i.e.

$$h = -D^2 J^* h$$

for all such $h \in H$. Taking into account (11) we see that A is an extension of $-D^2$. Finally observe that in this case

$$Eh = J^*h = t\int_0^1\int_0^s h(\tau)\,d\tau\,ds - \int_0^t\int_0^s h(\tau)\,d\tau\,ds. \tag{17}$$

Example 3. Take $X = C^1([0,1])$, $Y = X'$, $H = C([0,1])$, and $E\colon H \to V$ defined by

$$Eh(t) = \varphi(h) + \int_0^t h(s)\,ds,$$

where $\varphi \in H'$ is an arbitrary, fixed functional.

If we put $a(x,f) = f(x)$ for any $x \in X$, $f \in X' = Y$, then $a'(f,z) = z(f)$ for any $f \in X' = Y$ and $z \in X''$, and both forms a and a' are bounded and coercive; hence Theorems 1 and 2 apply.

Observe that $DE = I_H$, where $D\colon V \to H$ is the bounded derivative operator; hence $R(E') = H'$, and Theorem 4 applies as well. This means that A is the extension of E^{-1} over $U \subset X''$.

We have $E^{-1} = D|R(E)$.

It is easy to see that $R(E) \subset V$ is the set of all $v \in V$ such that the following boundary condition is satisfied:

$$v(0) = \varphi(Dv). \tag{18}$$

Indeed, if $v \in R(E)$, then $v(t) = \varphi(h) + \int_0^t h(s)\,ds$ and $v(0) = \varphi(h)$, and since $D^0 v = h$, (18) holds.

If, on the other hand, (18) holds for some $v \in V$, then $h = Dv$, $v \in V$ and

$$v(t) = v(0) + \int_0^t h(s)\,ds = \varphi(h) + \int_0^1 h(s)\,ds = Eh.$$

We can consider E^{-1} as the operator D over $R(E) \subset V$ with the boundary condition (18).

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Maximal operators defined by Fourier multipliers

by

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Abstract. The authors develop a linearization for maximal operators defined through Fourier multipliers, and establish for such operators transplantation and restriction theorems. Applications are discussed.

Introduction. Let λ be an $L^\infty(\boldsymbol{R}^n)$ function; define for each real number $R > 0$ an operator T_R on $L^2(\boldsymbol{R}^n)$ by $\widehat{T_R f}(\xi) = \lambda(\xi/R)\hat{f}(\xi)$ and $\tilde{T}_R$ on $L^2(\boldsymbol{T}^m)$ by $\widehat{\tilde{T}_R f}(n) = \lambda(n/R)\hat{f}(n)$. We say λ is *p-maximal* on $\boldsymbol{R}^n$ (or *weak p-maximal* on $\boldsymbol{R}^n$) if the operator T^* defined by $T^*f(x) = \sup_{R>0} |T_R f(x)|$ is bounded (or weakly bounded) on $L^p(\boldsymbol{R}^n)$; similarly for $\tilde{T}^*$ on $L^p(\boldsymbol{T}^m)$. The purpose of this note is to establish for p-maximal operators results on transplantation between T^* on $\boldsymbol{R}^n$ and $\tilde{T}^*$ on $\boldsymbol{T}^m$, and restriction of T^* and $\tilde{T}^*$ to subspaces. These results are similar to those of de Leeuw [3] for Fourier multipliers. The study of such transplantations was initiated by A. P. Calderòn [1] and by Coifman and Weiss [2].

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1. A linearization.

Lemma. *Fix p, $1 < p < \infty$. The function λ is p-maximal if and only if*

$$\Big\|\sum_k T_{R_k} f_k\Big\|_{p'} \leqslant c\Big\|\sum_k |f_k|\Big\|_{p'}$$

uniformly in all sequences of positive reals $\{R_k\}$.

