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The structure of polynomial ideals in the algebra of entire functions

by

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Abstract. We give a linear decomposition of polynomial ideals in differe algebras of entire functions of several complex variables. In particular, for any algebraic variety $V \subset \boldsymbol{C}^n$ the space $H(V)$ of all holomorphic functions on V has a linear extension operator $E: H(V) \to H(\boldsymbol{C}^n)$ to the space of all entire functions, i.e. $(Ef)|V = f$, $\forall f \in H(V)$, and its ideal $J(V) = \{f \in H(\boldsymbol{C}^n): f|V = 0\}$ with polynomial generators $\{Q_i\}_1^m$ has such a system of linear operators $\{R_i\}$ that $g = \sum_{i=1}^{m} R_i g \cdot Q_i$ for any $g \in J(V)$.

0. Introduction. Let $Q = \{Q_i(z)\}_1^p$ be a finite set of polynomials of n complex variables $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$. The polynomials $Q_1, \ldots, Q_p$ generate the ideal

$$J_Q = J(Q) = \Big\{f = \sum_{i=1}^{p} g_i Q_i\colon\ g_i \in H(\boldsymbol{C}^n),\ 1 \leqslant i \leqslant p\Big\} \tag{0.1}$$

in the algebra of all entire functions $H(\boldsymbol{C}^n)$; the ideals of this form will be called *polynomial ideals*. The $H(\boldsymbol{C}^n)$ will be regarded with the topology of uniform convergence on compact subsets of $\boldsymbol{C}^n$, it is then a nuclear Fréchet space. There are several linear problems (cf. [4]) of the complex analysis connected with the ideal J_Q: Is the ideal J_Q complemented as a linear (closed) subspace in $H(\boldsymbol{C}^n)$? Is it possible to determine the linear topological type of J_Q and the quotient space $H(\boldsymbol{C}^n)/J_Q$? Do there exist continuous linear operators $L_i\colon J_Q \to H(\boldsymbol{C}^n)$, such that $f = \sum_{i=1}^{p} L_i(f) Q_i$ for all $f \in J_Q$?

Similar problems have been solved (positively) in the paper of B. S. Mitiagin and G. M. Henkin [4] in the case of the space (algebra) $H(G)$ of all holomorphic functions on a strictly pseudoconvex domain $G \subset \boldsymbol{C}^n$ and an ideal J with finite number of "good enough" generators, and in the paper of W. Rudin and E. L. Stout [7] in the case of the space $H(D^n)$ of all holomorphic functions in the polydisc D^n. Recently V. P. Zahariuta [9] proved that the space $H(V)$ of all holomorphic functions on an algebraic variety $V \subset C^n$ is isomorphic to the space of entire functions $H(C^k)$, where $k = \dim_C V$: this solved positively the question 6.5 of [4].

In this paper, by an elementary examining the Taylor expansions, we obtain more general results on the structure of the polynomial ideals J_Q. The main theorem(1) (see Theorem 2) is formally similar to Theorem 1, II. 4, in the book of V. P. Palamodov [5]. We also consider special subsets of the set of all multi-indices Z_+^n (cf. [5]; II.3.1); but in our case of the space of entire functions the method developped in [5] does not work.

More generally, it is possible to investigate the structure of polynomial modules over the algebra $H(C^m)$, in particular to show that the free resolution of such a module splits (cf. Section 6). We investigate also the structure of ideals (modules) over algebras of entire functions with some bounds on the growth.

1. As usual, $Z_+ = \{0, 1, 2, \ldots\}$, $Z_+^n = \{\alpha = (\alpha_1, \ldots, \alpha_n)\colon \alpha_i \in Z_+, 1 \leqslant i \leqslant n\}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, where $z = (z_1, \ldots, z_n)$. We introduce an ordering in Z_+^n in the following way:

(1.1) $\alpha' < \alpha''$

$\Leftrightarrow |\alpha'| < |\alpha''|$ or $|\alpha'| = |\alpha''|$, but there exists k, $2 \leqslant k \leqslant n$, such that $\alpha'_k < \alpha''_k$ and $\alpha'_j = \alpha''_j$, $j = k+1, \ldots, n$.

Let I be an ideal in the algebra $C[z_1, \ldots, z_n]$ of all polynomials of n variables $z_1, \ldots, z_n$. Put

$$T = T(I) = \{\alpha \in Z_+^n\colon z^\alpha \in [z^\beta, \beta < \alpha] + I\}, \tag{1.2}$$

where the brackets $[\ldots]$ denote the linear hull of corresponding vectors. Evidently, the set T is monotonous in the sense that $T + Z_+^n \subset T$. This is a property of the ordering $<$: if $\alpha' < \alpha''$, then $\alpha' + \beta < \alpha'' + \beta$ for any $\beta \in Z_+^n$.

Put $S = Z_+^n \setminus T$; then every finite subset of the system $\{z^\sigma\colon \sigma \in S\}$ is linearly independent modulo I, and, for any $\tau \in T$, we have the unique representation

$$z^\tau = \sum_{\sigma < \tau, \sigma \in S} c_\sigma^\tau z^\sigma \mod I. \tag{1.3}$$

In the lattice Z_+^n we have the usual ordering: $\alpha \leqslant \beta \Leftrightarrow \alpha_i \leqslant \beta_i$, $1 \leqslant i \leqslant n$. It is easy to prove (cf. [5], Proposition 1, II.2) the following elementary

LEMMA 1. *Every monotonous subset of Z_+^n has finite number of minimal elements with respect to the ordering $\leqslant$.*

Let $\Gamma = \Gamma(T)$ be the set of all minimal elements in T. Then $T = \Gamma + Z_+^n$, i.e. for any $\tau \in T$ there exists $\gamma \in \Gamma$ and $\delta \in Z_+^n$, such that

$$\tau = \gamma + \delta. \tag{1.4}$$

(1) This theorem was announced in [11].

This representation is not unique, but we fix for any $\tau \in T$ one such representation. By (1.3) we have, for any $\gamma \in \Gamma$,

$$z^\gamma = \sum_{\sigma < \gamma, \sigma \in S} c_\sigma^\gamma z^\sigma + P_\gamma, \quad P_\gamma \in I. \tag{1.5}$$

It is easy to see that the system of polynomials $P = \{P_\gamma, \gamma \in \Gamma\}$ generates the original ideal I. We connect with this system also the ideal $J(P)$ of the type (0.1) generated by P in the algebra of all entire functions $H(C^n)$. The system $\{z^\alpha, \alpha \in Z_+^n\}$ is an absolute basis in $H(C^n)$. If $S \subset Z_+^n$, we denote by $H_S(C^n)$ the closed linear hull of the vectors $\{z^\sigma, \sigma \in S\}$, i.e.

$$H_S(C^n) = \Big\{\varphi \in H(C^n)\colon \varphi = \sum_{\alpha \in Z_+^n} \varphi_\alpha z^\alpha,\ \varphi_\alpha = 0 \text{ if } \alpha \notin S\Big\}. \tag{1.6}$$

Now we state our main result:

THEOREM 2. *Let $T = T(I)$, $S = Z_+^n \setminus T$; then there exist continuous linear operators $R_0, R_\gamma\colon H(C^n) \to H(C^n)$, $\gamma \in \Gamma$, such that*

(a) $f = R_0(f) + \sum_{\gamma \in \Gamma} R_\gamma(f) P_\gamma$ *for all* $f \in H(C^n)$;

(b) $\operatorname{Im} R_0 = H_S(C^n)$, $\ker R_0 = J(P)$, $R_0^2 = R_0$.

