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A mean value inequality for positive integral transformations with application to a maximal theorem

by

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Abstract. An integral mean value inequality of the Riesz type is derived and applied to obtain a sharp form of the maximal inequality of Jurkat and Troutman in dimension one generalizing the classical result of Hardy–Littlewood.

1. Introduction. In this paper we present conditions under which a positive integral transformation of the form

$$Kg(x, y) \overset{\text{def}}{=} \int_0^y k(x, t)g(t)\,dt \qquad (0 < y \leqslant x), \tag{1}$$

admits estimation from above by $[K\mathbf{1}(x, y)] \sup\limits_{0<v\leqslant y} Kg(v, v)$ where $\mathbf{1}$ denotes the unit function and K satisfies

$$K\mathbf{1}(x, x) = 1 \qquad (x > 0). \tag{2}$$

The method used is an integral analogue of that given for finite sums by Jurkat and Peyerimhoff in [1], [2] and the result constitutes a sharpening of the mean value inequality of M. Riesz.

The normalization (2) is automatically achieved for a kernel

$$k(x, t) = \varphi_x(t) = x^{-1}\varphi(tx^{-1})$$

when φ is positive with a unit integral over $J \equiv (0, 1)$ and vanishes elsewhere. Such kernels provide through convolution a standard approximation of the identity, and in a previous paper [4], we have shown that for any measurable $f \geqslant 0$, the associated maximal function,

$$M_\varphi f(x) \overset{\text{def}}{=} \sup_{h>0} \int_0^\infty \varphi_h(t)f(x-t)\,dt, \tag{3}$$

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satisfies the following inequality:

$$(4)\qquad (M_\varphi f)^*(\xi) \leqslant A\int_0^1 \varphi^*(t) f^*(t\xi)\,dt \qquad (\xi > 0),$$

where the asterisk denotes the formal decreasing rearrangement (defined below), and the constant $A \in [1, 6]$. For $\varphi = \mathbf{1}$ on J, this would be the classical inequality of Hardy–Littlewood [5], provided that the constant A — which arises from a well-known covering principle — can be taken as unity. In the concluding section of the present paper, we show that under some restrictions on φ (which still admit the Cesàro kernels $\varphi(t) = a(1-t)^{a-1}$, t, $a \in J$) our mean value theorem supplies a Vitali-like covering argument which yields the maximal inequality (4) with $A = 1$. That $A = 1$ is the best possible constant for this inequality was established in [4].

2. Notation. $\mathfrak{M}$, $(\mathfrak{M}^+)$ denotes the class of (non-negative) extended real valued Lebesgue measurable functions on $\boldsymbol{R}$, and L^∞ the usual equivalence class of essentially bounded functions. $|E|$ $(|E|_0)$ denotes the (outer) measure of a set $E \subseteq \boldsymbol{R}$ and the formal decreasing "rearrangement" of an extended real valued function F is defined by

$$F^*(\xi) = \inf\{\tau > 0:\ \big||F| > \tau\big|_0 \leqslant \xi\} \qquad \text{for} \qquad \xi > 0.$$

3. A mean value inequality.

THEOREM 1. *Let $k(\cdot,\cdot)$ be defined, positive, and for each x, be y-measurable in the triangle $\Delta_a = \{(x, y):\ 0 < y \leqslant x < a\}$ for some $a \in (0, \infty)$; and satisfy in addition the following conditions:*

(i) $\int_0^x k(x, t)\,dt = 1 \quad \forall x \in (0, a)$;

(ii) $\int_0^y |k(y, t) - k(x, t)|\,dt \to 0$ *as* $x \searrow y \in (0, a)$;

(iii) *for each* $(x, y) \in \Delta_a$, $k(x, \cdot)/k(y, \cdot)$ *decreases on* $(0, y)$.