Proof. Define the Banach space $L^p(G, l^\infty(\boldsymbol{Z}^+))$ for $G = \boldsymbol{R}^n$ or $G = \boldsymbol{T}^n$, as the collection of all sequences of $L^p(G)$ functions $\{f_k\}$ such that the norm $\|\sup_k |f_k|\|_p$ is finite. It is clear that λ is p-maximal if and only if the linear

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operator Γ: $L^p(G) \to L^p(G, l^\infty(\mathbf{Z}^+))$, defined for each sequence $\{R_k\}$ by $\Gamma(f) = \{T_{R_k} f\}$, are operators uniformly bounded for all sequences $\{R_k\}$. The operator Γ is bounded if and only if Γ^*: $L^{p'}(G, l'(\mathbf{Z}^+)) \to L^{p'}(G)$ defined by $\Gamma^*(\{f_k\}) = \sum_k T_{R_k} f_k(x)$ is bounded, that is,

$$\Big\| \sum T_{R_k} f_k \Big\|_{p'} \leqslant c \Big\| \sum |f_k| \Big\|_{p'}.$$

Remark. Similar results hold for $\tilde{T}^*$, and also for λ weak p-maximal, and also in the case that T^* is bounded from $H'(G)$ to $L\log^+ L$. In the latter cases, the linearization takes the form

$$\Big\| \sum T_{R_k} f_k \Big\|_{(p',p')} \leqslant c_p \Big\| \sum |f_k| \Big\|_{(p',1)}$$

or

$$\Big\| \sum T_{R_k} f_k \Big\|_{\mathrm{BMO}} \leqslant c \Big\| \exp\big(\sum |f_k|\big) \Big\|_1.$$

2. Transplantation. A function λ is *regulated* if every point of $\mathbf{R}^n$ is a Lebesgue point of λ [3]. For regulated λ, the multiplier properties of λ may be deduced from the multiplier properties of continuous approximations to λ. It is clear that the techniques of [3] for treating regulated multipliers extend to p-maximal or weak p-maximal λ; in the proofs which follow, we may therefore assume λ is continuous if it is regulated.

THEOREM 1. *Let λ be a regulated $L^\infty(\mathbf{R}^n)$ function. Fix p with $1 < p < \infty$. Then λ is p-maximal or weak p-maximal on $\mathbf{R}^n$ if and only if λ is p-maximal or weak p-maximal on $\mathbf{T}^n$.*

Proof. We establish the result only for the weak p-maximal case, as the proof in the p-maximal case is similar. From the linearization of Section 1, it suffices to show that the inequality

$$(1) \qquad \Big\| \sum T_{R_k} f_k \Big\|_{(p'p')} \leqslant c \Big\| \sum |f_k| \Big\|_{(p',1)}$$

uniformly in all sequences $\{R_k\}$ is equivalent to

$$(2) \qquad \Big\| \sum \tilde{T}_{R_k} f_k \Big\|_{(p',p')} \leqslant c \Big\| \sum |f_k| \Big\|_{(p',1)}$$

uniformly in all $\{R_k\}$. Assume (2) is valid for all $\{g_k\}$ in $L^{(p',1)}(\mathbf{T}^n, l'(\mathbf{Z}^+))$. To prove (1) it suffices to consider finite sequences $\{f_k\}$, where each f_k is in $C_0^\infty(\mathbf{R}^n)$. Let $f_{k,\varepsilon}(x) = \varepsilon^{-n} f_k(\varepsilon^{-1}x)$ and $\tilde{f}_{k,\varepsilon}(x) = \sum f_{k,\varepsilon}(x+m)$, where the sum extends over the lattice $\mathbf{Z}^n$. For each x in $\mathbf{R}^n$,

$$\lim_{\varepsilon\to 0} \varepsilon^n \sum_k \tilde{T}_{R_k/\varepsilon} \tilde{f}_{k,\varepsilon}(\varepsilon x) = \sum_k T_{R_k} f_k(x)$$

