Proof. Let \( P \) be a \( \Psi \)-hereditary function module property, \( J: C(K, X) \to C(L, X) \) an isometric isomorphism. By 2.9 and 2.4, \((K \times \hat{K}, (X_{(h, p)}(l)_{h\in K, p\in K} \times \hat{K})) \) and \((L \times \hat{L}, (X_{(h, p)}(l)_{h\in L, p\in L} \times \hat{L})) \) are equivalent so that, by definition, \( K \times P(\hat{K}, (X_{(h, p)}(l)_{h\in K, p\in K} \times \hat{K})) = P(K \times \hat{K}, (X_{(h, p)}(l)_{h\in K, p\in K} \times \hat{K})) \approx P(L \times \hat{L}, (X_{(h, p)}(l)_{h\in L, p\in L} \times \hat{L})) = L \times P(\hat{L}, (X_{(h, p)}(l)_{h\in L, p\in L} \times \hat{L})) \).

4.2. Examples. (a) \( \hat{K} \) is always contained in \( \mathcal{P}(\hat{K}, (X_{(h, p)}(l)_{h\in K, p\in K} \times \hat{K})) \) so that we always may conclude that \( K \times K \approx \hat{K} \times L \) (this has first been noted, for \( \mathcal{M} \)-finite Banach spaces, in [2], th. 4.1).

(b) For an \( \mathcal{M} \)-finite Banach space \( X = M_{\sigma_1} \oplus \cdots \oplus M_{\sigma_r} \), \( C(K, X) \approx C(L, X) \) implies that \( \{(\alpha, 1), (\beta, 1), \cdots, (\alpha, r)\} \times \hat{K} \approx \{(\beta, 1), (\alpha, 1), \cdots, (\beta, r)\} \times L \) (cf. 3.5). This is just the assertion of Theorem 4.4 in [2] for the compact case.

(c) \( X_{\alpha} \) has the Banach–Stone property for the class of all non-void compact Hausdorff spaces for every \( \alpha \in \{0, 1\} \). Note that \( \mathcal{P}(\hat{K}, (X_{(h, p)}(l)_{h\in K, p\in K} \times \hat{K})) \) consists of a single point for every function module representation \((\hat{K}, (X_{(h, p)}(l)_{h\in K, p\in K} \times \hat{K}) \times X_{\alpha} \).

Analogously one can treat the case of arbitrary \( G \)-spaces (cf. 2.5).

References


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The localization principle for double Fourier series

by

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Abstract. Definitive results are obtained for localization by sparse and rectangular sums for the Fourier series of functions of 2 variables. For this purpose functions of \( \mathcal{A} \) bounded variation, \( ABV \), and in particular, harmonic bounded variation, \( HBV \), are defined for functions of 2 variables. It is shown that if \( f \in HBV \), then localization holds for rectangular sums. However, if \( ABV \notin HBV \), there is no \( f \in ABV \) for which localization fails everywhere for square sums.

This contrasts with the 1 variable case, where localization holds for all summable functions. It differs as well from the case \( n > 3 \) where previously obtained definitive results are in a Sobolev space framework.

The Riemann localization principle for periodic functions of one variable asserts that if an integrable function vanishes identically on an open interval, then the partial sums of its Fourier series converge uniformly to zero on any compact subset of that interval. For functions of several variables, strong additional assumptions are required in order that the principle of localization may hold. Indeed, if we consider convergence of the rectangular partial sum series, \( S_n = S_{n_1} \cdots S_{n_m} \) of the Fourier series of an integrable function defined on \([-\pi, \pi]^m \), \( m > 1 \), localization may fail even if the function is continuous. Here, by convergence of \( (S_n) \), we mean the existence of \( \lim S_n \) as \( m \to \infty \).

There are various alternatives one may pursue to obtain localization theorems. Among these are:

1) to require that \( f = 0 \) on a larger set,
2) to make additional global requirements on \( f \),
3) to replace convergence by other limiting procedures,
4) to replace rectangular partial sums by other sums of terms of the Fourier series,
and various combinations of these [14], Chap. 17.

For example, if we require that \( f = 0 \) not only in the given interval, but on every line in the direction of a coordinate axis and intersecting

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the interval, a set which is called a cross-neighborhood, then localization holds for the rectangular partial sums in the original interval. On the other hand, if we replace convergence of the rectangular partial sums by \((G, 1)\)-summability, then localization holds for bounded functions.

More recently, Igar [6] has shown that, for \(s\) partial sums \((S_n)\) with \(n_1 = n_2 = \ldots = n_m\), the localization principle fails in the class of continuous functions, while for \((G, 1)\) means, the localization principle holds in \(L^p\) if \(p \geq m - 1\), but fails if \(p < m - 1\).