Proof. It is possible to choose constants $A_1, \ldots, A_n$ such that $2 < A_1 < \ldots < A_n$ and

$$\sum_{\sigma < \gamma, \sigma \in S} |c_\sigma^\gamma| A_1^{\sigma_2 - \gamma_2} \ldots A_{n-1}^{\sigma_n - \gamma_n} \cdot A_n^{|\sigma| - |\gamma|} < \tfrac{1}{2} \quad \text{for all } \gamma \in \Gamma; \tag{1.7}$$

here $\{c_\sigma^\gamma\}$ are the coefficients of representation (1.5). Indeed, since Γ is finite, we can choose $A_1 > 2$, such that

$$\sum_{\sigma \in M_1(\gamma)} |c_\sigma^\gamma| A_1^{\sigma_2 - \gamma_2} < \frac{1}{2n} \quad \text{for all } \gamma \in \Gamma, \tag{1.8}$$

where $M_1(\gamma) = \{\sigma \in S\colon |\sigma| = |\gamma|, \sigma_i = \gamma_i, 3 \leqslant i \leqslant n, \sigma_2 < \gamma_2\}$; next we choose $A_2 > A_1$, such that

$$\sum_{\sigma \in M_2(\gamma)} |c_\sigma^\gamma| A_1^{\sigma_2 - \gamma_2} A_2^{\sigma_3 - \gamma_3} < \frac{1}{2n} \quad \text{for all } \gamma \in \Gamma, \tag{1.9}$$

where $M_2(\gamma) = \{\sigma \in S\colon |\sigma| = |\gamma|, \sigma_i = \gamma_i, 4 \leqslant i \leqslant n, \sigma_3 < \gamma_3\}$. In an analogous way one can choose by induction constants $A_3, \ldots, A_{n-1}$; more exactly, let us put $\{\sigma \in S\colon \sigma < \gamma\} = \bigcup_{k=1}^n M_k(\gamma)$, where

$$M_k(\gamma) = \{\sigma \in S\colon |\sigma| = |\gamma|;\ \sigma_i = \gamma_i,\ k+2 \leqslant i \leqslant n;\ \sigma_{k+1} < \gamma_{k+1}\},$$
$$1 \leqslant k \leqslant n-1, \quad M_n(\gamma) = \{\sigma \in S\colon |\sigma| < |\gamma|\}.$$

If $A_1, \ldots, A_{k-1}$, $2 < A_1 < \ldots < A_{k-1}$, are chosen, we choose $A_k > A_{k-1}$ in such a way that

$$\sum_{\sigma \in M_k(\gamma)} |c_\sigma^\gamma| \cdot \Big(\prod_{i=1}^{k-1} A_i^{\sigma_{i+1} - \gamma_{i+1}} \Big) \cdot A_{k+1}^{\sigma_{k+1} - \gamma_{k+1}} < \frac{1}{2n} \quad \text{for all } \gamma \in \Gamma. \tag{1.10}$$

At the final step we choose $A_n > A_{n-1}$ in such a way that

$$\sum_{\sigma \in M_n(\gamma)} |c_\sigma^\gamma| \cdot \Big(\prod_{i=1}^{n-1} A_i^{\sigma_{i+1} - \gamma_{i+1}} \Big) A_n^{|\sigma| - |\gamma|} < \frac{1}{2n} \quad \text{for all } \gamma \in \Gamma. \tag{1.11}$$

The inequalities (1.7) follow immediately from (1.8)–(1.11).

Put $B_1 = A_n$, $B_i = A_{i-1} A_n$, $i = 2, \ldots, n$. Then by (1.7) the inequalities

$$\sum_{\sigma \in S, \sigma < \gamma} |c_\sigma^\gamma| B^{\sigma - \gamma} < \tfrac{1}{2} \quad \text{for} \quad \gamma \in \Gamma \tag{1.12}$$

hold, where $B = (B_1, \ldots, B_n)$, $B^{\sigma-\gamma} = B_1^{\sigma_1 - \gamma_1} \ldots B_n^{\sigma_n - \gamma_n}$.

Let $b = (b_1, \ldots, b_n)$, $b_i > 0$, $1 \leqslant i \leqslant n$. For any entire function $f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha z^\alpha$ we put $\|f\|_b = \sum_{\alpha \in \mathbf{Z}_+^n} |f_\alpha| b^\alpha$. Evidently, $|f_\alpha| \leqslant \|f\|_b \cdot b^{-\alpha}$ and $\|f \cdot g\|_b \leqslant \|f\|_b \cdot \|g\|_b$ for all $f, g \in H(\mathbf{C}^n)$. It is easy to see that the system of norms

$$\|f\|_{rB} = \sum_{\alpha \in \mathbf{Z}_+^n} |f_\alpha| (rB)^\alpha \quad \text{for} \quad f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha z^\alpha, \ r \geqslant 1, \tag{1.13}$$

generates the topology of $H(\mathbf{C}^n)$.

Now we show that, for any $\alpha \in \mathbf{Z}_+^n$, there exists a representation

$$z^\alpha = \psi^\alpha + \sum_{\gamma \in \Gamma} \varphi_\gamma^\alpha P_\gamma, \tag{1.14}$$

where ψ^α and φ_γ^α are polynomials, such that

(1) if $\alpha \in S$, then $\psi^\alpha(z) = z^\alpha$, $\varphi_\gamma^\alpha \equiv 0$;

(2) if $\alpha \in T$, then $\psi^\alpha \in H_s(\mathbf{C}^n)$, $\deg \psi^\alpha \leqslant |\alpha|$, $\deg \varphi_\gamma^\alpha < |\alpha|$;

(3) for every $r \geqslant 1$ the inequalities

$$\begin{aligned} \|\psi^\alpha\|_{rB} &\leqslant \|z^\alpha\|_{rB} = r^{|\alpha|} B^\alpha, \\ \|\varphi_\gamma^\alpha\|_{rB} &\leqslant \|z^\alpha\|_{rB} \quad \text{for all } \gamma \in \Gamma \end{aligned} \tag{1.15}$$

hold. Put $\psi^\alpha(z) = z^\alpha$, $\varphi_\gamma^\alpha = 0$ for $\alpha \in S$. Then (1) holds. For $\alpha \in \Gamma$ we use representation (1.5) putting

$$\psi^\alpha(z) = \sum_{\sigma < \alpha, \sigma \in S} c_\sigma^\alpha z^\sigma; \ \varphi_\alpha^\alpha(z) = 1; \ \varphi_\gamma^\alpha = 0 \quad \text{for} \quad \gamma \neq \alpha, \gamma \in \Gamma.$$

By (1.12), the conditions (2) and (3) hold for all $\alpha \in \Gamma$. For multiindices $\tau \in T \setminus \Gamma$ we use induction. Suppose that, for all $\alpha < \tau$, we have built the representations (1.14) with the properties (1)-(3). Then, by (1.4), $\tau = \delta + \gamma'$, and we have

$$\begin{aligned} z^\tau = z^\delta \cdot z^{\gamma'} &= z^\delta \Big(\sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} z^\sigma + P_{\gamma'} \Big) = \sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} z^{\delta + \sigma} + z^\delta \cdot P_{\gamma'} \\ &= \sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} \Big(\psi^{\delta + \sigma} + \sum_{\gamma \in \Gamma} \varphi_\gamma^{\delta + \sigma} \cdot P_\gamma \Big) + z^\delta \cdot P_{\gamma'} \\ &= \sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} \psi^{\delta + \sigma} + \sum_{\gamma \in \Gamma} \Big(\sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} \varphi_\gamma^{\delta + \sigma} + \Delta_{\gamma\gamma'} z^\delta \Big) P_\gamma, \end{aligned}$$

where $\Delta_{\gamma\gamma'} = 0$ for $\gamma \neq \gamma'$, and $\Delta_{\gamma\gamma'} = 1$ for $\gamma = \gamma'$. Put

$$\psi^\tau = \sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} \psi^{\delta + \sigma}, \tag{1.16}$$

$$\varphi_\gamma^\tau = \sum_{\sigma \in S, \sigma < \gamma'} c_\sigma^{\gamma'} \varphi_\gamma^{\delta + \sigma} + \Delta_{\gamma\gamma'} z^\delta, \quad \gamma \in \Gamma. \tag{1.17}$$

Evidently the formulas (1.16) and (1.17) imply that the conditions (2) hold for $\alpha = \tau$. Using (3) for $\alpha < \tau$, and the basic inequality (1.12) we obtain, for $\tau = \delta + \gamma'$,

$$\begin{aligned} \|\psi^\tau\|_{rB} &\leqslant \sum_{\sigma \in S, \sigma < \gamma'} |c_\sigma^{\gamma'}| \, \|\psi^{\delta + \sigma}\|_{rB} \leqslant \sum_{\sigma \in S, \sigma < \gamma'} |c_\sigma^{\gamma'}| r^{|\delta + \sigma|} B^{\delta + \sigma} \\ &= r^{|\tau|} B^\tau \sum_{\sigma \in S, \sigma < \gamma'} |c_\sigma^{\gamma'}| r^{|\sigma| - |\gamma'|} B^{\sigma - \gamma'} < \tfrac{1}{2} r^{|\tau|} B^\tau, \end{aligned}$$

and in an analogous way (since $B_j \geqslant A_n > 2$, $1 \leqslant j \leqslant n$):

$$\|\varphi_\gamma^\tau\|_{rB} < \tfrac{1}{2} r^{|\tau|} B^\tau + r^{|\delta|} B^\delta = r^{|\tau|} B^\tau \big(\tfrac{1}{2} + r^{-|\gamma'|} B^{-\gamma'} \big) \leqslant r^{|\tau|} B^\tau.$$

2. Construction of operators R in Theorem 2 with properties (a) and (b). Let $f \in H(\mathbf{C}^n)$, $f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha z^\alpha$. Then using (1.14) we obtain (formally)

$$f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha \Big(\psi^\alpha + \sum_{\gamma \in \Gamma} \varphi_\gamma^\alpha P_\gamma \Big) = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha \psi^\alpha + \sum_{\gamma \in \Gamma} \Big(\sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha \varphi_\gamma^\alpha \Big) P_\gamma. \tag{2.1}$$

