for all such \( h \in H \). Taking into account (11) we see that \( A \) is an extension of \( -D^2 \). Finally observe that in this case

\[
Eh = J^*h = t \int \frac{h(s)ds}{s} - \int \frac{h(s)ds}{s} \delta(s).
\]

**Example 3.** Take \( X = C^1([0,1]) \), \( Y = X' \), \( H = C([0,1]) \), and

\[
E: H \to V
\]

defined by

\[
Eh(t) = \varphi(h) + \int_0^t h(s)ds,
\]

where \( \varphi \in H' \) is an arbitrary, fixed functional.

If we put \( a(x,f) = f(x) \) for any \( x \in X \), \( f \in X' = Y \), then \( a'(f,x) = f(x) \) for any \( f \in X' = Y \) and \( x \in X' \), and both forms \( a \) and \( a' \) are bounded and coercive; hence Theorems 1 and 2 apply.

Observe that \( DE = fD \), where \( D: V \to R \) is the bounded derivative operator; hence \( R(E') = H' \), and Theorem 4 applies as well. This means that \( A \) is the extension of \( E^{-1} \) over \( U \subset X' \).

We have \( E^{-1} = D(R(E)) \).

It is easy to see that \( E(R(E)) \subset V \) is the set of all \( v \in V \) such that the following boundary condition is satisfied:

\[
\psi(0) - \varphi(D_v).
\]

Indeed, if \( v \in E(R(E)) \), then \( v(t) = \varphi(h) + \int_0^t h(s)ds \) and \( \psi(0) = \varphi(h) \), and since \( Dv = h \), (18) holds.

On the other hand, (18) holds for some \( v \in V \), then \( h = Dv \), \( v \in V \) and

\[
v(t) = v(0) + \int_0^t h(s)ds = \varphi(\psi + \int_0^t h(s)ds) = Eh.
\]

We can consider \( E^{-1} \) as the operator \( D \) over \( E(R(E)) \subset V \) with the boundary condition (18).

**References**


* Received October 35, 1977
* Revised version January 3, 1978

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**Maximal operators defined by Fourier multipliers**

by

**Carlos E. Kenig**

and

**Peter A. Tomas**

(Chicago, Ill.)

**Abstract.** The authors develop a linearization for maximal operators defined through Fourier multipliers, and establish for such operators transplantation and restriction theorems. Applications are discussed.

**Introduction.** Let \( \lambda \) be an \( L^p(\mathbb{R}^n) \) function; define for each real number \( R > 0 \) an operator \( T_R \) on \( L^p(\mathbb{R}^n) \) by \( \hat{T}_R(f) = \hat{f}(R) \) and \( \hat{T}_R(f) \) on \( L^p(\mathbb{T}^n) \) by \( \hat{T}_R(f) = \lambda(R)\langle f, \hat{f} \rangle \). We say \( \lambda \) is \( p \)-maximal on \( \mathbb{R}^n \) (or weak \( p \)-maximal on \( \mathbb{R}^n \)) if the operator \( T^p \) defined by \( T^p(f) = \sup R^p \hat{T}_R(f) \) is bounded (or weakly bounded) on \( L^p(\mathbb{R}^n) \); similarly for \( T^p \) on \( L^p(\mathbb{T}^n) \).

The purpose of this note is to establish for \( p \)-maximal operators results on transplantation between \( T_0 \) on \( \mathbb{R}^n \) and \( T_0 \) on \( \mathbb{T}^n \), and restriction of \( T_0 \) and \( T_0 \) to subspaces. These results are similar to those of de Leeuw [3] for Fourier multipliers. The study of such transplantations was initiated by A. P. Calderón [1] and by Coifman and Weiss [2].

The authors wish to express their gratitude to Antonio Cordoba, whose work has inspired us. We also thank R. Latter for his helpful comments.