Tonelli [10] introduced a notion of bounded variation which yielded a pointwise convergence theorem for functions of two variables. This theorem implies a pointwise localization principle: If a function of this class vanishes on an open set, then the rectangular partial sums of its Fourier series converge to zero at each point of the set. He obtains the usual (uniform) localization principle only with additional restrictive hypotheses which are unnecessary in view of the results of this paper.

Cesari [1] improved on Tonelli's results with his introduction of the notion of generalized bounded variation which guarantees localization and a.e. convergence and has had many other fruitful applications. This notion may be expressed as follows [5], [9] for \(f\) defined on an interval in \(\mathbb{R}^n\):

\[ f \text{ is measurable and, corresponding to each coordinate direction, there is an equivalent function which is} \]

\(\text{of bounded variation on a.e. line in that direction, and whose total variation on these lines is an integrable function of the remaining (m - 1) variables.}\)

This class contains the Sobolev space \(W^1\), which suggested to Goffman and Lin [3] an approach to obtaining a localization principle for \(m > 2\). They have shown that the localization principle for square partial sums holds in \(L^m\) for \(f \in W^p\) if \(p \geq m - 1\), but fails to hold if \(p < m - 1\). Lin [7] extended this to rectangular sums, but in this case localization holds if \(p > m - 1\) and fails if \(p \leq m - 1\).

Another way in which one may attempt to generalize the Cesari-Tonelli result is by enlarging the class of functions. A method which suggests itself is that of replacing ordinary bounded variation (on the lines in the coordinate directions) with other notions of bounded variation.

Let \(A = \{\lambda_m\}\) be a non-decreasing sequence of positive real numbers such that \(\sum_1^{\infty} \lambda_m\) diverges. A function \(g\) defined on \([a, b] \subset \mathbb{R}^n\) is said to be of \(A\)-bounded variation (ABV) if \(\sum_1^{\infty} |g(a_i) - g(b_i)|/\lambda_m\) converges for every sequence of non-overlapping intervals \([a_n, b_n] \subset [a, b]\). The supremum of such sums is the total \(A\)-variation of \(g\) on \([a, b]\). If \(\lambda_m = m\), we say that \(f\) is of harmonic bounded variation (HVB). The convergence and summability properties of Fourier series of functions of these classes have been studied recently [11], [15], [13]. In particular, we note that the conclusion of the Dirichlet-Jordan theorem holds for functions in HVB, but not for larger ABV classes. The class HVB contains properly the various classes of functions of generalized bounded variation introduced by Wiener, L. C. Young, Garcia and Sawyer, and Salem, for which a generalized Dirichlet-Jordan theorem was known to hold.

In §1 of this paper we shall show that if the Cesari variation is generalized by replacing ordinary variation by harmonic variation, then, in \(\mathbb{R}^n\), the localization principle for rectangular partial sums holds for integrable functions of that class.

In proving this result we will make certain measurability and continuity assumptions. In §2 we will show that these assumptions cause no loss in generality and that the measurable functions corresponding to each coordinate direction can be chosen so that the total variation on lines in that direction is minimized. We define the class \(V^p_{\infty}\) to consist of those \(f \in L^p\) if \(p \geq 1\), on an interval in \(\mathbb{R}^n\), to which there correspond equivalent \(f_i\) and \(s_i\), all \(m - 1\)-dimensional, that, on almost every line in the \(i\)th coordinate direction, \(f_i \in ABV\) and \(V_i\), the total \(A\)-variation of \(f_i\) on these lines, is in \(L^p\), \(a.e.\) as a function of the remaining \((m - 1)\)-variables. For \(m > 2\), we assume further that each \(f_i\), restricted to almost any line in the \(i\)th coordinate direction, has, at each point, a value between the upper and lower limits at that point. We also give an equivalent definition of the space which avoids the introduction of the \(m\) equivalent functions.

In §3 we show that a norm may be defined on \(V^p_{\infty}\) so that it is a Banach space with the property that \(f \in V^p_{\infty}\) if \(f = g\) a.e. if and only if \(g \in V^p_{\infty}\) and \(|f - g| = 0\). We are then able to show that, in \(\mathbb{R}^n\), the localization theorem of §1 is best possible in the sense that if \(ABV\) is not contained in HVB, then there is a function in \(V^p_{\infty}\), for which the localization principle for \(\mathbb{R}^n\) does not hold.

§1. If \(f(x, y)\) is a function on \(I = [-
\]
We assume that
(i) for a.e. \(y, g(x, y)\) is right continuous as a function of \(x\) in \([-\pi, \pi]\) and left continuous at \(x = \pi\),
(ii) \(V_0 [g(x, y), [a, b]]\) is a measurable function of \(y\) for any interval \([a, b] \subset [-\pi, \pi]\), and the analogous statements for the function \(h\).

