Constructions of singular measures with remarkable convolution properties

by

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Abstract. We prove for every I-group there is a probability measure $\mu$ with $\mu^2$ singular and $\mu \cdot \bar{\mu}$ absolutely continuous. We also prove for all nondiscrete LCA groups the existence of a nontrivial singular measure $v$ such that $\hat{v}$ is the Fourier transform of an absolutely continuous measure. Our results extend some work of Connelly and Williamson who obtained the above results for the real line.

Let $G$ be a nondiscrete LCA group with dual $\Gamma$ and let $M(G)$ be the convolution measure algebra of $G$ (cf. [6]). Given a measure $\mu$ in $M(G)$, we denote by $\hat{\mu}$ the Fourier transform of $\mu$ and by $\bar{\mu}$ the measure defined by the requirement that $\hat{\mu}(B) = \mu(-B)$ for all Borel sets $B$ in $G$.

For the real line $\mathbb{R}$, D.M. Connelly and J.H. Williamson [1] construct a probability measure $\mu$ with the remarkable property that $\mu^2 = \mu \mu$ is singular and $\mu \cdot \bar{\mu}$ is absolutely continuous (with respect to Haar measure). (In this connection, see also [5] and [9].) As an immediate consequence of this result, they prove the existence of a non zero singular measure $v$ such that $\hat{v}$ is the Fourier transform of an absolutely continuous measure. In the present paper we shall prove that their first result holds for all $I$-groups, and that their second result holds for all nondiscrete groups. Recall that $G$ is an $I$-group if every nonempty open set in $G$ contains an element of infinite order.

THEOREM 1. If $G$ is an $I$-group, then there exists a probability measure $\mu$ in $M(G)$ such that $\mu^2$ is singular and $\mu \cdot \bar{\mu}$ is absolutely continuous.

THEOREM 2. Suppose that $G$ is a nondiscrete LCA group. Then there exists a non zero singular measure $v$ in $M(G)$ such that $\hat{v}$ is the Fourier transform of an absolutely continuous measure.

In order to prove the above results, we need several lemmas. They will be stated in a generality greater than actually needed in the present paper. This is because we hope that they may be useful in some future study of the subject. Now let us introduce some notation. Let $\lambda = \lambda_G$ denote the Haar measure of the nondiscrete LCA group $G$. When $G$ is compact, we shall always normalize $\lambda$ so that $\lambda(G) = 1$. Define $\mathcal{M}_\lambda = \mathcal{M}_\lambda(G)$
and $M_r = M_r(\theta)$ to be the set of all absolutely continuous measures and the set of all singular measures in $M(\theta)$, respectively. We shall often identify $L^1(G)$ with $M_r(\theta)$ through the map $f \mapsto f_\theta$, and define $\|f + \mu\| = \|f_\theta + \mu\|$ for $f$ in $L^1(G)$ and $\mu$ in $M(\theta)$. Let $T = R(\mod 2\pi)$ denote the circle group, and $Z$ the group of integers identified with the dual of $T$. For a natural number $\tau \geq 2$, let $T(\tau)$ denote the cyclic subgroup of $T$ of order $\tau$. Thus the dual of $T(\tau)$ (isomorphic to) $Z(\tau) = R(\mod 2\pi)$. It is easy to see that the results of [1] continue to hold for $T(\tau)$.

Though our first lemma seems to be well-known, we do not know any appropriate references, and so, we give a complete proof.

**Lemma 1.** Let $H$ be a closed subgroup of $G$ whose annihilator $A$ is a discrete subgroup of $\Gamma$. Let $\tau$ be a probability measure in $M(\theta)$ such that $A \cap S = \emptyset$ and $A \cap (S - S)$ is a finite set, where $S = \{\gamma \in \Gamma : \hat{\tau}(\gamma) \neq 0\}$. Then there exists a unique linear map $J$ from $M(G/H)$ into $M(\theta)$ such that

$$
(J \mu)(\gamma) = \sum_{\alpha \in A} \hat{\mu}(\alpha) \hat{\tau}(\gamma - \alpha) \quad (\mu \in M(G/H), \gamma \in (A + S) \cup T),
$$

where $T$ denotes the interior of $\Gamma \setminus (A + S)$. Moreover, $J$ is an isometry, and preserves the positive and singularity of measures. If, in addition, $\tau$ is in $\mathcal{M}_r(\theta)$, then $J$ also preserves the absolute continuity of measures.

