$M$-structure and the Banach–Stone theorem

by

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Abstract. We prove that, under certain conditions, the existence of an isometric isomorphism between $C(K, X)$ and $O(L, X)$ implies that $A \times K$ is homeomorphic to $A \times L$ for $A$ in a family of completely regular spaces ($K$ and $L$ compact Hausdorff spaces, $X$ a real Banach space).

1. Introduction. Let $X$ be a real Banach space and $\mathfrak{F}$ a class of nonvoid compact Hausdorff spaces. We say that $X$ has the Banach–Stone property for $\mathfrak{F}$ if, for $K$ and $L$ in $\mathfrak{F}$, the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ (the spaces of continuous $X$-valued functions on $K$ resp. $L$) implies that $K$ and $L$ are homeomorphic. [3] contains a complete description of those Banach spaces $X$ which have the Banach–Stone property for all nonvoid compact Hausdorff spaces, provided that the centralizer of $X$ is finite dimensional (the $M$-finite Banach spaces).

In the following we consider Banach spaces for which the norm topology and the strong operator topology are identical on the centralizer (the $M$-finite Banach spaces are obviously contained in this class). For these spaces $X$ it is possible to calculate the most important $M$-structure properties of $C(K, X)$ (centralizer, function module representation) provided the corresponding properties of $X$ are known [we note that these results are a special case of theorems concerning the $M$-structure of tensor products in [4]]. Using this we get the following result: For every class of nonvoid compact Hausdorff spaces $\mathfrak{F}$ and every Banach space $X$ as described above there is a family of completely regular spaces such that the existence of an isometric isomorphism between $C(K, X)$ and $C(L, X)$ implies that $A \times K$ is homeomorphic to $A \times L$ for $A$ in this family ($K$ and $L$ in $\mathfrak{F}$). It turns out that this theorem is a strong generalization of parts of the results of the first-named author in [3].

2. Function modules. We will use the terminology of [3]. In particular, $X \cong Y$ for Banach spaces $X$ and $Y$ (resp. $K \cong L$ for topological spaces $K$ and $L$) means that $X$ and $Y$ are isometrically isomorphic (resp. $K$ and $L$ are homeomorphic).
Function modules have been introduced by Cunningham ([8], [9]). Note, however, that we use a slightly different definition.

2.1. Definition. ([1], [5]) Let \( X \) be a real Banach space. A linear operator \( T : X \to X \) is called \( M \)-bounded if there is a \( \lambda > 0 \) such that the following condition is satisfied:

\[
Tz \text{ is contained in every ball which contains } \pm \lambda x \text{ (all } x \in X) \nonumber
\]

\( Z(X) \), the centralizer of \( X \), means the collection of all \( M \)-bounded operators on \( X \). It is known that there exists an isometric algebra isomorphism between \( Z(X) \) and \( CK_X \) for a suitable compact Hausdorff space \( K_X \).

2.2. Definition. I. A function module is a triple \((\tilde{K}, (X_k)_{k \in K}, \tilde{X})\) where

(a) \( \tilde{K} \) is a nonvoid compact Hausdorff space;
(b) \((X_k)_{k \in K}\) is a family of Banach spaces, indexed by the points of \( \tilde{K} \);
(c) \( \tilde{X} \) is a closed subspace of \( \prod_{k \in K} X_k \) (the direct product of the \( X_k \), provided with the supremum norm);
(d) \( \tilde{X} \) and \((X_k)_{k \in K}\) have the following properties:

\( k \mapsto [x(k)] \) is upper semicontinuous for every \( x \in \tilde{X} \);
(ii) \( x \in \tilde{X} \) for \( x \notin CK_X \), \( x \in X \) (pointwise multiplication);
(iii) \( X_k = [x(k)] \) for \( x \in \tilde{X} \);
(iv) \( \tilde{X} = [x] \neq \emptyset \);
(v) \( T \in Z(\tilde{X}) \) iff there is a function \( h \in CK_X \) such that \( Tx = \lambda x \) for every \( \lambda \in \tilde{X} \).

II. Two function modules \((\tilde{K}_1, (X_k)_{k \in K_1}, \tilde{X}_1)\), \((\tilde{K}_2, (X_k)_{k \in K_2}, \tilde{X}_2)\) are called equivalent if there are

(i) an isometric isomorphism \( J : \tilde{X}_1 \to \tilde{X}_2 \);
(ii) a homeomorphism \( t : \tilde{K}_1 \to \tilde{K}_2 \);
(iii) isometric isomorphisms \( u_k : X_k \to X_{k_0} \) such that \( u_k(x(k)) = (Jx)(t(k)) \) for every \( x \in \tilde{X}_1, k \in \tilde{K}_2 \).