Proof of Theorem 1. Let \(D_m(s) = \sin (m + \frac{1}{2})s / \sin \frac{1}{2}s\). We must show that if \(f(x, y) = 0\) for \((x, y) \in [-\delta, \delta]^2\), where \(0 < \delta < \pi\), then for any \(\delta' \in (0, \delta),
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) D_m(s) \cdot D_m(t) \, ds \, dt \rightarrow 0
\]
uniformly for \((x, y) \in [-\delta', \delta']^2\). The integral in question can be written as the sum of integrals over the following domains:
\[
\begin{align*}
\{ & \pi > |s| > \delta \} \quad \{ \pi > |t| > \delta \} \quad \{ \delta > |s| > a \} \quad \{ \delta > |t| > a \} \\
\{ \pi > |s| > \delta \} \quad \{ a > |s| > 0 \} \quad \{ |s| > |t| > \delta \} \\
\{ a > |s| > 0 \} \quad \{ |s| > |t| > \delta \}
\end{align*}
\]
where \(a \in (0, \delta)\). For a fixed \(a\), we may show that the integral over \(E\), the union of the first three domains, tends uniformly to zero as a consequence of much the same arguments as are used to establish the localization principle for cross neighborhoods.

We have
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x + s, y + t) \psi \gamma(s, t) \sin (m + \frac{1}{2})s \sin (n + \frac{1}{2})t \, ds \, dt
\]
where \(\gamma(s, t)\) is of period \(2\pi\) in each variable and, in \([-\pi, \pi]^2\), equals \((\sin \frac{1}{2}s \sin \frac{1}{2}t)^{-1}\) in \(E\) and zero elsewhere. Proceeding in a manner similar to that employed in the one variable case [14], Chap. 2, Lemma 6.4, we can show that this integral tends uniformly to zero as \(m, n \rightarrow \infty\).

Our proof of the theorem will be complete when we show that \(\phi, M\) and \(N\) can be chosen to make the integrals over the remaining portions arbitrarily small uniformly for \((x, y) \in [-\delta', \delta']^2\) when \(m > M\) and \(n > N\).

We will discuss only the integrals over \([0, a] \times [a, \pi]\). The remainder can be divided into integrals over seven similar domains and can be estimated in the same way.

For \(x \neq x' < x + a\), write
\[
\int_{\pi}^{\pi} \int_{\pi}^{\pi} f(x + s, y + t) D_m(s) \cdot D_m(t) \, ds \, dt
\]
and the analogous statements for the function \(h\). Noting that \(g(x, y) = f(x, y)\) a.e., we will write
\[
\psi = g(x + s, y + t) - g(x', y + t),
\]
indicating the dependence of \(\psi\) on any of its variables only to the extent necessary to clarify the argument. Then, for \(a < 3\pi/(m + 1/2)\), we have
\[
\int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi D_m(s) D_m(t)| \, ds \, dt \leq 3\pi^2 \sup_{0 < c \leq a} \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt \leq 3\pi^2 V_a \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt
\]
for a.e. \(x\). For \(a > 3\pi/(m + 1/2)\), let \(k\) be the greatest integer such that \((2k + 1)\pi/(m + 1/2) \leq a\). Then
\[
\int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt = \int_{\pi}^{\pi} + \int_{(2k+1)\pi/(m+1/2)}^{(2k+1)\pi/(m+1/2)} + \int_{(2k+1)\pi/(m+1/2)}^{\pi} = I_1 + I_2 + I_3.
\]
For a.e. \(x\),
\[
|I_1| \leq \int_{\pi}^{\pi} \sup_{0 < c < a} \int_{\pi}^{\pi} \psi \, ds \, dt \leq 3\pi^2 V_a \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt
\]
and, similarly,
\[
|I_3| \leq 2\pi^2 V_a \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt
\]
and, similarly,
\[
|I_3| \leq 2\pi^2 V_a \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt
\]
and, similarly,
\[
|I_3| \leq 2\pi^2 V_a \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt
\]
and, similarly,
\[
|I_3| \leq 2\pi^2 V_a \int_{\pi}^{\pi} \int_{\pi}^{\pi} |\psi| \, ds \, dt
\]
Thus

\[ |J'_3| \leq 2 \sum_{\nu=0}^{k} \text{osc}_s \left( q(x+s, y+t), \left[ \frac{(2i-1)\pi}{m+\frac{1}{4}}, \frac{(2i+1)\pi}{m+\frac{1}{4}} \right] \right) \int_{2(2i-1)\pi}^{2(2i+1)\pi} \sin^{2m+1} \theta \, d\theta \]

\[ \leq 2 Y_s \left[ q(x+s, y+t), \left[ 0, \alpha \right] \right] \]

for a.e. \( t \). We have now shown that there is a constant \( C \) such that for a.e. \( t \), for every \( s' \in (x, x+\alpha) \), and for every \( m \),

\[ \left| \int_{0}^{\pi} \psi D_n(s) \, ds \right| < CV_s \left[ q(x+s, y+t), \left[ 0, \alpha \right] \right] . \]

