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An extension theorem of functionals on commutative semigroups

by

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Abstract. Generalizing the Sandwich Theorem, we give an extension theorem of additive functions into $[-\infty, +\infty]$ on commutative semigroups. Several results including some of R. Kaufman's results are derived from it.

The Hahn–Banach type theorem on commutative semigroups has been studied by many authors. The most general and efficient result might be what is called the *Sandwich Theorem*, which is a generalization of the Mazur–Orlicz theorem [7] and was proved originally by Kaufman [3] and established by Kranz [6] and Fuchssteiner [1]. Especially Fuchssteiner deduced many related results from it.

In this note we will generalize the Sandwich Theorem and give an extension theorem of functionals on commutative semigroups, from which several known results follow naturally.

Let G be a commutative semigroup with a compatible quasiorder σ, that is, σ is a reflexive and transitive relation satisfying that $x\sigma y \Rightarrow xz\sigma yz$ for $x, y, z \in G$. $\boldsymbol{R}$ denotes the real line and $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$ is the additive (partial) semigroup of the real numbers equipped with the negative and the positive infinite; the addition in $\boldsymbol{R}$ is extended naturally to $\overline{\boldsymbol{R}}$, but note that $(+\infty)+(-\infty)$ and $(-\infty)+(+\infty)$ are not defined.

An equation or an inequality in $\overline{\boldsymbol{R}}$ is understood to hold if it holds as far as every addition contained in it is defined. For example, we say that $a+b = c$ $(a, b, c \in \overline{\boldsymbol{R}})$ holds if either $a+b$ is defined and the equation holds or $a+b$ is not defined.

Let f be a function of G into $\overline{\boldsymbol{R}}$. f is called *additive* if $f(xy) = f(x)+f(y)$ for all $x, y \in G$ (as far as $f(x)+f(y)$ is defined; we will omit this kind of comment hereafter). f is called *subadditive* (resp. *superadditive*) if $f(xy) \leqslant f(x)+f(y)$ (resp. $f(xy) \geqslant f(x)+f(y)$) for all $x, y \in G$. f is called *monotone* if $x\sigma y \Rightarrow f(x) \leqslant f(y)$, for all $x, y \in G$.

The pointwise order of functions of G is denoted by $\leqslant$, that is, for functions f and g of G, $f \leqslant g$ means $f(x) \leqslant g(x)$ for all $x \in G$. The constant function with a value $a \in \overline{\boldsymbol{R}}$ is simply denoted by a.

The following is a slite generalization of the Sandwich Theorem of Fuchssteiner and would be proved in a similar way.

LEMMA (Sandwich Theorem). *Let G be a commutative semigroup with a compatible quasi-order σ. Let δ (resp. ω) be a sub (resp. super)-additive function of G. Then there exists a monotone additive function f of G such that $\omega \leqslant f \leqslant \delta$ if and only if*

$$(*) \qquad x\sigma y \Rightarrow \omega(x) \leqslant \delta(y) \quad \text{for} \quad x, y \in G.$$

THEOREM. *Let G be a commutative semigroup with a compatible quasi-order σ and let H be a subsemigroup of G. Let δ (resp. ω) be a sub (resp. super)-additive function of G and let f be an additive function of H. Assume that G has an identity element e which is contained in H and $f(e) = \delta(e) = \omega(e) = 0$. Then there exists a monotone additive function $\bar{f}$ of G such that $\bar{f}|_H = f$ and $\omega \leqslant \bar{f} \leqslant \delta$ if and only if*

$$(**) \qquad x_1h_1\sigma x_2h_2; \; x_1, x_2 \in G, \; h_1, h_2 \in H \Rightarrow \omega(x_1)+f(h_1) \leqslant \delta(x_2)+f(h_2).$$

Proof. Since the necessity of condition (**) is clear, we will prove only the sufficiency. We define functions p and q of G into $\bar{\boldsymbol{R}}$ as follows:

$$p(x) = \inf\{\delta(z)+f(h)-\omega(y) \mid xy\sigma zh; \; y, z \in G, \; h \in H, \; \delta(z)+f(h)-\omega(y) \text{ is defined}\},$$
$$q(x) = \sup\{\omega(z)+f(h)-\delta(y) \mid zh\sigma xy; \; y, z \in G, \; h \in H, \; \omega(z)+f(h)-\delta(y) \text{ is defined}\},$$

where $x \in G$.

(i) p is well-defined, that is, there are $y, z \in G$, $h \in H$ such that $xy\sigma zh$ and $\delta(z)+f(h)-\omega(y)$ is defined. This is assured by the relation $xe\sigma xe$. Similarly q is well defined.

(ii) $q \leqslant p$: Assume that $q(x) > p(x)$ for some $x \in G$. Then there are $y, z, y', z' \in G$ and $h, h' \in H$ such that $xy\sigma zh$, $z'h'\sigma xy'$ and

$$\omega(z')+f(h')-\delta(y') > \delta(z)+f(h)-\omega(y),$$

where the both sides are defined. Then we have

$$\omega(z'y)+f(h') > \delta(zy')+f(h),$$

where the both sides are defined again. On the other hand, we have by the compatibility of σ that $z'yh'\sigma zy'h$, this contradicts condition (**).