Proof. The uniqueness of $J$ is obvious since $(A + S) \cup T$ is dense in $\Gamma$. Notice also that the right-hand side of (1) is actually a finite sum for each $\mu$ in $M(G)$ and each $\gamma$ in $\Gamma$, since $A \cap S = \emptyset$ and $\hat{\tau}(0) = 1$. Now let $\lambda_0$ denote the Haar measure of the compact quotient group $G/H$ of norm one, and let $\pi : \Omega \to G/H$ denote the quotient map of $G$ onto $G/H$. We normalize the Haar measures $\lambda_0$ and $\hat{\lambda}_0$ so that

$$
\int_\Omega f \, d\lambda_0 = \int_H \int f(d\lambda_0(\overline{x})) f(\xi + \alpha) \, d\lambda_0(\alpha) = \|f\|
$$

holds for all $f$ in $L^1(G)$ (cf. 2.7.3 of [6]).

We first deal with the case where $\tau$ is in $M_r(\theta)$. Let $w \in L^1_+(\theta)$ denote the Radon–Nikodym derivative of $\tau$ with respect to $\lambda_0$. Then we have

$$
\int_\Omega \hat{w}(\xi + \alpha) \, d\lambda_0(\alpha) = 1 \quad (\lambda_0 \text{-a.s., } \xi \in G/H),
$$

since $\hat{w}(0) = \hat{\tau}(0) = 1$ and $\hat{\tau}(\alpha) = 1$ whenever $\gamma \in \Gamma$ and $\hat{\tau}(\gamma) \neq 0$. Given $f$ in $L^1(G/H)$, let us now define $(Jf)(\xi) = f(\overline{x}) w(\alpha)$. Then (1) and (2) yield

$$
\|Jf\| = \int_{G/H} \|f(\overline{x}) w(\alpha)\| \, d\lambda_0(\alpha) = \int_{G/H} \|f(\overline{x})\| \, d\lambda_0(\alpha) = \|f\|.
$$

Moreover, we claim that

$$
(Jf)(\gamma) = \sum_{\alpha \in A} \hat{\mu}(\alpha) \hat{\tau}(\gamma - \alpha) \quad (\gamma \in \Gamma),
$$

In fact, this is obvious when $f$ is a trigonometric polynomial. For a general $f$ in $L^1(G/H)$, take a sequence $(f_\alpha)$ of trigonometric polynomials on $G/H$ such that $\|f_\alpha - f\| \to 0$. Then we have $\|Jf_\alpha - f\| \to 0$ and

$$
\|J(f_\alpha - f)\|_{\infty} \leq \|J(f_\alpha - f)\| = 0.
$$

Since for each fixed $\gamma \in \Gamma$ the index $\alpha$ in the right-hand side of (3) essentially ranges over a finite set independent of $f$, it follows that $(Jf_\alpha)$ converges to the sum in the right-hand side of (3). This establishes our claim. By a weak* argument, it is now easy to show that $J$ extends to a norm-decreasing linear map from $M(G/H)$ into $M(\theta)$ for which (i) holds for all $\mu$ in $M(G/H)$ and all $\gamma$ in $\Gamma$. (Notice that $\tau$ vanishes at infinity; hence the sum in the right-hand side of (1) defines a continuous function on $\Gamma$ whenever $\mu$ is a bounded function on $\Omega$.)