Put

$$R_0(f) = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha \psi^\alpha, \tag{2.2}$$

$$R_\gamma(f) = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha \varphi_\gamma^\alpha, \quad \gamma \in \Gamma. \tag{2.3}$$

By (1.15) we have, for every $r \geqslant 1$,

$$\|R_0 f\|_{rB} \leqslant \sum_{\alpha \in \mathbf{Z}_+^n} |f_\alpha| \, \|\varphi^\alpha\|_{rB} \leqslant \sum_{\alpha \in \mathbf{Z}_+^n} |f_\alpha| \, \|z^\alpha\|_{rB} = \|f\|_{rB} \tag{2.4}$$

and in an analogous way

$$\|R_\gamma f\|_{rB} \leqslant \|f\|_{rB}, \quad \gamma \in \Gamma. \tag{2.5}$$

These inequalities prove that the operators R_0 and R_γ, $\gamma \in \Gamma$, are continuous and justify the formal operations in the formulas (2.1). Hence we obtain

$$f = R_0 f + \sum_{\gamma \in \Gamma} (R_\gamma f) P_\gamma \quad \text{for all } f \in H(\mathbf{C}^n). \tag{2.6}$$

By Definition (2.2), the image of R_0 is contained in $H_s(\mathbf{C}^n)$ and since (1) hold, we have $R_0 z^\sigma = z^\sigma$ for all $\sigma \in S$, therefore $R_0^2 = R_0$ and $\operatorname{Im} R_0 = H_s(\mathbf{C}^n)$. By the properties (2) the operators R_0 and R_γ, $\gamma \in \Gamma$, map the polynomials into polynomials and, moreover, their degrees satisfy the following inequalities:

$$\deg R_0 g \leqslant \deg g, \qquad \deg R_\gamma g < \deg g, \ \gamma \in \Gamma. \tag{2.7}$$

By (2.6) we obtain that $g \in I$ implies $R_0 g \in I$. Since the monomials z^σ, $\sigma \in S$, are linearly independent modulo I, we have $R_0 g = 0$ for all $g \in I$. Evidently, the closure $\bar{I}$ of the ideal I contains $J = J(P)$ and, since R_0 is continuous in $H(\mathbf{C}^n)$, we have $R_0|_{\bar{I}} = 0$ and $R_0|_J = 0$. On the other hand, if $R_0 h = 0$ for some function $h \in H(\mathbf{C}^n)$, then by (2.6) we get $h = \sum_{\gamma \in \Gamma} (R_\gamma h) \cdot P_\gamma \in J$. Consequently $J = \operatorname{Ker} R_0$ and therefore J is a closed subspace in $H(\mathbf{C}^n)$. ■

Let $H(\Delta_{rB})$ be the space of all holomorphic functions in the polydisc

$$\Delta_{rB} = \{z \in \mathbf{C}^n : |z_i| < rB_i, \ 1 \leqslant i \leqslant n\}, \qquad r > 1,$$

where $B = (B_1, \ldots, B_n)$ are the constants of Theorem 2 satisfying the conditions (1.12), with the topology of uniform convergence on compact subsets of Δ_{rB}. It is easy to see that the system of norms

$$\|f\|_{\varrho B} = \sum_{\alpha \in \mathbf{Z}_+^n} |f_\alpha| (\varrho B)^\alpha \quad \text{for} \quad f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha z^\alpha, \ 1 < \varrho < r,$$

define the same topology in $H(\Delta_{rB})$.

For any function $f \in H(\Delta_{rB})$, $f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha z^\alpha$, we define the operators R_0 and R_γ, $\gamma \in \Gamma$, using the formulas (2.2) and (2.3). Then, since (2.4) and (2.5) hold, we obtain the following

Corollary 3. *The operators R_0 and R_γ, $\gamma \in \Gamma$, act continuously in the space $H(\Delta_{rB})$, $r > 1$; moreover,*

(a) *$f = R_0 f + \sum_{\gamma \in \Gamma} (R_\gamma f) P_\gamma$ for all $f \in H(\Delta_{rB})$;*

(b) *$\operatorname{Im} R_0 = H_s(\Delta_{rB})$, $\operatorname{Ker} R_0 = J_p(\Delta_{rB})$, $R_0^2 = R_0$;*

where $H_s(\Delta_{rB}) = \{f \in H(\Delta_{rB}) : f = \sum_{\alpha \in \mathbf{Z}_+^n} f_\alpha z^\alpha, f_\alpha = 0$ for $\alpha \notin S\}$ and $J_p(\Delta_{rB})$ is the ideal generated by the system of polynomials $P = \{P_\gamma, \gamma \in \Gamma\}$ in the algebra $H(\Delta_{rB})$.

Corollary 4. *Let $Q = \{Q_i(z)\}_1^p$ be a finite set of polynomials and let $J = J(Q)$ be ideal (0.1) generated by Q in the space $H(\mathbf{C}^n)$. Then there exist linear continuous operators*

$$R_i : H(\mathbf{C}^n) \to H(\mathbf{C}^n), \qquad 0 \leqslant i \leqslant p,$$

such that

(a) *$f = R_0 f + \sum_{i=1}^p (R_i f) Q_i$ for all $f \in H(\mathbf{C}^n)$;*

(b) *$\operatorname{Im} R_0 = H_s(\mathbf{C}^n)$, $\operatorname{Ker} R_0 = J(Q)$, $R_0^2 = R_0$,*

where $S = \mathbf{Z}_+^n \setminus T$, and $T = T(I)$ corresponds to the ideal $I = I(Q)$, generated by the polynomials $Q_1, \ldots, Q_p$ in the algebra of all polynomials $\mathbf{C}[z_1, \ldots, z_n]$.

Proof. For a given ideal I we construct the sets T, S, Γ and the polynomials P_γ, $\gamma \in \Gamma$. Then we have

$$P_\gamma = \sum_{i=1}^p \pi_\gamma^i Q_i, \qquad \gamma \in \Gamma, \tag{2.8}$$

where π_γ^i are polynomials. Let R_0 and R_γ, $\gamma \in \Gamma$, be the corresponding operators of Theorem 2. Put

$$R_i f = \sum_{\gamma \in \Gamma} (R_\gamma f) \pi_\gamma^i, \qquad 1 \leqslant i \leqslant p. \tag{2.9}$$

By (2.6), we obtain

$$f = R_0 f + \sum_{i=1}^p (R_i f) Q_i \quad \text{for all } f \in H(\mathbf{C}^n).$$

Therefore the operators R_i, $0 \leqslant i \leqslant p$, satisfy the conditions (a) and (b). ■

It is easy to see that, by Corollary 3, the operators R_i, $0 \leqslant i \leqslant p$, act continuously in the space $H(\Delta_{rB})$. Hence Corollary 4 remains true if instead of $H(\mathbf{C}^n)$ one put the space $H(\Delta_{rB})$.

Remark 5. In Sections 5 and 6 we shall use some additional properties of the operators R_i, $0 \leqslant i \leqslant p$, which follow from their construction. First, observe that the relations (2.7) and (2.9) imply,

$$\deg R_i g \leqslant \deg g + d, \quad 0 \leqslant i \leqslant p, \tag{2.10}$$

where $d = \max\{\deg \pi_\gamma^i,\ \gamma \in \Gamma,\ 1 \leqslant i \leqslant p\}$, for any polynomial g.

Further, considering in the space $H(C^n)$ the system of norms

$$|f|_r = \sum_{\alpha \in Z_+^n} |f_\alpha| r^\alpha \quad \text{for} \quad f = \sum_{\alpha \in Z_+^n} f_\alpha z^\alpha,\ r \geqslant 1, \tag{2.11}$$

we conclude from (2.5) that

$$|R_\gamma f|_r \leqslant \|R_\gamma f\|_{rB} \leqslant \|f\|_{rB} \leqslant |f|_{ar}, \quad \gamma \in \Gamma, \tag{2.12}$$

where $a = B_n$. Using (2.9) we get

$$|R_i f|_r \leqslant C r^d |f|_{ar}, \quad 1 \leqslant i \leqslant p,\ r \geqslant 1, \tag{2.13}$$

where the constant C equals the sum of modulus of the coefficients of all polynomials π_γ^i, $\gamma \in \Gamma$, $1 \leqslant i \leqslant p$, appearing in expansion (2.8).