Then for each $g \in \mathfrak{M}[0, a)$ which is locally (essentially) bounded from below,

$$Kg(x, y) = \int_0^y k(x, t) g(t)\,dt$$

is defined and satisfies the inequality

$$Kg(x, y) \leqslant K\mathbf{1}(x, y) \sup_{0<v\leqslant y} Kg(v, v).$$

Proof. For $g \in L^\infty$, $|Kg(x, y)| \leqslant \|g\|_\infty K\mathbf{1}(x, y) \leqslant \|g\|_\infty$ from (i) when $0 < y \leqslant x < a$, and it is clear that for each $x \in (0, a)$, $Kg(x, \cdot)$ is continuous on $[0, x]$. The remaining integral requirement (ii) ensures that for

$x \geqslant y \geqslant z \in (0, a)$, $Kg(x, y) \to Kg(z, z)$ as $x \searrow z$. Indeed,

$$\begin{aligned}|Kg(x, y) - Kg(z, z)| &\leqslant |Kg(x, y) - Kg(x, z)| + |Kg(z, z) - Kg(x, z)|\\ &\leqslant \int_z^y k(x, t)|g(t)|\,dt + \int_0^z |k(z, t) - k(x, t)|\,|g(t)|\,dt\\ &\leqslant \|g\|_\infty\Big(\int_z^y k(x, t)\,dt + \int_0^z |k(z, t) - k(x, t)|\,dt\Big)\\ &\leqslant \|g\|_\infty \int_0^z 2[k(z, t) - k(x, t)]^+\,dt,\end{aligned}$$

by a double application of (i).

To prove the inequality, assume first that $0 \leqslant g \in L^\infty$ and that g vanishes indentically in $(0, x_0)$ for some $x_0 \in (0, a)$. Then for fixed $(x, y) \in \Delta_a$, introduce $w(\cdot) \overset{\text{def}}{=} k(x, \cdot)/k(y, \cdot)$ which decreases on $(0, y)$ by (iii) and is extended to vanish at y. Hence the Fubini theorem affords partial integration(s) which show that with $Y = (0, y]$,

$$\begin{aligned}G(x, y) \overset{\text{def}}{=} \frac{Kg(x, y)}{K\mathbf{1}(x, y)} &= \frac{\int_0^y w(t) k(y, t) g(t)\,dt}{\int_0^y w(t) k(y, t)\,dt}\\ &= \frac{\int_Y Kg(y, t)\,dw(t)}{\int_Y K\mathbf{1}(y, t)\,dw(t)} = \frac{\int_Y G(y, t) K\mathbf{1}(y, t)\,dw(t)}{\int_Y K\mathbf{1}(y, t)\,dw(t)}\\ &= G(y, y_1), \qquad \text{for some } y_1 \in [x_0, y]\end{aligned}$$

by the continuity of Kg (hence of $K\mathbf{1}$ and G) in its second argument, and the first law of the mean. If $y_1 = y$, the first step of the proof is complete; otherwise repetition of the above argument produces either the desired termination or a decreasing sequence $\{y_n\}$ with a limit $z \in [x_0, y)$ such that for each $n = 1, 2, \ldots$, $G(x, y) = G(y_n, y_{n+1})$. Thus, by the continuity property of Kg established above and extended to G, it follows from (i) that $G(x, y) = G(z, z) = Kg(z, z)$.

Next, each $g \in \mathfrak{M}^+(0, a)$ is the pointwise limit from below of a sequence of functions $0 \leqslant g_n \in L^\infty$, $n = 1, 2, \ldots$; each vanishing in a neighborhood of 0. Hence with (x, y) as above, there exist a sequence $z_n \in (0, y]$, $n = 1, 2, \ldots$, for which by standard monotonicity arguments,

$$\begin{aligned}0 \leqslant Kg(x, y) &= \lim_n Kg_n(x, y) = K\mathbf{1}(x, y) \lim_n Kg_n(z_n, z_n)\\ &\leqslant K\mathbf{1}(x, y) \sup_{0<v\leqslant y} Kg(v, v).\end{aligned}$$

Finally, when g is measurable and (essentially) bounded from below by $-m$ on $[0, a)$ so that $g+m\mathbf{1} \in \mathfrak{M}^+(0, a]$ we have

$$\frac{Kg(x,y)}{K\mathbf{1}(x,y)} = \frac{K(g+m\mathbf{1})(x,y)}{K\mathbf{1}(x,y)} - m$$
$$\leqslant \sup_{0<v\leqslant y} K(g+m\mathbf{1})(v,v) - m = \sup_{0<v\leqslant y} Kg(v,v),$$

which clearly implies the desired result.

Remark. When $g \in L^\infty_{\mathrm{loc}}[0, a)$ the corresponding two sided mean value inequality follows.