We now remove the assumption that $\tau$ is in $M_r(\theta)$. Take a net $(\tau_\alpha)$ of probability measures in $M_r(\theta)$ such that $\|\tau_\alpha - \tau_\beta\| \to 0$ on $\Gamma$ for all $\alpha$ and such that $\tau_\alpha$ converges $\tau_\beta$ uniformly on compact subsets of $\Gamma$. For each $\alpha$, let $J_\alpha$ denote the norm-decreasing linear map from $M(G/H)$ into $M(\theta)$ induced by $\tau_\alpha$ as in the preceding paragraph. Then one can easily check that given $\mu$ in $M(G/H)$, the net $(J_\alpha \mu)$ converges uniformly on compact subsets of $A + S \cup \Gamma$. Since $\|J_\alpha \mu\| \leq \|\mu\|$ for all $\alpha$ and since $(A + S) \cup \Gamma$ is a dense open subset of $\Gamma$, it follows at once that the net $(J_\alpha \mu)$ converges weak* to a measure $\mu_\alpha$ in $M(G)$ of norm $\leq \|\mu\|$, and that (i) holds. Since $(J_\alpha \mu) = \mu$ on $A + S$ and $\|\mu_\alpha\| \leq \|\mu\|$, $J$ is an isometry and preserves the positivity of measures. It only remains to show that $J$ maps $M_r(G/H)$ into $M_r(\theta)$. Suppose by way of contradiction that $\mu \notin M_r(\theta)$ for some $\mu \in M_r(G/H)$. Then there exists a measure $\nu$ in $M_r(\theta)$ such that $\|\nu - \tau_\alpha\| \leq \|\nu - \mu\|$ and $\sup\nu$ is compact. Let $P$ be the trigonometric polynomial on $G/H$ with $P = \nu$ on $\Gamma$. Then $\|\mu - \nu\| \leq \|\mu - \nu\|$. Since $\mu$ is in $M_r(G/H)$, this yields the desired contradiction. The proof is complete.

**Lemma 2.** Let $H \subset G$ and $A = \Gamma$ be as in Lemma 1 and let $\sigma \in M(G)$. Suppose that there are two non-negative measures $\tau_1$ and $\tau_2$ in $M(G)$, as in Lemma 1, such that $\hat{\tau}_1(\gamma) - \hat{\tau}_2(\gamma) = 1$ whenever $\gamma \in \Gamma$ and $\hat{\tau}_2(\gamma) \neq 0$. Then there exists a unique linear map $K$ from $M(G/H)$ into $M(G)$ such that

$$
(K \mu)(\gamma) = \sum_{\alpha \in A} \hat{\mu}(\alpha) \hat{\tau}_1(\gamma - \alpha) \quad (\gamma \in (A + S) \cup \Gamma),
$$

where $S = \{\gamma \in \Gamma : \hat{\tau}_1(\gamma) \neq 0\}$ and $\Gamma$ is the interior of $\Gamma \setminus (A + S)$. Moreover, we have

$$
\|K\| \leq 2 \|\tau_1\| + \|\tau_2\|.
$$

Since $\mu$ is in $M_r(G/H)$, this yields the desired contradiction. The proof is complete.
Proof. Notice that the condition $|v_j| = 1$ is not assumed. Applying the Hahn decomposition theorem to $Re \varphi$ and $Im \varphi$, we can write $\varphi = \varphi_1 + \varphi_2 - \varphi_3$, where $\varphi_2$ and $\varphi_3$ are nonnegative measures in $M(G)$ with $\|\varphi_2\| + \cdots + \|\varphi_3\| < 2|\varphi|$. Let $J_{j,k}$ be the map induced by $T_x \rho_k$ as in Lemma 1, so that $\|J_{j,k}\| = \|\varphi\|\|\rho_k\|$ for all $j = 1, 2$ and $k = 1, 2, 3, 4$. Then the required map $K$ may be defined by

$$K = \sum_{j=1}^2 \sum_{k=1}^4 (-1)^{j+k} J_{j,k}.$$ 

It is easy to confirm (i) and (ii), and the proof is complete.