3. The structure of the set $S = Z_+^n \setminus T$.

LEMMA 6. *If T is a monotonous subset of Z_+^n, i.e. $T = T + Z_+^n$, then the set $S = Z_+^n \setminus T$ can be represented as the union of a finite number of sublattices of type Z_+^m, $0 \leqslant m < n$.*

(Here by a sublattice of type Z_+^m we mean any subset of Z_+^n of the form

$$\alpha + Z_+^m(K), \quad \text{where} \quad Z_+^m(K) = \{\beta \in Z_+^n:\ \beta_i = 0 \text{ for } i \in K\}, \tag{3.1}$$
$$\alpha \in Z_+^n,\ K \subset \{1, 2, \ldots, n\},\ m + |K| = n.)$$

Proof. By Lemma 1, the set $\Gamma = \Gamma(T)$ is finite. Let $k(\Gamma) = \max\{\gamma_i: \gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma,\ 1 \leqslant i \leqslant n\}$ and choose $v > k(\Gamma)$. There are only finitely many points of Z_+^n belonging to the set $B(v) = \{x \in R^n:\ 0 \leqslant x_i \leqslant v,\ 1 \leqslant i \leqslant n\}$. Put, for any $\alpha \in B(v) \cap Z_+^n$,

$$K_\alpha = \{j \in \{1, 2, \ldots, n\}:\ \alpha_j < v\}.$$

Then we have the representation

$$S = \bigcup_{\alpha \in S \cap B(v)} \big(\alpha + Z_+^{m(\alpha)}(K_\alpha)\big). \tag{3.2}$$

Indeed, if $\sigma \in S$, then by the monotonicity of T, every point $\sigma' \leqslant \sigma$ belongs to S; in particular, the point $\tilde\sigma = (\tilde\sigma_i)_1^n$, where $\tilde\sigma_i = \min(\sigma_i, v)$, belongs to S. Hence

$$\sigma = \tilde\sigma + (\sigma - \tilde\sigma) \in \tilde\sigma + Z^{m(\tilde\sigma)}(K_{\tilde\sigma})$$

and $\tilde\sigma \in B(v)$. Relation (3.2) is proved.

Obviously $v\bar e \in T$, where $\bar e = (1, 1, \ldots, 1)$. Since $\alpha = v\bar e$ is the only point in $B(v)$ for which the set K_α is empty, we have in representation (3.2) only the sublattices $\alpha + Z_+^{m(\alpha)}(K_\alpha)$ with $m(\alpha) < n$. ∎

Put

$$m(T) = \max\{m(\alpha):\ \alpha \in S \cap B(v)\}; \tag{3.3}$$

this number does not depend on $v \geqslant K(\Gamma)$, but it depends on the monotonous set T.

Remark 7. It is easy to see that $m(T) = m$ if and only if, for any subset $K \subset \{1, 2, \ldots, n\}$ with $|K| = n - m - 1$, there exists an $\alpha \in T$, such that $\alpha_i = 0$ for $i \in K$, but for some $K \subset \{1, 2, \ldots, n\}$ with $|K| = n - m$, there is no $\alpha \in T$ with the property $\alpha_i = 0$ for $i \in K$.

The dimension q of the lattice L of type Z_+^q can be find from the asymptotic behavior of the number of points in subsets $L \cap \pi_k$, where $\pi_k = \{\nu \in Z_+^n:\ |\nu| \leqslant k\}$, $1 \leqslant k < \infty$. Namely, $|L \cap \pi_k| \sim k^q$, therefore

$$q = \lim_{k \to \infty} (\log |L \cap \pi_k| / \log k).$$

By representation (3.2) we obtain, for every point $\tilde\alpha$ such that $m(\tilde\alpha) = m(T)$, the following inequalities:

$$\big|(\tilde\alpha + Z_+^{m(\tilde\alpha)}(K_{\tilde\alpha})) \cap \pi_k\big| \leqslant |S \cap \pi_k| \leqslant \sum_{\alpha \in B(v) \cap S} \big|(\alpha + Z_+^{m(\alpha)}(K_\alpha)) \cap \pi_k\big|.$$

Obviously, there exist constants c, $c(\alpha)$, $C > 0$, such that the inequalities

$$c k^{m(T)} \leqslant |S \cap \pi_k| \leqslant \sum_{\alpha \in S \cap B(v)} C(\alpha) k^{m(\alpha)} \leqslant C \cdot k^{m(T)}$$

hold. Consequently we have

$$m(T) = \lim_{k \to \infty} (\log |S \cap \pi_k| / \log k). \tag{3.4}$$

Let I be an ideal in $C[z_1, \ldots, z_n]$. Put $m(I) = m(T(I))$. Formula (3.4) shows that the number $m(I)$ does not depend on a linear (affine) transformation of the coordinate system $z_1, \ldots, z_n$. Indeed, the number $|S \cap \pi_k|$ equals the dimension of the quotient space $P_n(k)/P_n(k) \cap I$, where $P_n(k)$ is the space of all polynomials with degrees $\leqslant k$.

LEMMA 8. *The correspondence $I \to m(I)$ has the following properties:*

(a) *if $I_1 \subset I_2$, then $m(I_1) \geqslant m(I_2)$;*

(b) $m(\operatorname{Rad} I) = m(I)$;

(c) *if $I = I_1 \cap I_2$, then $m(I) = \max\{m(I_1), m(I_2)\}$.*

(Here $\operatorname{Rad} I$ denotes the set of all polynomials $P \in C[z_1, \ldots, z_n]$, such that $P^k \in I$ for some integer k.)

Proof. (a) If $I_1 \subset I_2$, then $T(I_1) \subset T(I_2)$ and therefore $m(I_1) \geqslant m(I_2)$, i.e. (a) holds.

(b) Since $I \subset \operatorname{Rad} I$, we have $m(I) \geqslant m(\operatorname{Rad} I)$. On the other hand, if $\alpha \in T(\operatorname{Rad} I)$, we have $\big(z^\alpha - \sum_{\beta<\alpha} c_\beta^\alpha z^\beta\big)^k \in I$ for some integer k, and therefore $k\alpha \in T(I)$. Then by Remark 7, we get (b).

(c) Put $T_1 = T(I_1)$, $T_2 = T(I_2)$, $T = T(I)$, $T_{12} = \{\alpha \in Z_+^n: \alpha = \alpha_1 + \alpha_2,\ \alpha_1 \in T_1,\ \alpha_2 \in T_2\}$. It is easy to see that $T_{12} \subset T$. Indeed, if $\alpha_1 \in T_1$ and $\alpha_2 \in T_2$, we have

$$P_{\alpha_1} = z^{\alpha_1} - \sum_{\beta<\alpha_1} c_\beta^{\alpha_1} z^\beta \in I_1, \qquad Q_{\alpha_2} = z^{\alpha_2} - \sum_{\beta<\alpha_2} d_\beta^{\alpha_2} z^\beta \in I_2$$

for some constants $c_\beta^{\alpha_1}$, $d_\beta^{\alpha_2}$. Since $P_{\alpha_1}\cdot Q_{\alpha_2} \in I = I_1 \cap I_2$, we obtain $\alpha_1 + \alpha_2 \in T$. Evidently T_{12} is a monotonous set, and we have $T_{12} \subset T \subset T_1 \cap T_2$. Therefore $m(T_{12}) \geqslant m(T) \geqslant m(T_1 \cap T_2) \geqslant \max\{m(T_1), m(T_2)\}$. On the other hand, by Remark 7, we obtain $m(T_{12}) \leqslant \max\{m(T_1), m(T_2)\}$. Indeed, for every sublattice of form $Z_+^l(K_l)$, where $K_l \subset \{1, 2, \ldots, n\}$, $|K_l| = n-l$, $l > \max\{m(T_1), m(T_2)\}$, there exist $\alpha_1 \in T_1$ and $\alpha_2 \in T_2$, such that $\alpha_1, \alpha_2 \in Z_+^l(K_l)$. Consequently, $\alpha_1 + \alpha_2 \in Z_+^l(K_l)$, and by Remark 7, we obtain $m(T_{12}) < l$, whence it follows that $m(T_{12}) \leqslant \max\{m(T_1), m(T_2)\}$. Therefore

$$m(T) = \max\{m(T_1), m(T_2)\}, \quad \text{i.e.} \quad m(I) = \max\{m(I_1), m(I_2)\}. \ \blacksquare$$

Denote by $V(I)$ the algebraic variety corresponding to the ideal $I \subset C[z_1, \ldots, z_n]$, i.e.

$$V(I) = \{z \in C^n: P(z) = 0 \text{ for all } P \in I\}.$$

If $V \subset C^n$ is an algebraic variety, we put

$$(3.5) \qquad I^*(V) = \{P: P(z) = 0 \text{ for all } z \in V\};$$

this is the ideal, associated with V. By Hilbert's Nullstellensatz (cf. [10]) we have $I^*(V(I)) = \operatorname{Rad} I$.

Suppose A is an irreducible algebraic variety; then the ideal $I^*(A)$ is prime. By definition, the dimension of A ($\dim_C A$) is equal to the transcendence degree of the field $C(A)$ over C, where $C(A)$ is the field of fractions of the ring $C[z_1, \ldots, z_n]/I^*(A)$.