III. A function module \((\tilde{K}, (X_k)_{k \in K}, \tilde{X})\) is called a function module representation of a real Banach space \( X \) if \( X = \tilde{X} \).

2.3. Theorem.

(a) Two function modules \((\tilde{K}_1, (X_k)_{k \in K_1}, \tilde{X}_1)\) and \((\tilde{K}_2, (X_k)_{k \in K_2}, \tilde{X}_2)\) are equivalent iff \( \tilde{X}_1 \cong \tilde{X}_2 \).

(b) Every real Banach space \( X \) has a function module representation.

Proof. (a) Let \((\tilde{K}_1, (X_k)_{k \in K_1}, \tilde{X}_1)\), \((\tilde{K}_2, (X_k)_{k \in K_2}, \tilde{X}_2)\) be function modules, \( J : \tilde{X}_1 \to \tilde{X}_2 \) an isometric isomorphism. \( T \mapsto JTV^{-1} \) is an isometric algebra isomorphism from \( Z(\tilde{X}_1) \) onto \( Z(\tilde{X}_2) \) which induces an isometric algebra isomorphism \( J : CK_X \) onto \( CK_2 \) (2.2.1. (d) implies that \( CK \cong Z(X) \)). Note that \( J(hk) = (j(h))(j(k)) \) for \( h \in CK_1, k \in \tilde{K}_1 \). By the classical Banach–Stone theorem there exists a homeomorphism \( t : \tilde{K}_1 \to \tilde{K}_2 \) such that \( J(h)(t(k)) = (j(h))(j(k)) \) for every \( h \in CK_1, k \in \tilde{K}_1 \). Let \( \tilde{k} \in \tilde{K}_1 \) be fixed. We define \( u_{\tilde{k}} : X_{\tilde{k}} = \tilde{X}_{\tilde{k}} = X \) by

\[
u_{\tilde{k}}(x(k)) = (Jx)(t(k)) \quad (x \in \tilde{X}_1).
\]

\( \tilde{u}_{\tilde{k}} \) is well-defined, for let \( x \in \tilde{X}_1 \) be an element such that \( x(k_0) = 0 \) (we will show that \( Jx(k_0) = 0 \) in this case).

For \( \epsilon > 0 \), there is a neighbourhood \( U \) of \( k_0 \) such that \( |x(k)| < \epsilon \) for \( k \in U \). Choose a function \( h \in CK_X \) such that \( h(k_0) = 1 - \|x\|, h(k_0) = 0 \) for \( k \notin U \). It follows that

\[
|Jx(t(k))| \leq |Jx(k)| \leq U \|x(k)\| x(k_0) \leq \epsilon.
\]

Similarly, it can be shown that \( u_{\tilde{k}}(f(x(k))) = x(k) \). \( u_{\tilde{k}} \) is defined on all of \( \tilde{X}_1 \) by \( 2.2.1.(d) \) (iii) so that \( (u_{\tilde{k}})_{k \in K_1} \) is a family of isometric isomorphisms (the surjectivity of the \( u_{\tilde{k}} \) follows immediately from the surjectivity of \( J \) and \( 2.2.1.(d) \) (iii)). By definition, \( J, t, (u_{\tilde{k}})_{k \in K_1} \) define an equivalence. The reverse implication is valid by definition.

(b) The following conclusion is due to Cunningham. Let \( K = X \) be a compact Hausdorff space with \( Z(\tilde{X}) \cong CK_X \) and \( J_x : Z(\tilde{X}) \to CK_X \) an isometric algebra isomorphism. For \( x \in X \) we define \( \omega(x) : K \to \mathbb{R} \) (the norm resolution of \( x \)) by

\[
|\omega(x) : K \to \mathbb{R} || T \in Z(\tilde{X}) | J_x(T)(x(k)) = 1, J_x(T) > 0 \).
\]

It is easy to see that \( x \mapsto |\omega(x) : K \to \mathbb{R} | \) is a seminorm on \( X \) for every \( k \in K \).