(see [7], p. 266). Choose h in $C_0^\infty(\mathbf{R}^n)$ with $\|h\|_{(p,p)} = 1$; define $h^\varepsilon(x) = h(\varepsilon^{-1}x)$ and construct $\tilde{h}^\varepsilon$. Then

$$\int_{\mathbf{R}^n} \varepsilon^n \sum_k \tilde{T}_{R_k/\varepsilon} \tilde{f}_{k,\varepsilon}(\varepsilon x) h(x)\,dx = \int_{\mathbf{T}^n} \Big(\sum_k \tilde{T}_{R_k/\varepsilon} \tilde{f}_{k,\varepsilon}\Big)(x) \tilde{h}^\varepsilon(x)\,dx,$$

and employing (2),

$$\Big\| \sum_k T_{R_k} f_k \Big\|_{(p',p')} \leqslant \liminf_{\varepsilon\to 0} \Big\| \varepsilon^n \sum_k \tilde{T}_{R_k/\varepsilon} \tilde{f}_{k,\varepsilon}(\varepsilon x) \Big\|_{(p',p')}$$
$$\leqslant \liminf_{\varepsilon\to 0} \sup_{\|h\|_{(p,p)}=1} \Big\| \sum_k |\tilde{f}_{k,\varepsilon}| \Big\|_{(p',1)} \|\tilde{h}^\varepsilon\|_{(p,p)} = \Big\| \sum_k |f_k| \Big\|_{(p',1)}.$$

We now assume (1) holds for all $\{f_k\}$ in $L^{(p',1)}(\mathbf{R}^n, l'(\mathbf{Z}^+))$. To establish (2), it suffices to consider finite sequences $\{g_k\}$ where each g_k is a trigonometric polynomial. In the one-dimensional case, we proceed as follows. Let $\omega_\delta(y) = e^{-\pi\delta|y|^2}$ for $\delta > 0$ and y in $\mathbf{R}^n$. If Q is a trigonometric polynomial and $\alpha = 1/p'$, $\beta = 1/p$, then

$$(3) \qquad \lim_{\varepsilon\to 0} \varepsilon^{1/2} \int_{\mathbf{R}} \sum_k T_k(g_k \omega_{\varepsilon\alpha})(x) \bar{Q}(x) \omega_{\varepsilon\beta}(x)\,dx = \int_{\mathbf{T}} \sum_k \tilde{T}_{R_k} g_k(x) \bar{Q}(x)\,dx,$$

as in Stein and Weiss [7], p. 261. Equation (1) shows that (3) is majorized by

$$c \Big\| (\varepsilon^{\alpha/2} \omega_{\varepsilon\alpha}) \sum_k |g_k| \Big\|_{(p',p')} \|(\varepsilon^{\beta/2} \omega_{\varepsilon\beta}) \bar{Q}\|_{(p,1)}.$$

As ε tends to zero, the first factor tends to $\|\sum |g_k|\|_{(p',p')}$. To complete the proof it suffices to show that the second factor is majorized by $c_p \|Q\|_{(p,1)}$. This follows by reducing the problem to the case where Q is the characteristic function of a finite union of intervals. Then the non-increasing rearrangement of $(\varepsilon^{\beta/2} \omega_{\varepsilon\beta})Q$ may be computed explicitly. In higher dimensions these computations are cumbersome, and we employ the results of Coifman and Weiss [2]. They establish the result in the case that each $[\lambda(\xi/R_k)]^\wedge$ has compact support. To reduce the problem to this case, choose a sequence of functions φ_ε as in Lemma 3.4 of [2], so that $\varphi_\varepsilon \geqslant 0$, $\|\varphi_\varepsilon\|_1 = 1$ and define operators $S_{k,\varepsilon}$ by $\widehat{S_{k,\varepsilon} f}(\xi) = \varphi_\varepsilon * (\lambda(y/R_k)^\wedge)(\xi) \hat{f}(\xi)$. The $S_{k,\varepsilon}$ are then given by convolution with compactly supported kernels, and it is elementary to establish the following

LEMMA. *Let λ be a regulated $L^\infty(\mathbf{R}^n)$ function. λ is weak p-maximal if and only if*

$$\Big\| \sum S_{k,\varepsilon} f_k \Big\|_{(p',p')} \leqslant c \Big\| \sum |f_k| \Big\|_{(p',1)},$$

where c is independent of φ.