4. Application to a covering argument. To estimate the maximal function $M_\varphi f$ of (3), for fixed φ and f, it is known since F. Riesz [5], that a most fruitful approach is to obtain a precise estimate of the size of the sets $E_\tau \overset{\mathrm{def}}{=} \{x\colon M_\varphi f(x) > \tau\}$ $(\tau > 0)$ by means of intervals of size h for which $\varphi_h * f(x) \overset{\mathrm{def}}{=} \int_0^\infty \varphi_h(t) f(x-t)\,dt > \tau$. In this section we show that Theorem 1 affords such an estimate under the following conditions on φ: (Compare with [3], Theorem 4.3.)

(a) φ is positive, increasing, and differentiable on $J = (0, 1)$ and vanishes elsewhere;

(5) (b) $\int_0^1 \varphi(t)\,dt = 1$;

(c) $t\varphi'(t)/\varphi(t)$ increases on J;

(d) $(t-1)\varphi'(t)/\varphi(t)$ increases on J.

Here, "increasing" is to be interpreted as nondecreasing in each instance so that $\varphi = \mathbf{1}$ on J is admissible. Moreover, it is straightforward to verify that the conditions are satisfied by the important class of Cesàro kernels $\varphi(t) = a(1-t)^{a-1}$ for $t, a \in J$.

LEMMA. *If φ satisfies conditions* (5) *and $f \in \mathfrak{M}^+$ is bounded with compact support, then when $\tau > 0$, with each $x \in E_\tau$, $\exists v > 0$ such that $(x-v, x] \subseteq E_\tau$ and $\varphi_v * f(x) > \tau$.*

Proof. Under the hypotheses, $\varphi_h * f(x)$ is jointly continuous when $h > 0$, $x \in \boldsymbol{R}$ and so $M_\varphi f$ is measurable. Indeed, for each $\tau > 0$, E_τ is open and thus for each fixed $x \in E_\tau$ we have $(a, x] \subseteq E_\tau$ for a minimal a which we may suppose finite since otherwise the lemma is trivial; in particular $a \notin E_\tau$. Since $x \in E_\tau$, $\exists\, h > 0$ for which $\varphi_h * f(x) > \tau$ and we need only consider $h > y \overset{\mathrm{def}}{=} x - a$. Set $k(s, t) = \varphi_s(t)$ and observe that as $s \searrow z > 0$,

$$\int^z |\varphi_z(t) - \varphi_s(t)|\,dt = \int^1 \varphi(t)\,dt \to 0$$

since φ is increasing. Moreover, for $v \leqslant s$, φ_s/φ_v is decreasing on $(0, v)$ (an easily verifiable consequence of (5c)), and it follows that the kernel $k(\cdot,\cdot)$ so defined meets the conditions of Theorem 1 which with $g(t) = f(x-t)$ supplies the estimate

$$I \overset{\mathrm{def}}{=} \int_0^y \varphi_h(t) f(x-t)\,dt \leqslant \int_0^y \varphi_h(t)\,dt\,\Phi_y f(x)$$

where

$$\Phi_y f(x) \overset{\mathrm{def}}{=} \sup_{0<v\leqslant y} \varphi_v * f(x).$$

Next, with $u \overset{\mathrm{def}}{=} h - y$, it follows from (5d) that $\varphi_h(t+y)/\varphi_u(t)$ is also decreasing for $t \in (0, u)$; hence, as in the first step in the proof of Theorem 1, with $g(t) = f(a-t)$, we obtain the estimate

$$\mathrm{II} \overset{\mathrm{def}}{=} \int_0^u \varphi_h(t+y) f(a-t)\,dt \leqslant \int_0^u \varphi_h(t+y) \sup_{0<v\leqslant u} \left(\frac{\int_0^v \varphi_u(t) f(a-t)\,dt}{\int_0^v \varphi_u(t)\,dt} \right) dt.$$

and we may further bound each ratio within the parentheses exactly as above by $\Phi_u f(a)$ which cannot exceed τ since $a \notin E_\tau$.

Combining these estimates with obvious substitutions gives

$$\tau < \varphi_h * f(x) = \mathrm{I} + \mathrm{II}$$
$$\leqslant \Phi_y f(x) \int_0^y \varphi_h(t)\,dt + \tau \int_y^h \varphi_h(t)\,dt$$

which is only possible providing $\Phi_y f(x) > \tau$; i.e. $\varphi_v * f(x) > \tau$ for some $v \in (0, y]$ so that $(x-v, x] \subseteq (x-y, x] \subseteq E_\tau$ as desired.