Then

\[ |P| \leq C \int_{0}^{\pi} V_s \left[ q(x+s, y+t), \left[ 0, \alpha \right] \right] \, dt \]

\[ \leq \frac{\pi}{2} C \int_{0}^{n} \sup_{t} V_s \left[ q(x+s, y+t), \left[ 0, \alpha \right] \right] \, dt \]

since \( g \) is periodic. Now the integrand in this last expression is a non-negative measurable function of \( t \) bounded above by the integrable function \( V_s \left[ q(x+s, y+t), \left[ -\alpha, \alpha \right] \right] \). For a.e. \( t \), \( V_s \left[ q(x+s, y+t), \left[ 0, \alpha \right] \right] = 0 \) as \( \alpha \searrow 0 \) [13], Theorem 3. Hence, given \( \varepsilon > 0 \), there is an \( a(e, \alpha) > 0 \) such that

\[ |P| < \varepsilon \]

for every \( y, m \), and \( n \), if \( 0 < a < a(e, \alpha) \) and \( \alpha < x' < x + \alpha \).

If \( \text{osc}_s \left( g, \left[ a, b \right] \right) \) denotes the oscillation of \( g \) as a function of \( s \), over the interval \( [a, b] \), we have

\[ \int_{0}^{\pi} \left| \psi \left( \frac{s+2i\pi}{m+\frac{1}{4}}, \frac{s+(2i-1)\pi}{m+\frac{1}{4}} \right) \right| \sin^{2m+1} \theta \, d\theta \leq \text{osc}_s \left( g, \left[ \frac{(2i-1)\pi}{m+\frac{1}{4}}, \frac{(2i+1)\pi}{m+\frac{1}{4}} \right] \right) \int_{\frac{2(2i-1)\pi}{2m+1}}^{\frac{2(2i+1)\pi}{2m+1}} \sin^{2m+1} \theta \, d\theta \]

\[ = \frac{2m+1}{2i-1} \text{osc}_s(...) . \]
The localization principle

Lemma 2. If \( a: [a, b] \to \mathbb{R}^1 \) is in \( ABV \) and \( a(x) = a(\xi) \) at each point of continuity of \( a \) and has an internal saltus at each point of discontinuity, then \( a \) is in \( ABV \) and its total \( A \)-variation is less than that of \( a \).

Let \( I \) be an interval in \( \mathbb{R}^n \) and let \( \lambda_1 = \lambda_1(x_1, \ldots, x_{n-1}, x_{n-1} + \xi, \ldots, x_n) \) be the (non-empty) intersection of \( I \) and the line through \( (x_1, \ldots, x_{n-1}, 0, x_{n-1}, \ldots, x_n) \) in the direction of the \( n \)th coordinate axis. If \( f: I \to \mathbb{R}^1 \)

\[ V_f(I) \]

will denote the total \( A \)-variation of \( f \) on \( I \), so that \( V_f(I) \) is a function of the \((m-1)\) variables \( x_1, j \neq i \). We may set \( V_f(I) = \infty \) for those \( I \) on which \( f \) is not in \( ABV \). When \( f \) restricted to \( I_0 \) is continuous at \( x_i \), we will say that \( f \) is linearly continuous at \( x \) on \( I_0 \) or has \( x \) as a point of linear continuity. If \( f \) is right or left continuous at \( x_i \), or discontinuous at \( x \), analogous terminology is employed.

A function \( f \) is in class \( V_oI_1 \) if it is measurable and there exist corresponding functions \( f_i \), \( i = 1, \ldots, m \), equivalent to \( f \), such that \( V_o(f_i) < \infty \) a.e. for every \( i \).

A function \( f \) is in class \( V_nI_1 \), \( p \geq 1 \), \( a \geq 1 \), if \( f \in V_nI_1, f \in L_p, V_o(f_i) < \infty \) for every \( i \), and if, for \( m > 2 \), each \( f_i \) has an internal saltus at its points of linear discontinuity on each \( I_0 \) for which \( V_o(f_i) < \infty \).

Suppose now that \( f \) is a measurable function on \( I \) and \( V_o(f) < \infty \) a.e. Let \( L = \) the union of the \( I_0 \) on which \( V_o(f) = \infty \). Restricting \( f \) to an \( I_0 \) in \( L^p \), \( f \) has only simple discontinuities, each of which is an approximate discontinuity of \( f \) as a function of \( x_i \). It is known that the set of points in \( f \) at which a measurable function is not approximately continuous in a particular variable is a set of measure zero [2], 3. 14 (4). Letting \( A \) be the set of points at which \( f \) is not approximately continuous as a function of \( x_i \), we see that the points at which \( f \) is discontinuous as a function of \( x_i \) are contained in \( A \cup L \). We have, as a consequence, the following result.

Lemma 3. If \( f \) is measurable and \( V_o(f) < \infty \) a.e., then \( f \) is continuous a.e. as a function of \( x_i \).

The next result is essential to what follows, but its proof is quite technical and so we defer it to the end of this section. It must be emphasized that this result is for \( m = 2 \). At the end of its proof we will make some remarks on the case \( m > 2 \).