(iii) p (resp. q) is monotone and sub (resp. super)-additive: It is clear that p is monotone. Assume that p is not subadditive, that is, there are $x, x' \in G$ such that $p(x)+p(x')$ is defined and $p(x)+p(x') < p(xx')$. Choose $a, b \in \boldsymbol{R}$ so that $p(x) < a$, $p(x') < b$ and $a+b < p(xx')$. The first two inequalities mean that there are $y, z, y', z' \in G$ and $h, h' \in H$ such that $xy\sigma zh$, $x'y'\sigma z'h'$, $a > \delta(z)+f(h)-\omega(y)$ and $b > \delta(z')+f(h')-\omega(y')$, where the right-hand sides of these inequalities are defined. Hence

$$a+b > \delta(z)+\delta(z')+f(h)+f(h')-\omega(y)-\omega(y') \geqslant \delta(zz')+f(hh')-\omega(yy') \geqslant p(xx').$$

This is a contradiction, so p is subadditive.

(iv) $q|_H \leqslant f \leqslant p|_H$: This follows easily from condition (**) and the definitions of p and q.

(v) $q|_H \geqslant f \geqslant p|_H$: Let $h \in H$. The relation $he\sigma eh$ gives

$$p(h) \leqslant \delta(e)+f(h)-\omega(e) = f(h).$$

(vi) $p \leqslant \delta$ and $\omega \leqslant q$: Let $x \in G$. The relation $xe\sigma xe$ gives

$$p(x) \leqslant \delta(x)+f(e)-\omega(e) = \delta(x).$$

Thus we have proved that p (resp. q) is a monotone sub (resp. super)-additive function of G satisfying $\omega \leqslant q \leqslant p \leqslant \delta$ and $q|_H = f = p|_H$. By Lemma there exists a monotone additive function $\bar{f}$ of G such that $q \leqslant \bar{f} \leqslant p$, which is what we have desired.

Remark. The condition concerning the identity element in Theorem can be replaced by the following weaker condition

$$(***) \qquad \omega(h_0) \neq -\infty, \quad f(h_0) \neq +\infty, \quad f(h_0') \neq -\infty, \quad \delta(h_0') \neq +\infty, \quad \omega(x_0) \neq -\infty \quad \text{and} \quad \delta(x_0) \neq +\infty \quad \text{for some } h_0, h_0' \in H \text{ and } x_0 \in G.$$

In fact, let p and q be as in the proof of Theorem. The well-definedness of p and q follows from (***). Because we did not use the assumption concerning the identity in (ii), (iii) and (iv) in the proof of Theorem, p (resp. q) is a monotone sub (resp. super)-additive function of G satisfying $q \leqslant p$ and $q|_H \leqslant f \leqslant p|_H$. Now, let $\bar{p}$ and $\bar{q}$ be the functions of G defined as

$$\bar{p}(x) = \inf_n \frac{p(x^n)}{n}, \qquad \bar{q}(x) = \sup_n \frac{q(x^n)}{n}, \qquad x \in G.$$

It is not difficult to prove that $\bar{p}$ (resp. $\bar{q}$) is again a monotone sub (resp. super)-additive function of G and the inequalities in (v) and (vi) in the proof of Theorem hold for $\bar{p}$ and $\bar{q}$. Applying Lemma to $\bar{p}$ and $\bar{q}$, we can prove the remark above.

Let f be an additive function of a subsemigroup H of G. Then the function $f_{+\infty}$ (resp. $f_{-\infty}$) of G defined as follows is sub (resp. super)-additive:

$$f_{+\infty}(x) \; (\text{resp. } f_{-\infty}(x)) = \begin{cases} f(x) & \text{if} \quad x \in H, \\ +\infty \; (\text{resp. } -\infty) & \text{if} \quad x \in G \setminus H. \end{cases}$$

Applying Theorem to the case where $\omega = f_{-\infty}$, we get a result of Kaufman [2] (see also [1], Corollary 1.3). Considering the case where $\omega = 0$, we get an extension theorem of non-negative functions which is given by Kaufman ([4], Theorem 1). In particular, when the quasi-order σ is *positive*, that is, $x\sigma xy$ for all $x, y \in G$, we have

(1) *Any non-negative monotone additive function of H is extensible to G.*

This result is useful for studying archimedean commutative semigroups and gives another proof of Ross' extension theorem of semicharacters [8] (see [5]).

Finally, considering the case where H is a cyclic subsemigroup of G, we get the following generalization of a result of Kaufman ([4], Theorem 2).

(2) *Let δ (resp. ω) be a monotone sub (resp. super)-additive function of G. Let $a \in G$ and assume that $\delta(a) \neq +\infty$ and $\omega(a) \neq -\infty$. Then for $r \in \mathbf{R}$ there exists a monotone additive function f of G such that $f(a) = r$ and $\omega \leqslant f \leqslant \delta$ if and only if*

$$\delta(xa^n) \geqslant \omega(x) + nr, \quad \omega(xa^n) \leqslant \delta(x) + nr \quad \textit{for} \quad x \in G \textit{ and } n > 0.$$

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