Donc $\|\bar{z}_n - x_{n,\lambda}\|_n$ tend vers zéro (si $\lambda \to \infty$) uniformément par rapport à n — naturel, et la suite $\{\bar{z}_n\}$ est bien concordante. De plus, $\Psi_n(\bar{z}_n) = 0$, d'où $\Psi(\bar{z}) = \Psi(\{\bar{z}_n\}) = \lim_{n\to\infty} \Psi_n(\bar{z}_n) = 0$, donc $\bar{z} \in Z$. Soit z un autre élément de Z, $z = \{z_n\}$. On peut admettre, d'après le lemme (a), que $\Psi_n(z_n) = 0$, d'où il résulte, en vertu de 3.4.11, que $\Phi(z) = \lim_n \Phi_n(z_n) \geqslant \lim_n \Phi_n(\bar{z}_n) = \Phi(\bar{z})$. ∎

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Concerning some version of the Lax–Milgram Lemma in normed spaces

by

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Abstract. This paper contains a generalization of classical Lax–Milgram Lemma for the case of a complex bilinear form defined on a pair of linear normed spaces.

Theorems concerning existence and (local) unicity of solution for the subsequent variational and operator equations are given, as well as some information concerning the operator defined by the bilinear form (in the simplest case) is joint.

The Lax–Milgram Lemma is an useful and elegant tool in the theory of differential equations of elliptic type. It was discussed by many authors.

The classical version of this lemma tells that if a is a bounded and coercive bilinear form defined on a real Hilbert space V and L is a bounded linear functional over V, then there exists exactly one $v_0 \in V$ such that $a(v_0, v) = L(v)$ for any $v \in V$ ([3], pp. 92–95).

A theorem of similar kind for a pair of Hilbert spaces, especially convenient in applications to the theory of differential equations can be found in Lions' book [2]. The purpose of the present paper is to generalize ideas of [2].

Let X, Y be linear, complex, normed spaces. We denote by X', Y' their strong duals. If not necessary, we shall not distinguish in notation the norms in different spaces. We shall also equivalently use both notations for duality pairing for any $x \in X$ and $f \in Y'$:

$$f(x) = \langle x, f\rangle_X.$$

The following notation will be used in the text:

$\boldsymbol{R}$ — for real line,
$\boldsymbol{C}$ — for complex plane,
$D(A)$ — for the domain of the operator A,
$R(A)$ — for the range of the operator A,
I_X — for the identity operator over X;

Primes will be used for duality and stars for adjointness.

Consider a bilinear form

$$a\colon X \times Y \to \boldsymbol{C}$$

which we shall also call shortly the 'form'.

DEFINITION 1. (1) The form a is *X-bounded* iff

$$a_y = \sup_{\substack{\|x\|=1 \\ x\in X}} |a(x, y)| < \infty$$

for any $y \in Y$.

(2) The form a is *bounded* iff

$$\|a\| = \sup_{\substack{\|x\|=\|y\|=1 \\ x\in X, y\in Y}} |a(x, y)| < \infty.$$

LEMMA 1. *If a is X-bounded, then there exists a linear operator $F: Y \to X'$, in general unbounded, such that*

$$a(x, y) = \langle x, Fy\rangle_X$$

for any $x \in X$ and $y \in Y$.

Proof. For any fixed $y \in Y$, $a(\cdot, y)$ is the linear bounded functional over X. Hence

$$a(x, y) = \langle x, Fy\rangle_X$$

for any $x \in X$, where $F: Y \to X'$. From linearity of a follows linearity of F. ■

DEFINITION 2. Let $a: X \times Y \to C$ be an X-bounded form such that $a(x, y) = \langle x, Fy\rangle_X$ for any $x \in X$ and $y \in Y$. Put

$$a'(y, z) = \langle y, F'z\rangle_Y$$

for any $y \in Y$ and $z \in Z = D(F') \subset X''$, where $F': D(F') \to Y'$ is the dual operator (see [1], pp. 49–50). The form a' is called *dual with respect to the form a.*

LEMMA 2. *Let a be a bounded form, $a(x, y) = \langle x, Fy\rangle_X$. Then*

$$\|F\| = \|F'\| = \|a\| = \|a'\|,$$

and

$$Z = X''.$$

Proof. If a is bounded, then it is X-bounded and for any $x \in X$ $y \in Y$, $\|x\| = \|y\| = 1$,

$$|a(x, y)| = |\langle x, Fy\rangle_X| \leqslant \|x\| \, \|Fy\| \leqslant \sup_{\|y\|=1} \|Fy\| = \|F\|$$

and hence

$$\|a\| = \sup_{\substack{\|x\|=\|y\|=1 \\ x\in X, y\in Y}} |a(x, y)| \leqslant \|F\|.$$

Suppose $\|a\| < r < \|F\|$. It follows that there exists $y \in Y$, $\|y\| = 1$ such that

$$\|a\| < r < \|Fy\| = \sup_{\|x\|=1} |\langle x, Fy\rangle_X| \leqslant \|a\|,$$

which is impossible. Since in the considered case $F: Y \to X'$ is bounded, hence $F': X'' \to Y'$ is bounded too, and $\|F'\| = \|F\|$. Moreover, $Z = D(F') = X''$,

$$a'(y, z) = \langle y, F'z\rangle_Y$$

and by the same argument $\|a'\| = \|F'\| = \|F\| = \|a\|$. ■

Let $K: X \to X''$ be the canonical embedding, i.e. the linear isometry of X into its bidual X'' defined by the formula

$$(Kx)(f) = f(x)$$

for any $x \in X$ and $f \in X'$. If $a: X \times Y \to C$ is a bounded form, then

$$a(x, y) = \langle x, Fy\rangle_X = \langle Fy, Kx\rangle_{X'}.$$

Since $KX \subset X'' = D(F')$, we have

$$a(x, y) = \langle y, F'Kx\rangle_Y = a'(y, Kx).$$

This means that one can consider the dual form a' with respect to the bounded form a as some extension of the form a over the space X''.

