

(ii) \( x \in \mathfrak{Y} \) in \( Y \) with \( \|x(k)\|_{X_k} \neq \infty \) and \( \|x\|_\mathfrak{Y} = \|x(k)\|_{X_k} \) for each \( k \);

(iii) \( x(k) = \infty \) in \( X_k \) if \( x(k) \) is finite.

Since the equations in (i) and (ii) determine the element \( x \) uniquely, a lattice module carries the structure of a \( C(K) \)-module and \( \mathfrak{Y} \) is a norm for the lattice module \( M(K; \{X_k\}) \) if it is isometrically isomorphic (as a \( C(K) \)-module) to a lattice module in \( M(K; \{X_k\}) \).

Note. In the same way as \( [S] \) for \( [1] \) one can show that \( Y \) is a lattice module in \( M(K; \{X_k\}) \) which is complete in the norm, then \( Y \) is maximal amongst those subsets of \( M(K; \{X_k\}) \) for which (i) holds, and that we have equality in (ii) without taking the closure, i.e. if \( \mathfrak{Y} \) is in \( \mathfrak{Y} \), then there is an \( x \) in \( Y \) with \( x(k) = x_k \).

5.3. Theorem. Let \( X \) be a Banach space and \( \mathfrak{W} \) a complete uniformly decomposing projection algebra on \( X \) with Stonean space \( K \). There is then a Banach lattice function \( \mathfrak{Y} \) on \( K \) and Banach space \( X \) indexed by the points of \( K \) such that \( X \) has a representation in \( M(K; \{X_k\}) \).

Furthermore, if \( X \) also has a representation in \( N(K; \{X_k\}) \), then \( \mathfrak{Y} \cong N \) as Banach lattices and \( X \) is isometric to \( Y \) whenever both spaces are non-trivial.

Note. \( X \) is a \( C(K) \)-module by virtue of \( C(K) = \mathfrak{W} \), taking values in a lattice \( M \) of continuous numerical functions on \( X \) whose norm \( m \) is order-continuous. Let \( x = [a] \) be such a norm resolution. For each \( k \), \( \gamma \rightarrow \alpha(k) \) is a semi-norm on the subspace of \( X \) where it takes finite values. Let \( X \) be the Banach space obtained by factorising out the kernel of the semi-norm and completing the space in the resulting norm. The mapping \( x \rightarrow \alpha(k) \) from \( X \) into \( M(K; \{X_k\}) \) is defined as follows:

\[ \alpha(k) := 0 \quad \text{if} \quad \gamma \rightarrow \alpha(k) = +\infty. \]

Since \( \gamma \rightarrow \alpha(k) \) is simply \( \gamma \rightarrow \alpha(k) \), the functions \( \alpha \) all lie in \( M(K; \{X_k\}) \). It is easily verified that \( \alpha(k) \) is in \( X \) as a lattice module in \( M(K; \{X_k\}) \) and that \( x \rightarrow \alpha \) is an isometric isomorphism.

Suppose now that \( X \) has a representation in \( M(K; \{X_k\}) \) and also in \( N(K; \{X_k\}) \). Each \( x \) in \( X \) is represented as \( \langle \gamma \rangle \mathfrak{Y} \) in \( M(K; \{X_k\}) \) and as \( \langle \gamma \rangle \mathfrak{Y} \) in \( N(K; \{X_k\}) \). Writing \( \mathfrak{Y} := \langle \gamma \rangle \mathfrak{Y} \) in \( M(K; \{X_k\}) \) and similarly \( \mathfrak{Y} := \langle \gamma \rangle \mathfrak{Y} \) in \( N(K; \{X_k\}) \), we have two norm resolutions for \( X \) taking values in \( M \) and \( N \) respectively. Suppose that for a point \( k \) there is an \( x \) in \( X \) with \( \gamma \rightarrow \alpha(k) \) and \( \gamma \rightarrow \beta(k) \) both finite and non-zero. Then the identity

\[ \gamma \rightarrow \mathfrak{Y} = \lim_{\gamma \rightarrow \mathfrak{Y}} \|Ey\|/\|Ex\| = \mathfrak{Y} \]

implies that the quotient \( \gamma \rightarrow \mathfrak{Y} \) of \( \gamma \rightarrow \mathfrak{Y} \) is independent of the choice of \( x \). Set \( \mathfrak{Y} \) equal to this quotient. Then the \( \gamma \)'s are defined on a dense open subset of \( K \) and the mapping \( k \rightarrow \gamma \) is continuous since we can retain the same element \( x \) in a neighbourhood of \( k \) and the two norm resolutions are continuous. Let \( f \) be the unique continuous numerical function with \( f(k) = \gamma \) wherever this is defined. Then \( \gamma \rightarrow \gamma \mathfrak{Y} = \gamma \gamma \mathfrak{Y} \) for all \( x \) in \( X \).

Note that \( f \) is invertible since the \( \gamma \)'s are all non-zero. We claim that \( g \rightarrow f \) is a Banach lattice isometry of \( N \) onto \( M \). Clearly the mapping is linear and positive. Also, if \( m \) and \( n \) are the norms of \( M \) and \( N \), respectively, then \( m(\gamma \mathfrak{Y}) = m \mathfrak{Y} = m \gamma \mathfrak{Y} = n \mathfrak{Y} \) so that the mapping is an isometry for elements of the form \( \gamma \mathfrak{Y} \). However, these elements are
order-dense in \( N_4 \) and thus, by the order-continuity of \( s \), also norm-dense.