Let \( \tilde{X} \) be the associated Banach space, i.e. the completion of \( X | \omega(x) | = 0 \). The quotient provided with the norm \( ||x|| : = |\omega(x) : K \to \mathbb{R} | (k \in K \in X) \).

Define \( \omega : X \to \prod_{k \in K} X_k \) by

\[
\omega(x)(k) := \text{the equivalence class of } x \text{ in } X_k.
\]

Then \( X = \omega(X) \) is a function module representation of \( X \).

\( \omega(X) \cong \omega(X) \) and 2.2.1. (a), (b), (c) are easily verified. (d) is an immediate consequence of the definitions, (iii) is proved in [5], p. 621, (iv) follows from the surjectivity of \( J_x \). Note that \( T \mapsto \omega(T)x \) is an isometric isomorphism between \( Z(\tilde{X}) \) and \( Z(\omega(X)) \) and that \( \omega(T)x = J_x(T)x \) for all \( T \in Z(\tilde{X}) \) (\( x \in X \)) by which we get (d) and (v).
2.4. COROLLARY. Let $X$ and $Y$ be Banach spaces with function module representations $(\hat{X}, (X_{0})_{0\leq k\leq d}, \hat{X})$ resp. $(\hat{Y}, (Y_{0})_{0\leq k\leq d}, \hat{Y})$. If $X \cong Y$, then $(\hat{X}, (X_{0})_{0\leq k\leq d}, \hat{X})$ and $(\hat{Y}, (Y_{0})_{0\leq k\leq d}, \hat{Y})$ are equivalent.

2.5. EXAMPLES. (a) Let $L$ be a locally compact Hausdorff space.

$$X := C_{0}L = \{ f : f : L \to \mathbb{R}, f \text{ continuous}, f \text{ vanishes at infinity} \}.$$

In this case the centralizer of $X$ consists exactly of the mappings $M_{k} : f \to h$ with $h : L \to \mathbb{R}$ bounded and continuous. Then $(\beta L, (X_{0})_{0\leq k\leq d}, \mathbb{Z})$ is a function module representation of $X$, where $X_{0} = \mathbb{R}$ resp. $X_{0} = \{ 0 \}$ according to whether $k \in L$ or $k \in \beta L \setminus L$.

(b) Let $X$ be $M$-finite with canonical $M$-decomposition $X = \bigoplus_{q=0}^{\infty} M_{q}$... 

The operators in $Z(X)$ are just the operators of the form

$$(a_{1}^{1}, \ldots , a_{1}^{r}, a_{2}^{1}, \ldots , a_{2}^{r}, \ldots , a_{t}^{1}, \ldots , a_{t}^{r})$$

with $a_{i}^{q} \in \mathbb{R}$ for $q = 1, \ldots , r$, $i = 1, \ldots , n_{q}$. With $\hat{X} := \bigoplus_{q=1}^{t} (\{ 0 \}, 1, \ldots , (\delta, n_{q}), \text{ and } X_{0} := \bigoplus_{q=0}^{\infty} \mathbb{M}_{q}(\hat{X}, (X_{0})_{0\leq k\leq d}, \hat{X})$ is a function module representation of $X$.

(e) Let $e$ be a fixed real number, $0 < \| e \| \leq 1$, $X_{e} := \{ f : f \in \mathbb{C}[0,1], f(0) = e \}$. In this case $Z(X_{e})$ consists exactly of the operators of the form $f \to h$ with $h \in \mathbb{C}[0,1], h(0) = h(1)$.

2.7. EXAMPLES. (a) $C_{0}L$ has a cms iff $L$ is compact.

(b) Every $M$-finite Banach space (in particular every reflexive space [ex. 2 in [3], Section 1]) has a cms.

(c) $X$ has a cms for $0 < \| e \| \leq 1$.

2.8. THEOREM. Let $(\hat{X}, (X_{0})_{0\leq k\leq d}, \hat{X})$ be a function module, $K$ a nonvoid compact Hausdorff space. We define

$$X_{p}(p, k) := \{ (p, k) \in K \times \hat{X} : \| (h \otimes x)(p, k) \| \leq (\text{const}) \}$$

If $\hat{X}$ has a cms, then

(a) $(K \times \hat{X}, (X_{p}(p, k))_{0\leq k\leq d}, \hat{X})$ is a function module,

(b) $\hat{X} \cong C(K, \hat{X})$.

Proof. (a) We have to verify the properties listed in 2.2.1.

(a), (b), (c), (d), (iii), (iv) are trivially satisfied.