**Lemma 3.** Suppose that $n$ is a natural number and that there exists a probability measure $\varphi$ in $M(G)$ such that $\varphi^n \in M_\mu$ and $\rho_j \varphi \in M_\mu$. Let $f \in L^2(G)$, $\|f\| = 1$, and $\varepsilon > 0$. Then there is a probability measure $\mu$ in $M(G)$ with $\mu^n = M_\mu$, such that

$$\mu \in M_\mu, \quad \mu = \mu \in M_\mu,$$ 

and $|\|f^n \rho_j - \varphi^n\|\| \leq \varepsilon$.

**Proof.** Notice that $\tilde{g}$ is defined by $\tilde{g}(a) = \tilde{g}(a^{-1})$ for all $a \in G$. Choose a neighborhood $V$ of 0 in $G$ so that

$$(1) \quad |\|f^n \rho_j - \varphi^n\|\| \leq \varepsilon/2.$$

holds for every probability measure $\varphi$ in $M(G)$ concentrated on $V$. Let $W$ be any symmetric neighborhood of 0 such that $W + W = V$. By considering an appropriate translate of $\varphi$, we may assume that $\varphi(W) > 0$. Therefore, replacing $\varphi$ by $\varphi(W)^{-1}\varphi$, we may also assume that $\varphi$ is concentrated on $W$. Then (1) implies

$$(2) \quad |\|f^n \rho_j - \varphi^n\|\| \leq \varepsilon/2.$$ 

We now apply Theorem 2.4, its proof and Lemma 3.1 of [8] to find a probability measure $\varphi$ in $M(G)$ such that $\rho_j \varphi \in M_\mu$, $\varphi^n \varphi \in M_\mu$, and

$$(3) \quad |\|\rho_j \varphi^n \varphi - \varphi^n\|\| \leq \varepsilon/2.$$ 

(Notice that $\varphi^n \varphi$ is in $M_\mu$ by hypothesis.) Setting $\mu = \varphi^n \varphi$, we see that $\mu$ has all the required properties.

**Lemma 4.** Let $n$ be a natural number such that $\varphi^n \varphi \in M_\mu$ and $\rho_j \varphi \in M_\mu$ for some probability measure $\varphi$ in $M(G)$. Then, to each $\varepsilon > 0$, there corresponds a natural number $r$, with the following property:

For every natural number $r > r_1$, there exist two trigonometric polynomials $P$ and $Q$ on the cyclic group $T(2r+1)$ such that

(a) $P \geq 0$, $\hat{P}(0) = 1$, and $\|1 - P \varphi\| < \varepsilon$;

(b) $\hat{Q}(m) = \hat{Q}(m) = 0$ for all $m = \pm r_1, \pm (r_1 + 1), \ldots, \pm r \in Z(2r+1)$;

(c) $|Q|_\infty \leq 1$ and $\hat{Q}(0) = 0$;

(d) $\int_{T(2r+1)} Q \rho_j \varphi^\alpha d\alpha > 1 - \varepsilon$,

where $T(\alpha) = P \varphi \cdots P \varphi$ ($n$ times).

**Proof.** Let $\varepsilon > 0$ be given. By Lemma 3, we may assume that the measure $\varphi$ satisfies

$$(1) \quad |\|\varphi^n \varphi\| < \varepsilon/6.$$ 

Since $\varphi^n$ is a singular probability measure, there exists a trigonometric polynomial $Q_j$ on $T$ such that

$$(2) \quad |\|Q_j \varphi^n\| > 1 - \varepsilon,$$ 

$$(3) \quad |\|Q_j|_\infty \leq 1 \quad \text{and} \quad \hat{Q}_j(0) = 0.$$ 

To find such a $Q_j$, it suffices to apply Doss' result in [2]. Convolving $\varphi$ with an appropriate trigonometric polynomial, we then obtain a trigonometric polynomial $P_j$ on $T$ such that

$$(4) \quad P_j \geq 0, \quad \hat{P}_j(0) = 1, \quad \text{and} \quad |\|1 - P_j \varphi\| < \varepsilon/6,$$ 

$$(5) \quad \int_{T} Q_j \rho_j \varphi^\alpha d\alpha > 1 - \varepsilon.$$ 

We define $r_1$ to be any even integer such that

$$(6) \quad \hat{Q}_j(k) = \hat{Q}_j(k) = 0 \quad (k \in Z; |k| > r_1/2).$$ 

Now let $r$ be a given natural number $\geq r_1$. We apply Lemma 1 with $A = (2r+1)\varphi$ to find a probability measure $\varphi$ in $M(T)$ such that

$$(7) \quad \hat{\varphi}(2r+1) m + k = \hat{P}_j(k) \quad (m, k \in Z; |k| < r).$$ 