It is well known, that every algebraic variety V can be represented as a union of irreducible algebraic varieties;

$$V = A_1 \cup \ldots \cup A_s.$$

By definition, the dimension of V is

$$\dim_C V = \max\{\dim_C A_i,\ 1 \leqslant i \leqslant s\}.$$

THEOREM 9. *Let I be an ideal in $C[z_1, \ldots, z_n]$. Then $m(I) = \dim_C V(I)$.*

Proof. First, remark that it is enough to prove the statement in the case, where the ideal I is prime. Indeed, if I is not a prime ideal, we have $V(I) = A_1 \cup \ldots \cup A_s$, where $A_1, \ldots, A_s$ are irreducible algebraic varieties; then the associated ideals $I^*(A_1), \ldots, I^*(A_s)$ are prime. Suppose that $m(I^*(A_i)) = \dim_C A_i$, $1 \leqslant i \leqslant s$. Then $\operatorname{Rad} I = \bigcap_{i=1}^s I^*(A_i)$ and by Lemma 8 we get

$$\begin{aligned} m(I) = m(\operatorname{Rad} I) &= \max\{m(I^*(A_i)),\ 1 \leqslant i \leqslant s\} \\ &= \max\{\dim_C(A_i),\ 1 \leqslant i \leqslant s\} = \dim_C V(I). \end{aligned}$$

Consider the case when the ideal I is prime. Then, without loss of generality one may assume that $z_1, \ldots, z_n$ is a regular coordinate system for the ideal I (if not, using a linear transformation of coordinates we would get such a system — cf. [1], III. A, or [10]). "Regular" means that the following conditions hold:

(i) $I \cap C[z_1, \ldots, z_r] = 0$, where $r = \dim_C V$;

(ii) $C[z_1, \ldots, z_n]/I$ is integral over $C[z_1, \ldots, z_r]$.

Condition (ii) gives that there exists polynomials

$$w_j(z) = z_j^{a_j} + \sum_{i=0}^{a_j-1} w_{ij}(z') z_j^i, \qquad z' = (z_1, \ldots, z_r),\ r+1 \leqslant j \leqslant n,$$

such that $w_j \in I$. By (i), the monomials $z_1^{\alpha_1} \ldots z_r^{\alpha_r} = z'^\alpha$, $\alpha \in Z_+^r$, are linearly independent modulo I. Therefore

$$(3.6) \qquad |S \cap \pi_k| = \dim(P_n(k)/I \cap P_n(k)) \geqslant |Z_+^r \cap \pi_k| \geqslant \frac{1}{r!}(1+k)^r.$$

On the other hand, if $r+1 \leqslant j \leqslant n$, $a_j \leqslant b$, then one can represent z_j^b in the form

$$(3.7) \qquad z_j^b = \sum_{i=0}^{a_j-1} P_{ij}(z') z_j^i + t_{jb},$$

where $t_{jb} \in I$ and $\deg P_{ij} \leqslant M\cdot b$, $M = \max\{\deg w_{ij},\ 0 \leqslant i \leqslant a_j - 1,\ r+1 \leqslant j \leqslant n\}$. Indeed, if $b = a_j$, we have

$$z_j^{a_j} = -\sum_{i=0}^{a_j-1} w_{ij}(z') z_j^i + w_j(z).$$

Suppose the statement is true for some $b \geqslant a_j$. Then we obtain

$$\begin{aligned} z_j^{b+1} = z_j z_j^b &= \sum_{i=0}^{a_j-1} P_{ij}(z') z_j^{i+1} + z_j\cdot t_{jb} \\ &= \sum_{i=0}^{a_j-2} P_{ij}(z') z_j^{i+1} + P_{(a_j-1)j}(z')\Big(-\sum_{i=0}^{a_j-1} w_{ij}(z') z_j^i + w_j\Big) + z_j\cdot t_{jb}. \end{aligned}$$

Using this formula, one can easily show that (3.7) holds for $b+1$. Hence (3.7) holds for all $b \geqslant a_j$.

Now it is easy to see (using (3.7)) that for any $\beta \in \mathbf{Z}_+^n$ there exist constants d_σ^β, such that

$$z^\beta = \sum d_\sigma^\beta z^\sigma \mod I,$$

where $\sigma_1 + \ldots + \sigma_r \leqslant M|\beta|$, $0 \leqslant \sigma_j \leqslant a_j - 1$ for $j = r+1, \ldots, n$. Therefore

$$(3.8) \qquad |S \cap \pi_k| \leqslant \Big(\prod_{r+1}^{n} a_j\Big) |\mathbf{Z}_+^r \cap \pi_{Mk}| \leqslant \Big(\prod_{r+1}^{n} a_j\Big) \frac{1}{r!} (r+Mk)^r.$$

By (3.4), (3.6) and (3.8) we get

$$m(I) = \lim_{k\to\infty} (\log|S \cap \pi_k| / \log k) = r. \ \blacksquare$$

Lemma 10. *Let $T \subset \mathbf{Z}_+^n$ be a monotonous set and $S = \mathbf{Z}_+^n \setminus T$. Then the space $H_s(\mathbf{C}^n)$ is isomorphic to the space $H(\mathbf{C}^m)$, where $m = m(T)$.*

Proof. It is easy to see that representation (3.2) yields a decomposition of S into disjoint subsets. Since the system $\{z^\sigma, \sigma \in S\}$ is an absolute basis in $H_s(\mathbf{C}^n)$, the space $H_s(\mathbf{C}^n)$ can by represented as the direct sum of the subspaces $H_{s_\alpha}(\mathbf{C}^n)$, where $S_\alpha = \alpha + \mathbf{Z}_+^{m(\alpha)}(K_\alpha)$, $\alpha \in S \cap B(v)$. The operation of "dividing by z^α" is an isomorphism between the space $H_{s_\alpha}(\mathbf{C}^n)$ and the space $H(\mathbf{C}^{m(\alpha)})$ of all entire functions depending on $m(\alpha)$ variables (namely the variables z_i, $i \notin K_\alpha$). On the other hand, the space $H(\mathbf{C}^k) \oplus H(\mathbf{C}^l)$, $k \leqslant l$, is isomorphic to the space $H(\mathbf{C}^l)$ — this is a particular case of well-known statements about isomorphism between power series spaces (cf. [4], Prop. 18 or [6] § 9.3). Therefore $\bigoplus_\alpha H(\mathbf{C}^{m(\alpha)}) \simeq H(\mathbf{C}^m)$. ■

4. Ideals in $H(\mathbf{C}^n)$, connected with an algebraic variety in $\mathbf{C}^n$. Let $V \subset \mathbf{C}^n$ be an algebraic variety. It is natural to define the space $H(V)$ of holomorphic functions on V as the quotient space $H(\mathbf{C}^n)/J(V)$, where $J(V)$ is the ideal of all entire functions in $\mathbf{C}^n$, which equal zero on V, i.e.

$$J(V) = \{f \in H(\mathbf{C}^n) \colon f|_V = 0\}.$$

The associated with V ideal $I^*(V) = \{P \in \mathbf{C}[z_1, \ldots, z_n] \colon P|_V = 0\}$ has (as every ideal in $\mathbf{C}[z_1, \ldots, z_n]$) a finite system of generators $Q = \{Q_i\}_1^p$, i.e. it coincides with the ideal

$$I_Q = \Big\{P = \sum_{i=1}^{p} P_i Q_i \colon \ P_i \in \mathbf{C}[z_1, \ldots, z_n],\ 1 \leqslant i \leqslant p\Big\}.$$

The ideal

$$J_Q = \Big\{f = \sum_{i=1}^{p} f_i Q_i \colon f_i \in H(\mathbf{C}^n),\ 1 \leqslant i \leqslant p\Big\}$$

is generated by the same system of polynomials Q in the algebra of all entire functions $H(\mathbf{C}^n)$. Obviously the closure $\overline{I_Q}$ of the ideal I_Q in the space $H(\mathbf{C}^n)$ contains J_Q, therefore we have $J_Q \subset \overline{I_Q} \subset J(V)$.