**1. A linearization.**

**Lemma.** Fix \( p, 1 < p < \infty \). The function \( \lambda \) is \( p \)-maximal if and only if

\[
\left\| \sum_k T_{R_k} f_k \right\|_{L^p} \leq c \left\| \sum_k f_k \right\|_{L^p}
\]

uniformly in all sequences of positive reals \( \{R_k\} \).

**Proof.** Define the Banach space \( L^p(G, \mathcal{F}(\mathbb{R}^n)) \) for \( G = \mathbb{R}^n \) or \( G = \mathbb{T}^n \), as the collection of all sequences of \( L^p(G) \) functions \( \{f_k\} \) such that the norm \( \sup_k \|f_k\|_p \) is finite. It is clear that \( \lambda \) is \( p \)-maximal if and only if the linear

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* The first named author is a Victor J. Andrew Fellow at the University of Chicago; the second named author was supported by the National Science Foundation.
operator $T: L^p(G) \rightarrow L^p(G, l^\infty(Z^+))$, defined for each sequence $(R_k)$ by $T(f) = (R_k f_k)$, are operators uniformly bounded for all sequences $(R_k)$. The operator $T$ is bounded if and only if $T^*: L^p(G, l^\infty(Z^+)) \rightarrow L^p(G)$ defined by $T^*(f) = \sum \sum T_{R_k} f_k$ is bounded, that is,

$$\left\| \sum T_{R_k} f_k \right\|_{L^p} \leq c \sum \left\| f_k \right\|_{L^p}.$$ 

Remark. Similar results hold for $\hat{T}$ and also for $\lambda$ weak $p$-maximal, and also in the case that $T^*$ is bounded from $H^\infty(G)$ to $L^\infty(G)$. In the latter cases, the linearization takes the form

$$\left\| \sum T_{R_k} f_k \right\|_{L^{p,1}} \leq c \sum \left\| f_k \right\|_{L^{p,1}}$$

or

$$\left\| \sum T_{R_k} f_k \right\|_{H^\infty} \leq c \sum \left\| f_k \right\|_{H^\infty},$$

2. Transplantation. A function $\lambda$ is regulated if every point of $\mathbb{R}^n$ is a Lebesgue point of $1/\lambda$. For regulated $\lambda$, the multiplier properties of continuous approximations to $\lambda$. It is clear that the techniques of [3] for treating regulated multipliers extend to $p$-maximal or weak $p$-maximal $\lambda$; in the proofs which follow, we may therefore assume $\lambda$ is continuous if it is regulated.

Theorem 1. Let $\lambda$ be a regulated $L^\infty(\mathbb{R}^n)$ function. Fix $p$ with $1 < p < \infty$. Then $\lambda$ is $p$-maximal or weak $p$-maximal on $\mathbb{R}^n$ if and only if $\lambda$ is $p$-maximal or weak $p$-maximal on $T^*$. If $\lambda$ is $p$-maximal or weak $p$-maximal on $T^*$, then it is $p$-maximal or weak $p$-maximal on $\mathbb{R}^n$. If $\lambda$ is $p$-maximal or weak $p$-maximal on $\mathbb{R}^n$, then it is $p$-maximal or weak $p$-maximal on $T^*$.

Proof. We establish the result only for the weak $p$-maximal case, as the proof in the $p$-maximal case is similar. From the linearization of Section 1, it suffices to show that the inequality

$$\left\| \sum T_{R_k} f_k \right\|_{L^{p,1}} \leq c \sum \left\| f_k \right\|_{L^{p,1}}$$

uniformly in all sequences $(R_k)$ if only if

$$\left\| \sum T_{R_k} f_k \right\|_{L^{p,1}} \leq c \sum \left\| f_k \right\|_{L^{p,1}}$$

uniformly in all sequences $(R_k)$. Assume (2) is valid for all $(R_k)$. Then it suffices to consider finite sequences $(f_k)$, where each $f_k$ is in $L^\infty(\mathbb{R}^n)$. Let $f_{x,\delta}(x) = e^{-\delta x} f(x - \delta x)$ and $f_{x,\delta}(x) = \sum f_{x,\delta}(x + w)\delta^w$, where the sum extends over the lattice $\mathbb{Z}^n$. For each $x$ in $\mathbb{R}^n$,

$$\lim_{\delta \to 0} e^{-\delta x} \sum T_{R_k} f_{x,\delta}(x) = \sum T_{R_k} f_k(x)$$