Since the norm in a lattice is determined by the norm on the positive cone, \( g \to y_g \) is an isometry. That it is surjective follows from the invertibility of \( f \).

To show that \( X_3 \) and \( X_4 \) are isometric if they are non-trivial let \( k \) be a point with \( f(k) \) finite and \( X_3 = \emptyset \). Since either \( f \) or \( f^{-1} \) is always finite and the situation is symmetrical, this does not involve any loss of generality. Let \( D \) be a clopen neighbourhood of \( k \) on which \( f \) is finite, then \( f_I(K) \in C(K) \) for each \( \alpha \). For each \( \alpha \) in \( X_3 \) there is a \( \alpha \) in \( X \) with \( \sigma(\alpha)_M = \alpha \). Set \( T(\alpha) = \langle f_I(K) : \alpha \rangle \). \( T \) is a well-defined since \( \langle \sigma(\alpha)_M \rangle = 0 \) implies that \( \sigma(\alpha)_M = 0 \) and thus that \( \langle f_I(K) : \alpha \rangle \). \( T \) is clearly linear and the identity \( \langle \sigma(\alpha)_M \rangle = \sigma(\alpha)_M \) shows that it is an isometry. It remains to show that \( T \) is surjective. Let \( y \) be in \( Y \) with \( \|y\|_Y = 1 \). Choose an \( \alpha \) in \( \sigma \) with \( \alpha \subseteq D \) and \( \langle \sigma(\alpha)_M \rangle = 2 \) (since \( X_4 \) is non-trivial there is such an element), then \( \langle f_I(K) : \alpha \rangle = 2 \).

Let \( \alpha \subseteq D \) and \( \langle \sigma(\alpha)_M \rangle = \alpha \) and \( \langle \sigma(\alpha)_M \rangle = 2 \). Let \( \sigma \) be the directed set of those clopen subsets of \( D \) on which \( f \) is finite, ordered by inclusion. For \( \alpha \subseteq D \) we have \( \langle f^{-1}(\alpha)_M \rangle = \langle f^{-1}(\alpha)_M \rangle \) for \( \alpha \subseteq D \). Thus \( \langle f^{-1}(\alpha)_M \rangle \) is a Cauchy net in \( X \). Let \( \alpha \) be the limit of this net. Then \( \langle f^{-1}(\alpha)_M \rangle = \langle f^{-1}(\alpha)_M \rangle \) for \( \alpha \subseteq D \). Let \( \alpha \subseteq D \). Since \( \sigma \) is directed, \( \langle f^{-1}(\alpha)_M \rangle \) is a Cauchy net in \( X \). Let \( \alpha \) be the limit of this net. Then \( \langle f^{-1}(\alpha)_M \rangle = \langle f^{-1}(\alpha)_M \rangle \) for \( \alpha \subseteq D \). Thus \( T \) is surjective.

It is straightforward to show that the points of \( X \) for which the component spaces are trivial in all representations are those points of \( X \) any neighbourhood of which contains uncountably many pairwise disjoint open sets (the so-called intrinsic null points, see \[1], 3.11). Let us denote the set of these points by \( X_5 \). Note that \( X_5 \) is determined by the topology of \( X \) and thus by the Boolean algebra structure of \( \mathbb{F} \) and not by its action on \( X \).

Theorem 5.3 now shows that \( X \) and \( \mathbb{F} \) together define an unique non-trivial Banach space \( X_5 \) for each point \( k \) in \( X \).

6. Conclusion. In this paper we have generalised the integral module representation of \([5]\) to apply to a fairly large class of projection algebras. In particular, given a Banach space \( X \) and an embedding \( C(K) \to \mathcal{B}(X) \) we have a criterion for deciding whether \( X \) has a norm resolution over \( K \) taking values in a lattice with order-continuous norm and a method for constructing such a resolution and the associated representation. The key to this was the definition of uniform decomposition. A re-formulation of this property in terms of \( C(K) \) rather than the projection algebra \( \mathbb{F} \) would seem like a sound starting point for a general criterion to decide whether a space \( X \) has a norm resolution with respect to a given embedding \( C(K) \to \mathcal{B}(X) \). This general criterion would have to contain both the present theory and the \( \mathcal{M} \)-structure of Cunningham as special cases.

Theorem 5.3 raises another interesting point. Namely, suppose that \( X \) and \( Y \) are two Banach spaces with isomorphic complete uniformly decomposing projection algebras \( \mathbb{F}_X \) and \( \mathbb{F}_Y \), respectively. Then we have two uniquely determined Banach function lattices \( M_X \) and \( M_Y \) on the common Stonean space \( K \) and two families \( X_5 \) and \( Y_5 \) of non-trivial Banach spaces indexed by the points of \( K \setminus X_5 \). This raises the following question:

If \( X_5 = Y_5 \) for each \( k \) and \( M_X \cong M_Y \) (with respect to the common representation on \( K \)), does it follow that \( X \cong Y \), i.e. do the function lattices and the component spaces determine the space itself?

This problem and other allied ones, such as the sense in which the mapping \( k \to X(k) \) is 'continuous' seem almost insoluble even in fairly simple cases.

References


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