Let \( f_i, i = 1, 2 \), be the corresponding functions of \( f \in V_nI_1 \). We will say that a real number \( a \) has property \( A_i \) if one of the following is satisfied:

(i) on almost every \( I_i \), \( f_i \) is linearly right continuous at \( x_i = a \);

(ii) on almost every \( I_i \), \( f_i \) is linearly left continuous at \( x_i = a \).

Let \( V_o(f_i, [a, b]) \) denote the total \( A \)-variation of a function \( f \) on a segment \( a \leq x \leq b \) of an \( I_0 \).

Theorem 3.1. Let \( f \in V_nI_1, f_i, i = 1, 2 \), be the corresponding functions. If an \( f_i \) has an internal saltus at each of its points of linear discontinuity,

\[ \begin{aligned}
\int_{\xi}^{\xi+2d} D_n(t) dt = & \int_{\xi}^{\xi+2d} g(x, y, t) - g(x, y, t + \xi) + a(e, \xi) ds \leq \epsilon \\
\text{since } \xi & \leq x < \xi + a(e, \xi) \leq \xi + \frac{a(e, \xi)}{2} \text{ and } x < 2d \leq a(e, \xi)/2. \\
\end{aligned} \]
The localization principle

Proof. Without loss of generality, we may set \( i = 1 \) and write \( x = a, y = (a_2, \ldots, a_m) \).

There is a set \( Z \) of \((m-1)\)-dimensional measure zero such that, for \( y \notin Z \), there is a function \( g_y(x) \in ABV \) and \( f(x, y) = g_y(x) \) a.e. (w).

We may assume that \( V_i(f) = \infty \) if \( y \in Z \).

Let \( [a, b] \) be the projection of \( I \) on the \( x \)-axis. For \( y \in Z \), let \( f(x, y) = f(x, y) \). For \( y \notin Z \), let

\[
\tilde{f}(x, y) = \begin{cases} 
  g_y(x), & x = a, \\
  g_y(b), & x = b, \\
  \frac{1}{2}[g_y(x-, y) + g(x+, y)], & x \in (a, b).
\end{cases}
\]

If \( I_{a,b}([a-h, a+h]) \cap [a, b], h > 0 \), we see that, for \( y \notin Z \) and \( I_k = I_k(x) \),

\[
\tilde{f}(x, y) = \lim_{h \to 0} \frac{1}{|I_k|} \int_{I_k} g_y(t) \, dt = \lim_{h \to 0} \frac{1}{|I_k|} \int_{I_k} f(t, y) \, dt.
\]

Suppose \( y \notin Z \) and \( f \) is approximately continuous in \( x \) at \((a_0, y)\). Then there is a set \( E \subseteq B^1 \) having density one at \( a_0 \), and such that

\[
g_y(t) = f(a_0, y) + o(1)
\]

as \( t \to a_0 \) within \( E \). Since \( g_y(t) \) is bounded, as \( h \to 0 \) we have, for \( I_k = I_k(x) \),

\[
\frac{1}{|I_k|} \int_{I_k} g_y(t) \, dt = \frac{1}{|I_k|} \int_{I_k} g_y(t) \, dt + \cdots + \frac{1}{|I_k|} \int_{I_k} g_y(t) \, dt = f(a_0, y) + o(1).
\]

Thus \( \tilde{f}(x, y) = f(x, y) \). Since \( f \) is a.e. approximately continuous in \( x \) and \( [a, b] \times Z \) has measure zero, we have \( \tilde{f} = f \) a.e. For \( y \notin Z \), we note that \( \tilde{f}(x, y) = g_y(x) \) at each point of continuity of \( g_y(x) \) and is either continuous or has an internal saltus elsewhere. By Lemma 2, \( V_i(f) = V_i(g_y) \), the total \( \Lambda \)-variation of \( g_y \), for \( y \notin Z \).

The next theorem shows that we can dispense with the notion of “corresponding functions” in the definition of \( V_i \) and \( V_i^* \).

Theorem 2.3. \( f \in V_i \) if and only if \( f \) is measurable and, for each \( i, i = 1, \ldots, m \), on almost every \( y \), \( f \) is equivalent to a function in \( ABV \).

\( f \in V_i^* \) if and only if \( f \) is a.e. equivalent to a function in \( ABV \) having, if \( m > 2 \), an internal saltus at points of discontinuity whose total \( \Lambda \)-variation as a function of the remaining \((m-1)\)-variables, is in \( L^p \).
Proof of Theorem 2.3. It is clear that functions in $V_2$ or $V_{\infty}$ have the indicated properties. We will show the converse.

If, on almost every $I$, $f$ is equivalent to a function in $ABV$, then Lemma 4 asserts the existence of $f_i = f \text{ a.e.}$ in $I$ such that $V(f_i) < \infty$ a.e. Thus $f_i, i = 1, \ldots, m$, are corresponding functions and $f = f_i \in V_2$.