Lemma 11. *Under the assumptions and notations of this section we have $J_Q = \overline{I_Q} = J(V)$.*

Proof. This statement is a particular case of Cartan's Theorem B (cf. [1], Theorem VIII. A. 14). Indeed, let us denote by $\mathscr{H}$ the sheaf of germs of holomorphic functions over $\mathbf{C}^n$ and by $\mathscr{J}(V)$ the sheaf of ideals of the algebraic variety V, i.e. the subsheaf of $\mathscr{H}$, consisting of germs of holomorphic functions, which are zero on V. Consider the $\mathscr{H}$ homomorphism of sheaves

$$q\colon \mathscr{H}^p \to \mathscr{J}(V)$$

defined by the formula $(g_i)_1^p \in \mathscr{H}_z^p \xrightarrow{q} \sum_{i=1}^{p} g_i Q_i \in \mathscr{J}(V)_z$. The homomorphism q is surjective (see [8], Prop. 4). Therefore the sequence of sheaves

$$0 \to \operatorname{Ker} q \xrightarrow{j} \mathscr{H}^p \xrightarrow{q} \mathscr{J}(V) \to 0$$

is exact. Consider the corresponding exact sequence of the cohomology groups:

$$0 \to H^0(\mathbf{C}^n, \operatorname{Ker} q) \xrightarrow{j_0} H^0(\mathbf{C}^n, \mathscr{H}^p) \xrightarrow{q_0} H^0(\mathbf{C}^n, \mathscr{J}(V)) \xrightarrow{\delta} H^1(\mathbf{C}^n, \operatorname{Ker} q) \to \ldots$$

Since the sheaf $\ker q$ is coherent (see [1], Theorem IV. D. 2 and Prop. IV. B. 12) by Cartan's Theorem B we have $H^1(\mathbf{C}^n, \operatorname{Ker} q) = 0$. Therefore the homomorphism q_0 is surjective. But $H^0(\mathbf{C}^n, \mathscr{H}^p) = H(\mathbf{C}^n)^p$, $H^0(\mathbf{C}^n, \mathscr{J}(V)) = J(V)$ and q^0 acts in the following way: $(f_1, \ldots, f_p) \in H(\mathbf{C}^n)^p \xrightarrow{q_0} \sum_{i=1}^{p} f_i Q_i \in J_Q$. Evidently the image of q_0 is the ideal J_Q, hence $J_Q = J(V)$. ■

Proposition 12. *Let V be an algebraic variety in $\mathbf{C}^n$ and let the system of polynomials $Q = \{Q_i\}_1^p$ generates the ideal $I^*(V)$. Then there exist a set $S \subset \mathbf{Z}_+^n$ of type (3.2) with a characteristic number*

$$\max_{\alpha \in S \cap B(v)} m(\alpha) = \dim_{\mathbf{C}} V$$

and continuous linear operators $R_i\colon H(\mathbf{C}^n) \to H(\mathbf{C}^n)$, $0 \leqslant i \leqslant p$, such that

(a) $f = R_0 f + \sum_{i=1}^{p} (R_i f) Q_i$ *for all* $f \in H(\mathbf{C}^n)$;

(b) $\operatorname{Im} R_0 = H_s(\mathbf{C}^n)$, $\operatorname{Ker} R_0 = J(V)$, $R_0^2 = R_0$.

Proof. By Theorem 9 and Lemma 11 the statement is a particular case of Corollary 4.

COROLLARY 13. *The ideal $J(V)$ is a complemented subspace in $H(C^n)$.*

Indeed, consider the operator P defined by the formula

$$Pf = (1-R_0)f = \sum_{i=1}^{p} (R_i f) Q_i \quad \text{for all } f \in H(C^n).$$

Since $R_0^2 = R_0$, we have $P^2 = P$ and $\operatorname{Im} P = \operatorname{Ker} R_0 = J(V)$, i.e. P is a projector on $J(V)$.

COROLLARY 14. *The space $H(V)$ of holomorphic functions on the algebraic variety $V \subset C^n$ is isomorphic to the space $H(C^m)$, where $m = \dim_C V$.*

Proof. Consider the operator

$$A\colon H_S(C^n) \to H(V), \quad Af = f|_V \quad \text{for all } f \in H_S(C^n).$$

The operator A is surjective. Indeed, for any function $\varphi \in H(V) = H(C^n)/J(V)$ there exists an entire function $\tilde{\varphi} \in H(C^n)$, such that $\tilde{\varphi}|_V = \varphi$. Then

$$\tilde{\varphi} = R_0\tilde{\varphi} + P\tilde{\varphi},$$

and we have $P\tilde{\varphi}|_V = 0$, $R_0\tilde{\varphi}|_V = \tilde{\varphi}|_V = \varphi$, i.e. $\varphi = Af$, where $f = R_0\tilde{\varphi} \in H_S(C^n)$. Therefore $\operatorname{Im} A = H(V)$.

On the other hand, A is an injective operator. For if $Af = f|_V = 0$ and $f \in H_S(C^n)$, then $f \in (\operatorname{Im} R_0) \cap (\operatorname{Ker} R_0) = \{0\}$. Since A is a continuous linear bijective mapping between two Fréchet spaces, by Open Mapping Theorem it is a linearly-topological isomorphism. Now we have $H(V) \overset{A}{\simeq} H_S(C^n)$ and, by Theorem 9 and Lemma 10, $H_S(C^n) \simeq H(C^m)$, where $m = \dim_C V$. Hence $H(V) \simeq H(C^m)$.

COROLLARY 15. *There exists a continuous linear operator $E\colon H(V) \to H(C^n)$ of extention of holomorphic functions from the algebraic variety V on the entire space C^n, i.e. $E\varphi|_V = \varphi$ for all $\varphi \in H(V)$.*

Proof. Indeed, put $E = jA^{-1}$, where A^{-1} is the inverse operator to the operator A from Corollary 14, and $j\colon H_S(C^n) \to H(C^n)$ is the imbeding of the subspace $H_S(C^n)$ into $H(C^n)$. Since $A^{-1}\varphi|_V = \varphi$, we have $E\varphi|_V = \varphi$ for all $\varphi \in H(V)$. ■

Let us remark, that the Corollaries 13 and 15 are equivalent, but in the proof our argument gives not only the existence of suitable operators but also some information on their structure.

Let us also observe that Proposition 6.4 of [4] states that if there exists an isomorphism $H(V) \simeq H(C^m)$, then one can prove the existence of a continuous linear operator of extention $E\colon H(V) \to H(C^n)$. Therefore Corollary 14 implies Corollary 15. But the proof, sketched in [4] requires an analysis of the solutions of $\bar{\partial}$-problem and cohomologies with estimates, which is, in fact, not necessary in this situation.

The ideal $J(V)$, as every polynomial ideal J_Q, has the same linear topological structure as $H(C^n)$. Indeed, we have

COROLLARY 16. *There exists an isomorphism $J_Q \simeq H(C^n)$.*

Proof. Evidently,

$$J_Q = \operatorname{Im}(1-R_0) = \operatorname{Ker} R_0 \simeq H(C^n)/\operatorname{Im} R_0 \simeq H_T(C^n),$$

where $T = T(J_Q)$ is the monotonous set (1.2). Since

$$T = \bigcup_{\gamma\in\Gamma} (\gamma + Z_+^n),$$

where $\Gamma = \Gamma(T)$ is the (finite) set of the minimal elements of T, we have the representation $T = \bigcup_{\beta\in F} L(\beta)$, where $|F| < \infty$ and $L(\beta)$, $\beta \in F$, are disjoint sublattices of type Z_+^q, $\max_{\beta\in F} q(\beta) = n$. Using the same argument as in Lemma 10 we get $J_Q \simeq H(C^n)$.

COROLLARY 17 (cf. [4], Theorem 4.1). *Let $J = J_Q$ be a polynomial ideal in $H(C^n)$, generated by the system of polynomials $Q = \{Q_i\}_1^p$. Then there exist continuous linear operators $L_i\colon J \to H(C^n)$, $1 \leqslant i \leqslant p$, such that*

$$f = \sum_{i=1}^{p} L_i(f) Q_i \quad \textit{for all } f \in J.$$

Proof. Indeed, since $R_0|_J = 0$, it is enough to put (by Corollary 4) $L_i = R_i|_J$.

5. Spaces of entire functions with bounds on the growth. We have investigated in details the structure of polynomial ideals in the space of all entire functions $H(C^n)$. Using the properties of the constructed operators R one can obtain similar results for some special spaces of entire functions. We recall that operators R, which have been built in Theorem 2 and Corollary 4, for the ideal J_Q, satisfy (by (2.13)) the estimates

(5.1) $\quad |R_i f|_r \leqslant C_1 r^d \cdot |f|_{ar}, \; r \geqslant 1, \; 0 \leqslant i \leqslant p, \quad$ for all $f \in H(C^n)$,

where C_1, d, a are suitable positive constants. The norm $|\cdot|_r$ have been defined in (2.11).