(d), (i), (ii) are easy consequences of the well-known identity

$$C(K \times \hat{X}) = (\{ (h_{1} \otimes h_{2})_{1 \leq k \leq d}, h_{1} \in C(K), h_{2} \in C(L) \})^{-1} \cong C(K) \hat{\otimes} C(L).$$

It remains to prove that $(d)$ is true. Clearly, for $h \in C(K \times \hat{X})$, $z \mapsto h_{z}$ is in the centralizer of $\hat{X}$ (this mapping is well-defined by $(d)(iv)$, and it satisfies the defining property for $M$-bounded operators with $\lambda = \| h \|$).

Conversely, let $T : \hat{X} \cong \hat{X}_{p}$ be $M$-bounded. For $p \in K$, define $T_{p} \hat{X} \cong \hat{X}$ by $T_{p}(x)(h) := (T(h \otimes x))(p, k) \in \hat{X}$ (note that $T_{p} \hat{X}$ since $\hat{X}$ is complete and $e(p, k) \in \hat{X}$ for $x \in \mathbb{C}[0,1], x \in \hat{X}$). It is easy to see that every $T_{p} \hat{X}$ is $M$-bounded (in fact, $T_{p}$ satisfies 2.1 for the same values as $T$ does). Consequently there exists, for $p \in K$, a function $h_{z} \in C(K)$ such that $T_{p}x = h_{z}x$ for every $x \in \hat{X}$. Define $h_{z} : K \times \hat{X} \rightarrow \hat{X}$ by $h_{z}(p, k) := h_{z}(k)$. We will show that $h_{z}$ is continuous and that $T_{p}x = h_{z}x$ for $x \in \hat{X}_{p}$. Let $x \in \hat{X}_{p}$, then $x \in \mathbb{C}[0,1], x \in \hat{X}, x \in \mathbb{C}[0,1], x \in \hat{X}$.

Thus, for $h \in C(K, \hat{X})$, $x \in \hat{X}$,

$$T(h \otimes x)(p, k) = h(p, k)h_{x}(k_{x})$$

$$= t(x)(h \otimes x)(p, k) = \| (h \otimes x)(p, k) \| \leq (\text{const}) \}$$

2.9. DEFINITION. (cf. [4]) Let $X$ be a real Banach space. A centraliser norming system (cms) is a finite family $x_{1}, \ldots , x_{n}$ in $X$ such that there exists a number $r > 0$ for which $\max(\| x_{1} \|, \ldots , \| x_{n} \|) > r \| x \|$ (all $x \in Z(X)$). Obviously, $X$ has a cms iff the norm topology and the strong operator topology coincide on $Z(X)$. It is easy to see that, for function modules $(\hat{X}, (X_{0})_{0\leq k\leq d}, \hat{X})$, $X$ has a cms iff there are $x_{1}, \ldots , x_{n} \in \hat{X}$ such that $\inf_{n} \max_{k \leq d} \| x_{i} \| > 0$. 

Thus for $h \in C(K, \hat{X})$, $x \in \hat{X}$,

$$T(h \otimes x)(p, k) = h(p, k)h_{x}(k_{x})$$

$= t_{x}(h \otimes x)(p, k) = \| (h \otimes x)(p, k) \| . (\text{const})$. 

$$= \| (h \otimes x)(p, k) \| \leq (\text{const}) \} \leq (\text{const}) \}$$

$$= \| (h \otimes x)(p, k) \| \leq (\text{const}) \} \leq (\text{const}) \}$$

$$= \| (h \otimes x)(p, k) \| . (\text{const}) \} \leq (\text{const}) \}$$

$$= \| (h \otimes x)(p, k) \| \leq (\text{const}) \} \leq (\text{const}) \}$$
for every \((p_n, k_n) \in K \times \tilde{X}\) (since \([k \otimes x] = (k \otimes x)\) for every \(x \in \tilde{X}\), note that \(k\) is bounded by \((T)\)). For the proof of the continuity of \(h_n\) we first note that it is sufficient to show that \(p \mapsto T_p\) is a continuous mapping from \(K\) to \(\tilde{X}\) (since \(\tilde{X} = C(K, \tilde{X})\)). Let \(x_1, \ldots, x_n \in \tilde{X}, r > 0\) as in 2.1. For every \(x \in \tilde{X}\), \(p \mapsto \|p(x)\|\) is continuous from \(K\) to \(\tilde{X}\) (this follows at once from the definition of \(\tilde{X}\)). In particular, for \(p_n \in K, r > 0\), there is a neighbourhood \(U\) of \(p_n\) such that

\[
\|T(1 \otimes x_n)(p_n) - T(1 \otimes x_n)(p)\| < \epsilon\]  

for \(i = 1, \ldots, n, p \in U\). This yields \(\|x_n - x\| < \epsilon\) for every \(p \in U\) so that \(p \mapsto T_p\) is continuous at \(p_n\).