If we assume also that the total $A$-variation $V_1$ of the function in $ABV$, which is equivalent to $f$ on $I$, is in $L^1$, then by Lemma 4, if $m = 2$, $V(f_i) \leq V_1$ a.e. and Theorem 2.1 implies that $V(f_i) = V_1$ is measurable. Hence $V(f_i)_{a.e.} \leq V_1$. If $m > 2$, then Lemma 1 implies that $V(f_i) = V_1$ and, therefore, $V(f_i) \in L^1$.

We turn now to the proof of Theorem 2.1.

A function $f$ will be said to be almost continuous at a point if there is a set of measure zero such that the restriction of $f$ to the complement of that set is continuous at the point. Almost lower (and upper) semicontinuity are similarly defined.

**Lemma 5.** If $f$ is measurable, equivalent to $g$ in $I$, and $V_g(g) < \infty$ a.e., then $f$ is almost continuous in $a_i$ a.e.

**Proof.** By Lemma 3, $g$ is continuous in $a_i$ a.e., say on the set $A$. Let $B = \{x \mid f(x) = g(x)\}$. Let $B$ be the union of the $I_i$ on which $f(x) = g(x)$ a.e. If $x \in A \cap B \cap C$, a set of full measure, then $f$ is almost continuous at $x$ in the variable $a_i$.

**Corollary.** A function $V$ is almost continuous in each $a_i$ a.e.

**Lemma 6.** Let $f \in V_2$ on $I = I_1 \times I_2$ with corresponding functions $f_i, i = 1, 2$. If $f_i$ has an interval at each of its points of linear discontinuity on the $I_1$ for which $V(f_i) < \infty$ a.e., then for $[a, b] \in I_2$, and such that, on a.e. $I_1$, $f_i$ is right continuous at $x$ and left continuous at $b$, the function $V(f_i([a, b]))$ is almost lower semicontinuous a.e.

**Proof.** Let us choose $i = 1$ and write $(x, y) = (a_1, a_2)$ and $g(x, y) = f_1(a_1, a_2)$. Let $A = I_2 \cap B_2$ be the set of $y$ such that, as a function of $x$, $g(x, y) \in ABV$, is right continuous at $x = a$ and left continuous at $y = b$, and, as a function of $y$, $g(x, y)$ is continuous at $y = b$ for almost every $x$. By the corollary to Lemma 5, we see that $A$ has full measure. Fix $y_b \in A$. Then

$$V(y_b) = V(f_1([a_1, b], [a_2, b])) < \infty.$$

For any $\varepsilon > 0$, according to Lemma 1 there are non-overlapping intervals $[a_j, b_j] = [a, b], j = 1, \ldots, m$, such that $g(x, y)$ is continuous in $x$ and almost continuous in $y$ at each $(a_j, b_j), (b_j, y_b)$ and

$$\sum |g(a_j, y) - g(b_j, y)|/b_j > V(y_b) - \varepsilon/2.$$

There is a $\delta$ and a set $Z \subset I_2$ of measure zero such that, if $B = \{y \mid V(y) < \infty\}$, then $y \in B - Z$ and $|y - y_b| < \delta$ imply

$$V(y) \geq \sum_{j=1}^m |g(a_j, y) - g(b_j, y)|/b_j \geq V(y_b) - \varepsilon.$$

Thus $V$ is almost lower semicontinuous at $y_b$.

**Lemma 7.** If a function $g$ is defined on an interval in $R^2$ and is almost lower semicontinuous a.e. then $g$ is measurable.

**Proof.** Suppose $g$ is almost lower semicontinuous except on $Z$, a set of measure zero. Let $(I_n)$ be the intervals with rational endpoints. Suppose $a \in R^2$ and $x_n \notin Z$ such that $g(x_n) > k$. Then for some $n = n(k, x_n)$, there is a set $Z_n$ of measure zero such that $g(x) > k$ for $x \in I_n - Z_n$. Thus, for $A = \{n(k, x) \mid g(x) > k\}$ and $Z_n' = \{x \mid g(x) > k\} \cap Z_n$, then

$$\{x \mid g(x) > k\} = Z \cup \bigcup (I_n - Z_n).$$

which is measurable.

**Proof of Theorem 2.1.** We use the notation of the proof of Lemma 6 and consider $g = f_i$. If we suppose that for almost every $y$ for which $V_y(g) < \infty$, $g(x, y)$ as a function of $x$, is right continuous at $(a, y)$ and left continuous at $(b, y)$, then by Lemma 6, $V_y(g, (a, b))$ is almost lower semicontinuous a.e. and is, therefore, by Lemma 7, also measurable. There are three other cases. We will consider only one; the others may be treated analogously.