The result now follows from the work of Coifman and Weiss [2].

Remark. These methods also show that $\|\tilde{T}^* f\|_{L\log^+ L} \leqslant c \|f\|_{H'}$ on $\mathbf{T}^n$ implies the corresponding inequality on $\mathbf{R}^n$.

THEOREM 2. *Let λ be a regulated function in $L^\infty(\mathbf{R}^n)$; fix $1 < p < \infty$. For each y in $\mathbf{R}^m$, define λ_y in $L^\infty(\mathbf{R}^{n-m})$ by $\lambda_y(x) = \lambda(x, y)$. If λ is p-maximal or weak p-maximal on $\mathbf{R}^n$, then for each y in $\mathbf{R}^m$. λ_y is p-maximal or weak p-maximal on $\mathbf{R}^{n-m}$.*

Proof. By Theorem 1, it suffices to show λ_y is p-maximal or weak p-maximal for $\boldsymbol{T}^{m-m}$. We linearize the problem as above; for the p-maximal case the techniques of de Leeuw [3] apply to the linearized problem. In the weak case, it suffices to show that a function $g(\theta)$ on $\boldsymbol{T}^{m-m}$, extended to $g(\theta,\varphi)$ on $\boldsymbol{T}^n$ by $\tilde{g}(\theta,\varphi) = g(\theta)$, enjoys the property $\|\tilde{g}(\theta,\varphi)\|_{(p,q)} = \|g(\theta)\|_{(p,q)}$, which is a triviality as $\boldsymbol{T}^m$ is compact. We remark that the proof establishes an analogue of Theorem 2 for $\boldsymbol{T}^n$.

3. Applications.

(1) On $\boldsymbol{T}^n$, the finiteness almost everywhere of $\tilde{T}^*f$ for all f in L^p $1 \leqslant p \leqslant 2$ implies that λ is weak p-maximal; see Stein [6]. If we let $\lambda(x) = 1$ when $|x| < 1$ and zero otherwise, the pointwise summability of Fourier series on $L^p(\boldsymbol{T}^n)$ by spherical means is equivalent to the weak p-maximality of λ on $\boldsymbol{T}^n$, which by Theorem 1 is equivalent to the weak p-maximality of λ on $\boldsymbol{R}^n$. If $n = 1$, that λ is p-maximal on $\boldsymbol{T}$ is a deep result of Carleson and Hunt [5]; it is folk result that their methods apply to $\boldsymbol{R}^1$, while the results of this paper show the transplantation is trivial. But in higher dimensions, no methods have been developed even to compute $\hat{\lambda}$ on $\boldsymbol{T}^n$, whereas on $\boldsymbol{R}^n$ it is relatively simple to show $\hat{\lambda}(x) = J_{n/2}(2\pi|x|)/|x|^{n/2}$. Thus the almost everywhere summability of spherical means for $L^2(\boldsymbol{T}^n)$ is equivalent to the weak 2-maximality of λ on $\boldsymbol{R}^n$.

(2) Results similar to the above may be established for suprema of operators T_k defined by $\widehat{T_k f}(\xi) = \lambda(\xi/2^k)\hat{f}(\xi)$. If λ is as in the preceeding remark, the inequality

$$\|\sup_k |T_k f(x)|\|_{L\log^+L} \leqslant c\|f\|_{H'}$$

is valid for $\boldsymbol{T}$ (see [8]). The methods of this paper allow the transplantation of this inequality to $\boldsymbol{R}$. In higher dimensions, these methods show λ is weak p-maximal on $\boldsymbol{T}^n$ if and only if $p = 2$, as C. Fefferman [4] has shown λ is not p-maximal on $\boldsymbol{R}^n$.

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