(b) This is an immediate consequence of \(C(K, \tilde{X}) \cong C \otimes \tilde{X}\) (explicitly: \(C(K, \tilde{X}) = (\text{lin}(k \otimes x) \mid k \in K, x \in \tilde{X})^\sim\), whereby in this case \((k \otimes x)(p) = h(p, x)\).

2.9. COROLLARY. Let \(K\) be a real Banach space having a cns. If \((\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) is a function module representation of \(K\), then \((X, (X_{\alpha})_{\alpha \in \Lambda, X})\) is a function module representation of \(C(K, \tilde{X})\).

3. Function module properties.

3.1. DEFINITION. A function module property \(P\) is a rule which determines for every function module \((\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) a subset \(P(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) of \(\tilde{K}\) such that the following property holds:

If \(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}\) and \(\tilde{K}, (X'_{\alpha})_{\alpha \in \Lambda, \tilde{X}'}\) are equivalent function modules, the equivalence being defined by \(f, (\omega_{\alpha})_{\alpha \in \Lambda, \tilde{X}}\) (cf. 2.2), then

\[
P(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) = \{ \tilde{P}(\tilde{K}, (X'_{\alpha})_{\alpha \in \Lambda, \tilde{X}'}) \}.
\]

For a function module \((\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) let \(\varphi(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) be the collection of all sets \(P(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) for arbitrary function module properties \(P\).

3.2. NOTE. It is easy to see that \(\varphi(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) is always a complete Boolean algebra of subsets of \(\tilde{K}\).

3.3. EXAMPLES. (a) For function modules \((\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) we define

\[
P_{\varphi}(x, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) := \{ k \in \tilde{K}, x_{\alpha} \in \tilde{X} \}
\]

\((x_{\alpha} a \text{ fixed Banach space})\),

\[
P_{\text{cont,n}}(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) := \{ k \in \tilde{K}, 1 \mapsto \|x\| \text{ is continuous at } k \text{ for every } x \in \tilde{X}\}.
\]

For every \(x \in \tilde{X}\), (a) \((a \in [0, 1]\) a fixed number),

\[
P_{\text{cont,lin}}(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) := \{ k \in \tilde{K}, \text{ the neighbourhood filter of } k \text{ has a countable basis} \}.
\]

In all these cases we define function module properties. This follows immediately from 2.2.

(b) Let \(n_1, \ldots, n_r\) be natural numbers, \(M_1, \ldots, M_r\), real nonzero Banach spaces, \(\tilde{K} := \bigcup_{\alpha \in \Lambda} \{ (\epsilon, 1), \ldots, (\epsilon, n_r) \}\), \(X_{\alpha}: = M_\alpha, \epsilon = 1, \ldots, n_r\), \(\tilde{X} := \prod_{\alpha \in \Lambda} X_{\alpha}\).

In this case \(\varphi(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) is the Boolean algebra generated by the sets \(\{ (\epsilon, 1), \ldots, (\epsilon, n_r) \}\), \(\epsilon = 1, \ldots, r\).

3.4. DEFINITION. Let \(\varphi\) be a class of nonvoid compact Hausdorff spaces. A function module property \(P\) is called \(\varphi\)-hereditary if

\[
P(\tilde{K} \times \tilde{X}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) = \tilde{K} \times P(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})
\]

for every \(K \in \varphi\) and every function module \((\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) for which \(K\) has a cns (cf. 2.9).

\[
\varphi(x, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) := \{ P(x, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) \}
\]

is a Boolean sub-algebra of \(\varphi(\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\).

3.5. EXAMPLES. \(P_{\varphi}, P_{\text{cont,n}}\) are \(\varphi\)-hereditary with \(\varphi = \"\text{the class of all nonvoid compact Hausdorff spaces}\"\).

\(P_{\text{cont,lin}}, P_{\text{cont,lin}}\) is \(\varphi\)-hereditary with \(\varphi = \"\text{the class of all nonvoid first countable compact Hausdorff spaces}\"\).