Suppose that, for almost every $y$ for which $V_y(g) < \infty$, $g(x, y)$, as a function of $y$, is right continuous at $(a, y)$ and at $(b, y)$. By the corollary to Lemma 3, the set $A$ of $x$ for which $g(x, y)$ is continuous in $x$ for almost every $y$ has full measure. Choose $b_2 \in A$, $n = 1, 2, \ldots$, so that $b_2 \notin b$. For each $n$, $V_n(g, (a, b_n))$ is a measurable function of $y$. Let $Z_n \subset I_2$ be the set of measure zero consisting of those $y$ for which $g(x, y)$ is not continuous in $x$ at $x = b_n$. Let $Z_n \subset I_2$ be the set of measure zero consisting of those $y$ for which $g(x, y)$ is not right continuous at $x = b_n$. Then if $y \notin Z_n$, we have $V_n(y) \leq V(y)$; hence $V$ is measurable. 

A result analogous to Theorem 2.1, but for $m > 2$, would require a condition implying that the function is almost continuous as a function of $(m-1)$ variables on almost all coordinate hyperplanes. Such a condition has appeared in [3] and [4] in work related to absolutely continuous functions, i.e., to functions in Sobolev spaces $W_{m}$. Functions in $W_{m}$ are $(m-1)$ continuous if $p > m-1$. In our present setting it is not known what further conditions on $f$ imply almost $(m-1)$ continuity.
§ 3. In the first section of this paper we showed that the localization principle for rectangular partial sums hold for the class \( V_{p,1} \) in \( R^2 \) \((H = \{ n \})\). The following theorem asserts that this result is best possible in a certain sense.

**Theorem 3.1.** If \( \text{ABV} \) is not contained in \( \text{HBV} \), then the localization property for square partial sums does not hold for the class \( V_{p,1} \) in \( R^2 \).

In proving this result we use the fact that \( V_{p,1} \), with a suitable norm, is a Banach space. This is a consequence of our final result.

For \( f \in V_{p,1} \), choose corresponding functions \( f_i, i = 1, \ldots, m \), to be real continuous for \( -\pi \leq a_i < \pi \) and left continuous at \( a_i = \pi \) on almost every \( f_i \).

\[
\|f\|_{V_{p,1},a} = \|f\|_p + \sum_{i=1}^m \|V_i f_i\|_a.
\]

Note that if \( f \in V_{p,1} \), then \( f \) is chosen and \( g = f \) a.e., then the \( f_i \) are suitable corresponding functions for \( g, g \in V_{p,1} \), and \( \|f - g\|_{V_{p,1},a} = 0 \). Conversely, if \( g \in V_{p,1} \) and \( \|f - g\|_{V_{p,1},a} = 0 \), then \( f = g \) a.e. and \( f \in V_{p,1} \).

We see then that we may consider the elements of \( V_{p,1} \) to be equivalence classes of a.e. equal functions.

Our final result is the following.

**Theorem 3.2.** \( V_{p,1} \) is a Banach space with norm \( \| \cdot \|_{V_{p,1}} \).

We turn now to the proof of Theorem 3.1 assuming Theorem 3.2 to be valid. The following lemmas have been established recently [8]. We include its proof for the sake of completeness.

**Lemma 8.** If \( \text{ABV} \) is not contained in \( \text{HBV} \), then \( \sum_{i=1}^n 1/k \neq O \left( \sum_{i=1}^n 1/\lambda_i \right) \).

**Proof.** The hypothesis implies that there is a real sequence \( (a_n)_n \), \( a_n > 0 \), such that \( \sum_{i=1}^n a_i = \infty \), but \( \sum_{i=1}^n a_i/n = \infty \). \( \sum_{i=1}^n 1/k \leq O \left( \sum_{i=1}^n 1/\lambda_i \right) \) for all \( n \), then

\[
\sum_{i=1}^n a_i/k \leq C \sum_{i=1}^n 1/\lambda_i,
\]

implying \( \sum_{i=1}^n a_i/n \infty \), contrary to our hypothesis. \( \blacksquare \)

**Proof of Theorem 3.1.** Let \( I \) denote the square \([-\pi, \pi] \times I' \), the square \([-\pi/2, \pi/2] \times \), and \( k \) the characteristic function of \( I-I' \).

If \( S_m(X, f) \) is the \( m \)th square partial sum of the Fourier series of \( f \) at \( X = (x, y) \), define a linear operator on \( V_{p,1} \) by

\[
T_n(g) = S_m(0, k; g).
\]

If the localization principle for square partial sums holds in \( V_{p,1} \), then \( T_n(g) \to 0 \) as \( n \to \infty \) for every \( g \in V_{p,1} \), implying that \( (T_n) \) is a bounded sequence of operators. Let

\[
f_n = \text{signum} D_k(x) D_k(y).
\]

Then

\[
\|f_n\|_{V_{p,1}} = 4\pi^3 + 8\pi \sum_{i=1}^n 1/k,
\]

\[
\|f_n\|_{V_{p,1},a} = 4\pi^3 + 8\pi \sum_{i=1}^n 1/\lambda_i.
\]