Let $H_\varrho(C^n)$, $H^\varrho(C^n)$ denote the algebras of all entire functions of order ϱ, which are of minimal or finite type, respectively, i.e.

$$H_\varrho = \lim_{D\to 0} \operatorname{proj} H_{\varrho D}, \quad \text{or} \quad H^\varrho = \lim_{D\to\infty} \operatorname{ind} H_{\varrho D},$$

where $H_{\varrho D} = H_{\varrho D}(C^n)$ is the Banach space of all entire functions f, such that $|f|_r \leqslant C \exp Dr^\varrho$ for some $C > 0$ and every $r \geqslant 1$, with the norm

$$\|f|_{H_{\varrho D}}\| = \inf\{C\colon |f|_r \leqslant C \exp Dr^\varrho, \; r \geqslant 1\}.$$

If $f \in H_{\varrho D}$, we obtain

(5.2) $$|Rf|_r \leqslant C_1 r^d \cdot |f|_{ar} \leqslant C_1 C r^d \cdot \exp D(ar)^\varrho \leqslant C_1 C(D) \cdot \exp(2Da^\varrho \cdot r^\varrho),$$
$$\text{for } R = R_i,\ 0 \leqslant i \leqslant p,$$

where $C(D) = C \sup_{r \geqslant 1} r^d \cdot \exp(-Da^\varrho \cdot r^\varrho)$. Therefore every operator

$$R\colon H_{\varrho D} \to H_{\varrho D_1}, \quad D_1 = 2Da^\varrho,$$

is continuous and its norm $\leqslant C_1 C(D)$, where C_1 is the constant of (5.1). Since D_1 is proportional to D, the operators R act continuously in both the spaces H_ϱ and H^ϱ, $\varrho > 0$. Therefore we have

PROPOSITION 18. *Let $Q = \{Q_i\}_1^p$ be a system of polynomials, and let $J_\varrho(Q)$ (resp. $J^\varrho(Q)$) be the ideal, generated by Q in the algebra H_ϱ (resp. H^ϱ). Then this ideal is a (closed) complemented linear subspace of H_ϱ (resp. H^ϱ) and the linear operators R_i, $0 \leqslant i \leqslant p$, act continuously in H_ϱ (resp. H^ϱ); moreover,*

(a) $f = R_0 f + \sum_{i=1}^p (R_i f) Q_i$ *for all* $f \in H_\varrho$ *(resp. H^ϱ);*

(b) $R_0^2 = R_0$, $\operatorname{Ker} R_0 = J_\varrho(Q)$ *(resp. $J^\varrho(Q)$).*

In particular, if V is an algebraic variety and Q is a system of generators of the associated ideal $I^*(V)$, then we have

(5.3) $$\{f \in H_\varrho\colon f|_V = 0\} = \{\sum_{i=1}^p h_i Q_i\colon h_i \in H_\varrho(C^n),\ 1 \leqslant i \leqslant p\}.$$

The same is true, if one puts the space H^ϱ instead of H_ϱ.

Proof. The relations (a) and (b) have been proved in Theorem 2 (or Corollary 4), and factors $R_i f$, $0 \leqslant i \leqslant p$, belong to the space H_ϱ or H^ϱ because the estimates (5.2) hold. These relations imply that the ideal $J_\varrho(Q)$ (resp. $J^\varrho(Q)$) is a closed, complemented subspace of H_ϱ (resp. H^ϱ), and that the ideals (5.3) coincide. Indeed, if $f \in H_\varrho$ and $f|_V = 0$, then by Lemma 11 (or Prop. 12) $R_0 f = 0$, therefore $f = \sum_{i=1}^p (R_i f) Q_i$. ■

(Let us emphasize, that we don't use cohomologies with bounds to prove (5.3); we only remark, that the operators R act continuously in H_ϱ (or H^ϱ) and therefore factors $R_i f$ belong to the corresponding space.)

Similar statements hold for any algebra of entire functions of the type

(a) $$H_M(C^n) = \lim_{K\to\infty} \operatorname{proj} H_{M_K}$$

or

(b) $$H^M(C^n) = \lim_{K\to\infty} \operatorname{ind} H_{M_K},$$

where $M = \{M_K\}$ is a system of functions $M_K\colon (1, \infty) \to (1, \infty)$ such that the following conditions are fulfilled:

(a) $\forall\, d, D, K\ \exists\, K', C \mid r^d M_{K'}(Dr) \leqslant C M_K(r), \quad r \geqslant 1,$

or

(b) $\forall\, d, D, K\ \exists\, K', C \mid r^d M_K(Dr) \leqslant C M_{K'}(r), \quad r \geqslant 1,$

and $H_{M_K} = H_{M_K}(C^n)$ is the Banach space of all entire functions such that $|f|_r \leqslant C M_K(r)$ for some C and every $r \geqslant 1$, with the norm

$$\|f | H_{M_k}\| = \sup_{r \geqslant 1} |f|_r / M_k(r).$$

6. Results about polynomial modules. Let P_n denote the algebra of all polynomials $C[z_1, \ldots, z_n]$. The cartesian products P_n^N and $H(C^n)^N$ of N copies of P_n (respectively $H(C^n)$) are modules over P_n and $H(C^n)$, respectively. To every $(N_0 \times N)$-matrix of polynomials $Q = (Q_{ij})$ we attach the module homomorphisms

$$Q\colon P_n^N \to P_n^{N_0} \quad \text{and} \quad \tilde{Q}\colon H(C^n)^N \to H(C^n)^{N_0},$$

which are defined to map $(g_i)_{j=1}^N$ into $\big(\sum_{j=1}^N Q_{ij} g_j\big)_{i=1}^{N_0}$. Such homomorphisms will be called *polynomial homomorphisms*. Evidently, the vectors $Q_j = (Q_{ij})_{i=1}^{N_0}$, $1 \leqslant j \leqslant N$, generate the submodules $\operatorname{Im} Q$ and $\operatorname{Im} \tilde{Q}$, where

$$\operatorname{Im} Q = \Big\{f \in P_n^{N_0}\colon f = \sum_{j=1}^N g_j Q_j,\ g_j \in P_n,\ 1 \leqslant j \leqslant N\Big\},$$

$$\operatorname{Im} \tilde{Q} = \Big\{f \in H(C^n)\colon f = \sum_{j=1}^N g_j Q_j,\ g_j \in H(C^n),\ 1 \leqslant j \leqslant N\Big\}.$$

Modules of this form will be called *polynomial modules*.

Now consider the submodule $\operatorname{Ker} Q \subset P_n^N$. Since the polynomial ring P_n is Noetherian (cf. [10]), $\operatorname{Ker} Q$ has a finite system of generators $\{Q_k^1 \in P_n^N,\ 1 \leqslant k \leqslant N_1\}$. The polynomial vectors $Q_k^1 = (Q_{jk}^1)_{j=1}^{N_1}$, $1 \leqslant k \leqslant N_1$, determine the matrix $Q^1 = (Q_{jk}^1)$. Consider the corresponding module homomorphisms Q^1 and $\tilde{Q}^1$; then we have the sequences

(6.1) $$P_n^{N_1} \xrightarrow{Q^1} P_n^N \xrightarrow{Q} P_n^{N_0}$$

and

(6.2) $$H(C^n)^{N_1} \xrightarrow{\tilde{Q}^1} H(C^n)^N \xrightarrow{\tilde{Q}} H(C^n)^{N_0}.$$

By the construction of Q^1, $\operatorname{Ker} Q = \operatorname{Im} Q^1$, i.e. sequence (6.1) is exact. It is natural to ask if the same is true for sequence (6.2), i.e. if $\operatorname{Ker} \tilde{Q} = \operatorname{Im} \tilde{Q}^1$. The answer is affirmative (see Lemma 7.6.4 in [2]). Therefore we have

LEMMA 19. *Let $\tilde{Q}\colon H(C^n)^N \to H(C^n)^{N_0}$ be a polynomial homomorphism. Then $\operatorname{Ker} \tilde{Q}$ is a polynomial module, i.e. there exists a polynomial homomorphism $\tilde{Q}^1\colon H(C^n)^{N_1} \to H(C^n)^N$, such that $\operatorname{Ker} \tilde{Q} = \operatorname{Im} \tilde{Q}^1$.*

THEOREM 20. *Let $\tilde{Q}\colon H(C^n)^N \to H(C^n)^{N_0}$ be a polynomial homomor-*

phism. Then $\operatorname{Ker}\tilde{Q}$ *and* $\operatorname{Im}\tilde{Q}$ *are complemented subspaces in* $H(C^n)^N$ *and* $H(C^n)^{N_0}$, *respectively. In particular, every polynomial module is a complemented subspace.*

Proof (Sketch). By Lemma 19, it is enough to prove that $\operatorname{Im}\tilde{Q}$ is a complemented subspace. The detailed proof would require to repeat many constructions of Sections 1–4. We shall only underline those points, where some modifications are needed.