Notice that (7) implies that $\varphi$ is concentrated on the subgroup $T(2r+1)$ of $T$. In fact, we have $\varphi = P_0$, where $P$ is the restriction of $P_j$ to $T(2r+1)$. Then $\varphi$ is the Haar measure of $T(2r+1)$ regarded as an element of $M(T)$. It is obvious that $P \geq 0$ and $\hat{P}(0) = 1$. To confirm that $|\|1 - P \varphi\| < \varepsilon$, let $R_m$ be the Fejér kernel on $T$ ($m = 0, 1, \ldots$). Setting $\tau_r = \tau_r(\rho_j)$ and $\tau_2 = 2\tau_r$, we see that $\tau_r(k) = \tau_r(k) = 1$ for all $k = 0, \pm 1, \ldots, \pm r_1/2$. It follows from (4), (6), (7) and Lemma 2 that

$$(8) \quad |\|1 - P \varphi\| < 2|\|1 - P_0 \varphi\| (\tau_r + |\tau_2|) < \varepsilon.$$ 

This establishes part (a). Now define $Q$ to be the restriction of $Q_j$ to $T(2r+1)$. 

Then (b) and (c) follow from (3), (6) and (7). Finally we have

\[ \int_{D^{(n)}} QP_{0}(d\lambda) = \int_{D^{(n)}} Qd\mu^n = \sum_{k=0}^{\infty} \hat{Q}(k) k^n(k) \]

\[ = \sum_{k=0}^{\infty} \hat{Q}(k) P_{k}^{(n)}(k) = \int_{D} QP_{0}(d\lambda) d\mu > 1 - \varepsilon \]

by (5), (6) and (7). This establishes part (d) and the proof is complete.

**Lemma 5.** Let \( G \) be a compact abelian group of unbounded order, and let \( n \) be a natural number, as in Lemma 4. Then there exists a probability measure \( \mu \) in \( M(G) \) such that \( \mu^n \in M_n \) and \( \mu^n \in M_n \).

**Proof.** By induction, we shall construct two sequences \((P_m)\) and \((Q_m)\) of trigonometric polynomials on \( G \) as follows. Set \( P_0 = 1 \) and \( Q_0 = 0 \), and assume that \( P_0, Q_1, \ldots, P_{m-1}, Q_{m-1} \) have been defined for some natural number \( m \). Let \( S_{m-1} = \{ g \in G : \hat{P}(g) \neq 0 \} \), where \( R_{m-1} = P_{m-1}, \ldots, P_{m-1} \), and \( \gamma_m \) be any sufficiently large natural number. Since \( G \) is of unbounded order and since \( S_{m-1} \) is a finite set, there exists an element \( \theta_m \) of \( G \) such that

\[ \{ f \theta_m : \theta = \pm 1, \pm 2, \ldots, \pm \gamma_m \} \cap (S_{m-1} - S_{m-1}) = \emptyset. \]

By Lemma 4 (and its proof), there exist two trigonometric polynomials \( P_m, Q_m \) on \( G \) subject to the following conditions:

1. \( P_m \geq 0, \hat{P}_m(0) = 1 \), and \( \|P_m - P_m\| < 2^{-m} \|R_{m-1}\| \);
2. \( \supp(\hat{P}_m) \in \{ \theta_m : \theta = \pm 1, \pm 2, \ldots, \pm \gamma_m \} \); and \( \hat{Q}_m(0) = 0 \);
3. \( \|Q_m\| \leq 1 \); and \( \hat{Q}_m(0) = 0 \);
4. \( \int_{D} Q_m P_{0}(d\lambda) d\mu > 1 - 2^{-m} \).