Let \( g_n = f_n/\|f_n\|_{V_{p,1}} \). There is a \( C > 0 \) such that

\[
T_n(f) = \frac{1}{n} \int_{-n}^{n} \int_{-n}^{n} f(x, y) D_k(x) D_k(y) dx \, dy \geq C \log n
\]

for large \( n \). Thus

\[
T_n(g_n) \geq \frac{C \log n}{4\pi^3 + 8\pi \sum_{i=1}^n 1/\lambda_i} \neq O(1)
\]

as \( n \to \infty \), in view of Lemma 8. Thus \( (T_n) \) is not a bounded sequence. \( \blacksquare \)

**Proof of Theorem 3.2.** We will show only that \( V_{p,1} \) is complete. It is clear that \( V_{p,1} \) is a linear space and verification of the properties of the norm is straightforward. We treat the case \( m = 2 \), since it is entirely typical. The facts used concerning the space \( \text{ABV} \) are known [11], [13]. Consider \( (f_n) \) Cauchy convergent in \( V_{p,1} \) on \( I = [-\pi, \pi]^2 \). Let \( g_n = (f_n), h_n = (f_n) \) be the corresponding functions, chosen as above. If \( f \) is an \( \bar{F} \)-limit of \( (f_n) \), \( (F_n) \) be a subsequence of \( (f_n) \) converging a.e. to \( f \) and \( (G_n) \) the corresponding subsequence of \( (g_n) \).

For every \( k \in \mathbb{Z}^+ \) there is a least \( n_k \in \mathbb{Z}^+ \) such that

\[
\int V_1(G_n - G_m) dy < 1/3^k \quad \text{if} \quad m, n \geq n_k.
\]

Thus

\[
\int V_1(G_{n_k} - G_m) dy < 1/3^k \quad \text{for} \quad k = 1, 2, \ldots
\]

Let

\[
E_k = \{ y \mid V_1(G_{n_k} - G_m) > 2^{-n_k/3}\}
\]
Then
\[ |E_k| < (2/3)^k \]
and
\[ (*) \quad \sum_{k=N}^{\infty} V_1(G_{n_{k+1}} - G_{n_k}) \]
converges of \( y \notin \bigcup_{k=N}^{\infty} E_k \) for some \( N \). Since \( \bigcup_{k=N}^{\infty} E_k \to 0 \) as \( N \to \infty \),
the series (**) converges for almost every \( y \). Hence, for \( k > j \) and almost every \( y \), as \( k, j \to \infty \),
\[ V_j(G_{n_k} - G_{n_j}) \leq V_j(G_{n_{j+1}} - G_{n_j}) + \ldots + V_j(G_{n_k} - G_{n_j}) \to 0. \]
Now \( P_n \to f \) a.e. implies \( G_n \to f \) a.e. Thus, for almost every \( y \), \( \{G_n\} \) converges for some \( s \), which, with (**), implies that \( \{G_n\} \) converges in \( ABV \) as a function of \( x \), for almost every \( y \), to a \( ABV \) function \( g(x, y) \). For such \( y \), \( g \) is right continuous for \( -\pi < s < \pi \) and left continuous at \( s =\pi \), since converge in \( ABV \) implies uniform convergence. Since, for almost every \( y \),
\[ \lim_{k \to \infty} V_1(g_n - G_{n_k}) = V_1(g - g) \]
and
\[ \lim_{k \to \infty} V_1(G_{n_k}) = V_1(g), \]
we see that \( V_1(g_n - g) \) and \( V_1(g) \) are measurable functions. Given \( \varepsilon > 0 \), we have
\[ \int V_1^*(g_n - g) \, dy \leq \liminf_{k \to \infty} \int V_1^*(g_n - G_{n_k}) \, dy < \varepsilon \]
if \( n \) is sufficiently large. We must show that \( V_j(g) \in L^s \), but
\[ \int V_j^*(g) \, dy \leq \liminf_{k \to \infty} \int V_j^*(G_{n_k}) \, dy < \infty \]
since
\[ \int |V_j(G_{n_k}) - V_j(G_{n_j})|^s \, dy \leq \int V_j^*(G_{n_k} - G_{n_j}) \, dy \to 0 \]
as \( k, j \to \infty \), implying that \( \{V_j(G_{n_k})\} \) converges in \( L^s \).
In the same fashion, we may show that there is an \( h = f \) a.e., with
the appropriate continuity properties and such that
\[ ||V_1(h_n - h)||_s \to 0 \]
and \( V_1(h) \in L^s \). Then \( f \in V_{h,n}^* \) with corresponding functions \( g \) and \( h \),
\[ ||f||_{\lambda, R, s} = ||f||_R + ||V_1(g)||_s + ||V_1(h)||_s \]
and
\[ ||f_n - f||_{\lambda, R, s} \to 0, \]
as was to be shown. 