We begin with some constructions, connected with the module $M = \operatorname{Im}\tilde{Q}$. The system $\{z(\alpha, i),\ \alpha \in Z_+^n,\ 1 \leqslant i \leqslant N_0\}$, where $z(\alpha, i)_{i'}$ equals z^α if $i' = i$, and 0 if $i' \neq i$, is the natural basis in the space $H(C^n)^{N_0}$. We introduce a linear ordering in the set of all indices $\{(\alpha, i)\}$ in the following way:

$$(\alpha', i') < (\alpha'', i''), \quad \text{if} \quad \alpha' < \alpha'' \text{ or } \alpha' = \alpha'' \text{ and } i' < i'',$$

where for multi-indices $\alpha', \alpha'' \in Z_+^n$ the relation of ordering "<" is the same, as in (1.1).

Let $T = T(M)$ be the set of all indices (α, i) such that the basis element $z(\alpha, i)$ belongs modulo M to the linear hull of the preceding basis elements $z(\beta, j)$, i.e.

$$T = T(M) = \{(\alpha, i):\ z(\alpha, i) \in [z(\beta, j):\ (\beta, j) < (\alpha, i)] + M\}.$$

Then $T = \bigcup_{j=1}^{N_0} T_j$ where $T_j = \{(\alpha, i) \in T:\ i = j\}$. We can consider T_j as a subset of the jth copy of the lattice Z_+^n, $1 \leqslant j \leqslant N_0$, and T as a subset in their union $\bigcup_{j=1}^{N_0} Z_+^n$. Obviously every subset T_j is monotonous and, by Lemma 1, the set of its minimal elements Γ_j is finite. Therefore the set $\Gamma = \bigcup_{j=1}^{N_0} \Gamma_j$ is also finite. Let us fix, for every element $\tilde{\tau} = (\tau, i) \in T$, some representation (analogously to (1.4)) $\tau = \gamma + \delta$, $\delta \in Z_+^n$, $\gamma \in \Gamma_i$.

For every $\tilde{\gamma} = (\gamma, i) \in \Gamma$, we consider the corresponding expansion

$$z(\tilde{\gamma}) = \sum_{\tilde{\sigma} < \tilde{\gamma},\ \tilde{\sigma} \notin T} C_{\tilde{\sigma}}^{\tilde{\gamma}} z(\tilde{\sigma}) + P(\tilde{\gamma}), \quad P(\tilde{\gamma}) \in M.$$

The system $\pi = \{P(\tilde{\gamma}),\ \tilde{\gamma} \in \Gamma\}$ generate the module M.

We shall not give all constructions, which have to be done analogously to the constructions in Sections 1 and 2. Instead of this we shall only remark that they imply the following statement:

PROPOSITION 21. *Under the assumptions and notations of this section there exist continuous linear operators*

$$R_0:\ H(C^n)^{N_0} \to H(C^n)^{N_0},$$

and

$$R_{\tilde{\gamma}}:\ H(C^n)^{N_0} \to H(C^n, L(N_0, C)), \quad \tilde{\gamma} \in \Gamma,$$

where $H(C^n, L(N_0, C))$ *is the space of all holomorphic functions on* C^n *with values in the space of all complex* $(N_0 \times N_0)$*-matrices such that*

(a) $f = R_0 f + \sum_{\tilde{\gamma} \in \Gamma} (R_{\tilde{\gamma}} f) P_{\tilde{\gamma}}$ *for all* $f \in H(C^n)^{N_0}$;

(b) $\operatorname{Im} R_0 = H_S$, *where* H_S *is the closed linear hull of the system* $\{z(\alpha, i):\ (\alpha, i) \in S\}$, $S = \left(\bigcup_{n=1}^{N_0} Z_+^n\right) \setminus T$; $\operatorname{Ker} R_0 = M(\pi) = M$, $R_0^2 = R_0$;

(c) *The operators* $R_0, R_{\tilde{\gamma}}$, $\gamma \in \Gamma$, *map the polynomial elements into polynomial elements; moreover, for their degrees we have*

$$\deg Rf \leqslant \deg f + \bar{d},$$

where $\bar{d}$ *is a constant, depending only on the module* M. (The element $f = (f_i)_1^{N_0}$ is said to be a *polynomial element* if f_i, $1 \leqslant i \leqslant N_0$, are polynomials. We define $\deg f = \max_{1 \leqslant i \leqslant N_0} \deg f_i$.)

(d) *For every operator* R $(R_0, R_{\tilde{\gamma}},\ \tilde{\gamma} \in \Gamma)$ *we have*

$$|Rf|_r \leqslant C r^{\bar{d}} \cdot |f|_{Ar} \quad \text{for all } f \in H(C^n),\ r \geqslant 1,$$

where $|f|_r = \max_i |f_i|_r$ *for* $f = (f_i)_1^{N_0}$.

Point (b) of this proposition gives the statement of Theorem 20 on the module $M = \operatorname{Im} Q$. ■

Evidently, if $M \subset H(C^n)^{N_0}$ is a polynomial module, one can construct (using Lemma 19) an exact sequence

$$(6.3) \quad \longrightarrow H(C^n)^{N_k} \xrightarrow{\tilde{Q}_k} H(C^n)^{N_{k-1}} \xrightarrow{\tilde{Q}_{k-1}} \ldots \longrightarrow H(C^n)^{N_1} \xrightarrow{\tilde{Q}_1} H(C^n)^{N_0},$$

where $M = \operatorname{Ker}\tilde{Q}_1$ and $\tilde{Q}_k$, $k = 1, 2, \ldots$, are polynomial homomorphisms. Every such sequence is called a *polynomial free resolution* of M.

COROLLARY 22. *Every polynomial free resolution of a polynomial module* $M \subset H(C^n)^{N_0}$ *splits.*

Proof. Indeed, "splits" means that there exist continuous linear operators S_k, which are right-hand-inverse to $\tilde{Q}_k$, $k = 1, 2, \ldots$, i.e. $\tilde{Q}_k S_k = 1_{\operatorname{Im}\tilde{Q}_k}$. Since $\operatorname{Im}\tilde{Q}_k$, $\operatorname{Ker}\tilde{Q}_k$, $k = 1, 2, \ldots$, are complemented subspaces, one can easily construct such operators.

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Corrigendum and addendum to the paper "A simple diophantine condition in harmonic analysis"

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by

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1. Lemma 2.3 in [1] is misstated and should be replaced by:

LEMMA 2.3. *Let Γ be a discrete (not necessarily countable) abelian group. Let $\{F_j\}_{j=1}^{\infty}$ be a family of finite and mutually independent sets ($0 \notin F_j$), i.e., $\mathrm{gp}(F_i)\cap\mathrm{gp}(F_j) = \{0\}$ whenever $i \neq j$. Then, $\{F_j\}$ is a* sup-*norm partition for $\bigcup_j F_j$.*

Victimized by the misstatement of Lemma 2.3, the proof of Theorem C contains an error: We can conclude only that the S_N's are independent in the sense that whenever $\gamma_i \in S_N$, $i = 1, \ldots, k$, $N_i \neq N_{i'}$, if $i \neq i'$, then $\{\gamma_i\}_{i=1}^{k}$ is an independent set. But, we cannot conclude that $\mathrm{gp}(S_N)\cap\mathrm{gp}(S_M) = \{0\}$ whenever $N \neq M$, and therefore we are unable to apply the (correctly stated) Lemma 2.3. We are unable to supply a correct proof of Theorem C. The above error does not affect the main results of the paper.

2. Our diophantine condition is necessarily satisfied by $E = \bigcup F_j$, where $\{F_j\}$ is as in Lemma 2.3: Without loss of generality, we assume that $\bigcup F_j \subset \oplus\Gamma_j = \Gamma$, where $\Gamma_j = \mathrm{gp}(F_j)$ and $\Gamma_j^{\wedge} = G_j$. Let D_j, as usual be a dense countable subgroup of G_j, and write $D = \oplus D_j$, which is, then, a dense countable subgroup of $\otimes G_j = \Gamma^{\wedge}$. The proof of the following proposition is a routine verification.

PROPOSITION. *$\Phi_D(\bigcup F_j)$ accumulates precisely at* 0 (Φ_D: $\oplus\Gamma_j \to \otimes D_j^{\wedge}$).

Again, as at the end of [1], we note that the independence condition in the above proposition is sharp in the following sense: A sequence of disjoint and mutually lacunary blocks of integers, $\{F_j\}$, can be constructed so that $\Phi_D(\bigcup F_j)$ is dense in $\hat{D}$, for all $D \leqslant \Gamma$. To see this, we mimic the construction at the end of [1], and add the requirement that $\|h_j\|_A = 1$. It then follows (see Lemma 1.2 in [2]) that $\bigcup \mathrm{spect}\, h_j$ is dense in $\tilde{\boldsymbol{Z}}$, the Bohr compactification of $\boldsymbol{Z}$. Our claim now follows from the observation that if $E \subset \boldsymbol{Z}$ is dense in $\tilde{\boldsymbol{Z}}$, then $\overline{\Phi_D(E)} = \hat{D}$, for all $D \leqslant \Gamma$.