This completes the induction. Notice that every \( R_{m} \) is non-negative by (2), that (1), (2) and (3) imply

\[ \hat{R}_{m}(g_1 + \ldots + g_m) = \hat{P}_m(g_1) \ldots \hat{P}_m(g_m) \quad (m \leq N) \]

whenever \( g_1 \in \supp \hat{P}_m, \ldots, g_m \in \supp \hat{P}_m \), and that

\[ \hat{R}_{m} = 0 \quad \text{on} \quad D - \{ \supp \hat{P}_m + \ldots + \supp \hat{R}_m \}. \]

It follows that the sequence \( (R_m) \) converges weak* to a probability measure \( \hat{\mu} \) in \( M(G) \) such that

\[ \hat{\mu}(g_1 + \ldots + g_m) = \hat{P}_m(g_1) \ldots \hat{P}_m(g_m) \]

for all \( g_1 \in \supp \hat{P}_m, \ldots, g_m \in \supp \hat{P}_m \) and such that

\[ \hat{\mu} = 0 \quad \text{on} \quad D - \{ \supp \hat{P}_m + \ldots + \supp \hat{P}_m \}. \]

Thus \( \mu \) is a kind of Bessel product (cf. [4]).

In order to prove that \( \mu^n \) is in \( M_n \), take any finite set \( K \) in \( D \). Then \( \{ f \theta_m : \theta = \pm 1, \ldots, \pm \gamma_m \} \) is disjoint from \( K \) whenever \( m \) is large enough. Moreover, (3), (6) and (5) yield

\[ \int_{D} Q_m d\mu^n \leq \int_{D} Q_m d\mu^n = \sum_{\gamma_m} Q_m(-\gamma) \hat{\mu}^n(\gamma) = \int_{D} Q_m d\mu^n d\mu > 1 - 2^{-m} \]

for all \( m \). It follows from (3), (4) and Doss' theorem [2] that \( \mu^n \) is a singular measure.

To confirm that \( \mu^n \mu^n \) is in \( M_n \), notice that

\[ R_{m} \hat{R}_{m} = P_{m} \hat{P}_{m} \ldots P_{m} \hat{P}_{m} \quad (m \geq 0) \]

by (1) and (3). It follows from (3) that

\[ \|R_{m} \hat{R}_{m} - R_{m} \hat{R}_{m} - R_{m} \hat{R}_{m} \| \leq \|R_{m} \hat{R}_{m} - R_{m} \hat{R}_{m} \| < 2^{-m} (m \geq 0). \]

Therefore the sequence \( (R_{m} \hat{R}_{m}) \) converges in norm to some \( f \in L^1(D) \). Since

\[ f(\gamma) = \lim_{\gamma \to \infty} \hat{R}_{m}(\gamma) \| \hat{\mu}(\gamma) \|^2 \quad (\gamma \in D), \]

we conclude that \( \mu^n \mu^n \) is absolutely continuous. This completes the proof.

**Proofs of Theorems 1 and 2.** Let \( G \) be a nondiscrete LCA group.

To prove the required results, we may replace \( G \) by any open subgroup of \( G \). Thus, by the well-known structure theorem [6], we may also assume that \( G \) has the form \( \mathbb{R}^n \times K \) for some integer \( n \geq 0 \) and some compact abelian group \( K \). If \( n > 1 \), then our results follow easily from the results in [1]. So assume that \( G \) is compact. If \( G \) is also an I-group, then \( G \) must be of unbounded order. In the last case, Lemma 5 with \( a = 2 \) yields both Theorems 1 and 2. Consequently it will suffice to prove that the conclusion of Theorem 2 holds for every infinite compact abelian group of bounded order. Let \( G \) be such a group. Then \( G \) has the form \( \prod_{\alpha} G_{\alpha} \), where every \( G_{\alpha} \) is an infinite compact abelian group (see (A.25) of [3]).