(see [7], p. 266). Choose $h$ in $C_0^\infty(\mathbb{R}^n)$ with $\left\| \hat{h} \right\|_{L^\infty} = 1$; define $h'(x) = h(e^{-\delta x})$ and construct $h'$. Then

$$\int \sum_{\mathbb{R}^n} e^{i x \cdot \xi} \hat{T}_{R_k} f_{x,\delta}(x) h'(x) dx = \int \sum \sum_{\mathbb{R}^n} e^{i x \cdot \xi} \hat{T}_{R_k} f_k(x) h'(x) dx,$$
Proof. By Theorem 1, it suffices to show \( \lambda \) is \( p \)-maximal or weak \( p \)-maximal for \( T^{n-m} \). We linearize the problem as above; for the \( p \)-maximal case the techniques of de Leeuw \([3]\) apply to the linearised problem. In the weak case, it suffices to show that a function \( g(\theta) \) on \( T^{n-m} \), extended to \( g(\theta, \phi) \) on \( T^n \) by \( g(\theta, \phi) = g(\theta) \), enjoys the property \( \|g(\theta, \phi)\|_{L^p_{\theta, \phi}} = \|g(\theta)\|_{L^p_{\theta}} \), which is a triviality as \( T^n \) is compact. We remark that the proof establishes an analogue of Theorem 2 for \( T^n \).

3. Applications.

(1) On \( T^n \), the finiteness almost everywhere of \( T^nf \) for all \( f \) in \( L^p \) \( 1 \leq p \leq 2 \) implies that \( \lambda \) is weak \( p \)-maximal; see Stein \([6]\). If we let \( \lambda(x) = 1 \) when \( |x| < 1 \) and zero otherwise, the pointwise summability of Fourier series on \( L^p(T^n) \) by spherical means is equivalent to the weak \( p \)-maximality of \( \lambda \) on \( T^n \), which by Theorem 1 is equivalent to the weak \( p \)-maximality of \( \lambda \) on \( R^n \). If \( n = 1 \), that \( \lambda \) is \( p \)-maximal on \( T \) is a deep result of Carleson and Hunt \([2]\); it is folk result that their methods apply to \( R^1 \), while the results of this paper show the transplantation is trivial. But in higher dimensions, no methods have been developed even to compute \( \lambda \) on \( T^n \), whereas on \( R^n \) it is relatively simple to show \( \lambda(x) = \sum_{k=0}^{\infty} (2\pi |x|)^{-k} \). Thus the almost everywhere summability of spherical means for \( L^p(T^n) \) is equivalent to the weak \( 2 \)-maximality of \( \lambda \) on \( R^n \).

(2) Results similar to the above may be established for suprema of operators \( T_\lambda \) defined by \( T_\lambda f(\xi) = \lambda(\xi) |\xi|^m f(\xi) \). If \( \lambda \) is as in the preceding remark, the inequality

\[
\|\sup_{k} |T_\lambda f(\xi)|\|_{L^q_{\xi}} \leq c \|f\|_{L^p}
\]

is valid for \( T \) (see \([8]\)). The methods of this paper allow the trans- plantation of this inequality to \( R \). In higher dimensions, these methods show \( \lambda \) is weak \( p \)-maximal on \( T^n \) if and only if \( p = 2 \), as C. Fefferman \([4]\) has shown \( \lambda \) is not \( p \)-maximal on \( R^n \).

References


Received November 7, 1977 (1302)