Under the conditions of the preceding lemma, when $-\infty < a \notin E_\tau$ we have from continuity that for each $x \in (a, b] \subseteq E_\tau$,

$$h(x) = \max_{0<v\leqslant x-a} \{v\colon \varphi_v * f(x) \geqslant \tau\}$$

is well-defined and positive. This defines on $(a, b]$ a function h such that $0 < h(x) \leqslant x - a$ and $\varphi_h * f(x) \geqslant \tau$ $\forall x \in (a, b]$. With this h as choice function, a straightforward application of transfinite induction on the countable ordinals (utilizing the fact that an uncountable family of positive numbers cannot be assigned a finite sum) shows that $(a, b]$ may be represented as the union of a countable family of disjoint intervals $I_n = (x_n - h_n, x_n]$ for which $\varphi_{h_n} * f(x_n) \geqslant \tau$ $\forall n = 1, 2, \ldots$

5. The sharp maximal inequality ($A = 1$).

THEOREM 2. *If φ satisfies conditions* (5) *and $f \in \mathfrak{M}^+$, then the maximal function* (3) *satisfies the following inequality*:

$$(M_\varphi f)^*(\xi) \leqslant \int_0^1 \varphi^*(t) f^*(t\xi)\,dt \qquad (\xi > 0).$$

Proof. The proof is essentially that of Theorem 1 of our earlier paper [4], and only the significant features will be indicated here. We first consider a bounded f with compact support and restrict attention to $h \leqslant \delta$ for some $\delta > 0$. The associated sets $E_\tau^{(\delta)}$ are open and bounded for each $\tau > 0$, and hence, by standard techniques utilizing the covering argument of the preceding section, admit for each $N = 1, 2, \ldots$, approximation in measure within $1/N$ from within by a finite sequence of disjoint half-open intervals $I_n(N)$ of lengths $h_n(N)$ satisfying the same inequality as above. Moreover, we can arrange that as $N \nearrow +\infty$, $H_N \overset{\text{def}}{=} \sum_n h_n(N) \nearrow |E_\tau^{(\delta)}|$. Introducing

$$\varphi_N(y) = \sum_{n \leqslant N} \varphi\big(h_n^{-1}(x_n - y)\big)$$

and observing that for $\xi > 0$, $\varphi_N^*(\xi) = \varphi^*(\xi/H_N)$, we obtain exactly as in [4], the estimates

$$H_N \leqslant \tau^{-1} \int \varphi_N(y) f(y)\,dy \leqslant \tau^{-1} \int_0^\infty \varphi_N^*(t) f^*(t)\,dt;$$

$$\tau \leqslant \int_0^1 \varphi^*(t) f^*(t|E_\tau^{(\delta)}|)\,dt;$$

$$(M_\varphi^{(\delta)} f)^*(\xi) \leqslant \int_0^1 \varphi^*(t) f^*(t\xi)\,dt \qquad (\xi > 0).$$

The restrictions on h and f are removed by standard approximation and monotonicity arguments.

Remark. If $f \in \mathfrak{M}^+(J)$, and vanishes elsewhere, it was shown in [4] that

$$\int_0^1 \varphi^*(t) f^*(t\xi)\,dt = M_\varphi f^*(\xi), \qquad \xi \in J.$$

Inasmuch as the maximal inequality (4) is also valid in $\boldsymbol{R}^n$ for an appropriate constant A ([4], Theorem 1), it would be desirable to find the best constants or more appropriate multidimensional analogues for the results of this paper.

References

[1] W. B. Jurkat and A. Peyerimhoff, *Mittlewertsätze bei Matrix- und Integral-Transformationen*, Math. Z. 55 (1951), 92–108.
[2] —, — *Mittlewertsätze und Vergleichsätze für Matrixtransformationen*, ibid. 56 (1952), 152–178.
[3] —, — *Inclusion theorems and order summability*, J. Approx. Theory 4 (1971), 245–262.
[4] W. B. Jurkat and J. L. Troutman, *Maximal inequalities related to generalized a. e. continuity*, Trans. Amer. Math. Soc. 252 (1979), 49–64.
[5] F. Riesz, *Sur un théorème de maximum de M. M. Hardy et Littlewood*, J. London Math. Soc. 7 (1932), 10–13.

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