By Theorem 2.4 of [8] and its proof, there exist two symmetric probability measures \( \mu_{\alpha}, \nu_{\alpha} \) in \( M(G_{\alpha}) \) such that \( \mu_{\alpha} \in M_{\alpha}, \nu_{\alpha} \in M_{\alpha}, \mu_{\alpha} - \nu_{\alpha} \in M_{\alpha}, \)

\[ \|1 - \mu_{\alpha}^\alpha - 1 - \nu_{\alpha}^\alpha\| < 2^{-n}. \]

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Setting \( \lambda_n = (1 + i)^{-1}(\mu_n + \phi_n) \), we then have

\[
\lambda_n^2 = \mu_n \ast \phi_n + (2 + i)(\mu_n^* - v_n^*),
\]

(2)

\[
\lambda_n \ast \lambda_n = 2^{-1}(\mu_n^* + v_n^*).
\]

(3)

Now define \( \phi_n \in \mathcal{M}(G) \) by setting

\[
\phi_n = \lambda_1 \times \ldots \times \lambda_n \times \prod_{k=1}^{n-1} (\mu_k \ast \phi_k) \quad (n = 1, 2, \ldots).
\]

(4)

It follows from (1) and (2) that the sequence \( \{\phi_n\} \) converges in norm to a complex measure \( \phi \in \mathcal{M}(G) \). Since every \( \phi_n \) is a singular measure, so is \( \phi \). Moreover, (3) and (4) imply that

\[
|\phi| = \left[ \prod_{n=1}^{\infty} 2^{-1}(\mu_n^* + v_n^*) \right]^{-1}.
\]

It follows at once that \( |\phi| \) is in \( \mathcal{M}(G) \). This completes the proof.

Remarks. (a) Applying the method in Section 6 of [3], we can generalize Theorem 1 as follows. Let \( G \) be an I-group and let \( r \) be a natural number. Then there exist \( r \) probability measures \( \mu_1, \ldots, \mu_r \) in \( \mathcal{M}(G) \) such that

\[
\mu_1 \ast \mu_2 \ast \ldots \ast \mu_r \in \mathcal{M}_r(G),
\]

and such that

\[
\mu_1^* \ast \mu_2^* \ast \ldots \ast \mu_r^* \in \mathcal{M}_r(G),
\]

for every \( r \)-tuple \( (n_1, \ldots, n_r) \) of non-negative integers with \( \min\{n_j : 1 \leq j \leq r \} \leq 2 \).

(b) Suppose that \( G \) contains an open subgroup isomorphic to \( \mathbb{Z}(2)^n \) for some infinite cardinal \( \alpha \). Then it is easy to show that there exists no probability measure in \( \mathcal{M}(G) \) as in the conclusion of Theorem 1.

(c) Let \( \Sigma_\alpha \) denote the set of all symmetric complex homomorphisms of \( \mathcal{M}(G) \). Then \( \mu \in \mathcal{M}(G) \) with \( \mu \ast \mu \in \mathcal{M}_2 \) has Gelfand transform vanishing on \( \Sigma_\alpha \). Moreover, it is not difficult to prove that if \( G \) is a group as in (b) and if \( \mu \in \mathcal{M}(G) \) has Gelfand transform vanishing on \( \Sigma_\alpha \), then \( |\mu^*| \ast |\mu^*| = O(1) \) as \( n \to \infty \) (cf. Problem (ii) on p. 233 of [7]). Consequently, for any group \( G \) as in (b), there exists no nonzero measure \( \mu \in \mathcal{M}(G) \) such that \( \mu \ast \mu \in \mathcal{M}_2 \) and \( \mu^* \in \mathcal{M}_1 \) for all \( n \geq 1 \).

(d) Does there exist a sequence \( \{P_n\} \) of non-negative trigonometric polynomials on \( T \) such that \( P_n(0) = 1 \) for all \( n \),

\[
\sum_{n=1}^{\infty} |P_n \ast \hat{P}_n - 1| < \infty \quad \text{and} \quad \limsup_{n \to \infty} |P_n(0)^2 - 1|^{1/n} > 0
\]

If such a sequence exists, we can prove that every I-group has a prob-