This follows at once from the definition of \((K \times \tilde{X}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\).

Note that \(\varphi(x, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}}) = \varphi(x, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) in the case of Example 3.3(b) (\(\varphi\) an arbitrary class of nonvoid compact Hausdorff spaces).


4.1. THEOREM. Let \(\tilde{X}\) be a real Banach space having a cns, \(\varphi\) a class of nonvoid compact Hausdorff spaces, \((\tilde{K}, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) a function module representation of \(K\). Then, for \(K, L\) in \(\varphi\), the existence of an isometric isomorphism between \(C(K, \tilde{X})\) and \(C(L, X)\) implies that \(K \times X \cong L \times X\) for every \(x \in \tilde{X}\).

In particular, \(\tilde{X}\) has the Banach–Stone property for \(\varphi\) if \(\varphi(x, (X_{\alpha})_{\alpha \in \Lambda, \tilde{X}})\) contains an element consisting of a single point.
Proof. Let \( P \) be a \( \Psi \)-hereditary function module property, \( J: C(K, X) \to C(L, X) \) an isometric isomorphism. By 2.9 and 2.4, \((K \times \hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}_K))\) and \((L \times \hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}_L))\) are equivalent so that, by definition, \( K \times P(\hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}_L)) \cong P(K \times \hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}_K)) \cong P(L \times \hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}_L)) \cong L \times P(\hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}_K)). \)

4.2. Examples. (a) \( \hat{K} \) is always contained in \( \mathcal{P}(\hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}) \) so that we always may conclude that \( \hat{K} \times K \cong \hat{K} \times L \) (this has first been noted, for \( M \)-finite Banach spaces, in [2], th. 4.1).

(b) For an \( M \)-finite Banach space \( X = M_{\gamma_1} \oplus \cdots \oplus M_{\gamma_l} \), \( C(K, X) \cong C(L, X) \) implies that \( \{(\gamma, 1, \ldots, \gamma, n_0) \times K \cong \{(\gamma, 1) \times L \} \times \cdots \times L \} \) (cf. 3.5). This is just the assertion of Theorem 4.4 in [2] for the compact case.

(c) \( X \) has the Banach–Stone property for the class of all nonvoid compact Hausdorff spaces for every \( |a| \in [0, 1] \). Note that \( P(\hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}) \) consists of a single point for every function module representation \( \hat{K}, (X_{(a,b)}(\alpha,\delta), \hat{X}) \) of \( X_a \) and \( a = z \).

Analogously one can treat the case of arbitrary \( G \)-spaces (cf. 2.5).

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The localization principle for double Fourier series

by

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Abstract. Definitive results are obtained for localization by square and rectangular sums for the Fourier series of functions of 2 variables. For this purpose functions of \( A \)-bounded variation, \( ABV \), and in particular, harmonic bounded variation, \( HBV \), are defined for functions of 2 variables. It is shown that if \( f \in ABV \), then localization holds for rectangular sums. However, if \( ABV \neq HBV \), there is an \( f \in ABV \) for which localization fails even for square sums.

This contrasts of course with the 1 variable case, where localization holds for all summable functions. It differs as well from the case \( n > 3 \) where previously obtained definitive results are in a Sobolev space framework.

The Riemann localization principle for periodic functions of one variable asserts that if an integrable function vanishes identically on an open interval, then the partial sums of its Fourier series converge uniformly to zero on any compact subset of that interval. For functions of several variables, strong additional assumptions are required in order that the principle of localization may hold. Indeed, if we consider convergence of the rectangular partial sums, \( S_n = S_{n_1,n_2}, \ldots, n_t \) of the Fourier series of an integrable function defined on \([-\pi, \pi]^d \) in \( L_1 \), localization may fail even if the function is continuous. Here, by convergence of \( (S_n) \), we mean the existence of \( \lim S_n \) as \( n \to \infty \).

There are various alternatives one may pursue to obtain localization theorems. Among these are:

1. to require that \( f = 0 \) on a larger set,
2. to make additional global requirements on \( f \),
3. to replace convergence by other limiting procedures,
4. to replace rectangular partial sums by other sums of terms of the Fourier series, and various combinations of these [14], Chap. 17.

For example, if we require that \( f = 0 \) not only in the given interval, but on every line in the direction of a coordinate axis and intersecting

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