The notion of coerciveness is in general referred to the bilinear forms in Hilbert spaces [2]. Now we need some extension of this notion.

DEFINITION 3. The form $a: X \times Y \to C$ is *coercive* iff there exist: positive constants $\beta, \gamma < \infty$;
the function $\varphi: X \to Y$ such that

$$\|\varphi(x)\| \leqslant \beta \|x\|$$

and

$$\gamma \|x\|^2 \leqslant |a(x, \varphi(x))|$$

for any $x \in X$.

EXAMPLE 1. A bilinear form $a: X \times X' \to C$, defined by $a(x, f) = \langle x, f\rangle_X$ for any $x \in X$ and $f \in X'$ is clearly bounded and coercive. Indeed, by the Hahn-Banach Theorem, for any $x \in X$ there exists $f_x \in X'$ such that $\|f_x\| = 1$ and $f_x(x) = \|x\|$. If we take

$$\varphi(x) = \|x\| f_x,$$

then

$$\|\varphi(x)\| = \|x\|$$

and

$$\|x\|^2 = \langle x, f_x\rangle_X \cdot \|x\| = \langle x, \varphi(x)\rangle_X = |a(x, \varphi(x))|. \ ■$$

LEMMA 3. *Let $a: X \times Y \to C$ be an X-bounded and coercive form such that $a(x, y) = \langle x, Fy\rangle_X$ for any $x \in X$ and $y \in Y$. If, for the canonical embedding $K: X \to X''$,*

$$R(K) \subset Z = D(F') \subset X''$$

holds, then the operator

$$(F'K)^{-1}\colon R(F'K) \to \begin{array}{c} X \\ \cap \\ Y' \end{array}$$

is bounded.

Proof. If a is X-bounded and coercive, then

$$\begin{aligned}\gamma\|x\|^2 &\leqslant |a(x, \varphi(x))| = |\langle x, F\varphi(x)\rangle_X| = |\langle F\varphi(x), Kx\rangle_{X'}| \\ &\leqslant \|\varphi(x)\|\cdot\|F'Kx\| \leqslant \beta\|x\|\,\|F'Kx\|\end{aligned}$$

for the constant β and γ, and the function φ from Definition 3 because $Kx \in D(F')$. Hence

$$\|F'Kx\| \geqslant \frac{\gamma}{\beta}\|x\|$$

for any $x \in X$. ∎

Corollary. *Let $a\colon X\times Y \to C$ be bounded and coercive. Then $KX \subset Z = X''$ and Lemma 3 holds.*

Lemma 4. *Let $a(x, y) = \langle x, Fy\rangle_X$. Assume that a' is coercive. Then*

$$F^{-1}\colon R(F) \to \begin{array}{c} Y \\ \cap \\ X' \end{array}$$

is bounded.

Proof. If a' is coercive, then there exists positive constants β, γ and the function $\varphi\colon Y \to Z = D(F')$ such that

$$\begin{aligned}\gamma\|y\|^2 &\leqslant |a'(y, \varphi(y))| = |\langle y, F'\varphi(y)\rangle_Y| \\ &= |\langle Fy, \varphi(y)\rangle_{X'}| \leqslant \|Fy\|\beta\|y\|,\end{aligned}$$

and hence

$$\|Fy\| \geqslant \frac{\gamma}{\beta}\|y\|$$

for any $y \in Y$. ∎

Lemma 5. *If F and F^{-1} are bounded, then the form $a(x, y) = \langle x, Fy\rangle_X$ is bounded and the form a' is bounded and coercive.*

Proof. Let F^{-1} be bounded; then for some positive constant α $\|Fy\| \geqslant \alpha\|y\|$ holds for any $y \in Y$. From the Hahn–Banach Theorem, for any $y \in Y$ there exists $z_y \in X''$, such that $\|z_y\| = 1$ and

$$\langle Fy, z_y\rangle = \|Fy\|.$$

Put

$$\varphi(y) = \|Fy\|z_y;$$

then by boundedness of F

$$\varphi\colon Y \to Z = X'',$$

$$\|\varphi(y)\| \leqslant \|F\|\,\|y\|\,\|z_y\| = \|F\|\,\|y\|$$

and

$$|a'(y, \varphi(y))| = |\langle Fy, \varphi(y)\rangle_{Y'}| = \|Fy\|^2 \geqslant \alpha^2\|y\|^2.$$

On the other hand,

$$a(x, y) = \langle x, Fy\rangle_X,$$

and by Lemma 2 $\|a\| = \|F\| < \infty$. ∎

Let now X, Y and H be normed spaces, and

$$E'\colon Y \to H'$$

a linear operator. Consider an X-bounded form

$$a\colon X\times Y \to C,$$

$$a(x, y) = \langle x, Fy\rangle_X$$

and its' dual

$$a'\colon Y\times Z \to C, \qquad Z = D(F') \subset X'',$$

$$a'(y, z) = \langle y, F'z\rangle_Y.$$

Define the subspace $U \subset Z \subset X''$ as follows: $u \in U$ iff both conditions (i) and (ii) are satisfied:

(i) $E'y = 0$ implies $a'(y, u) = 0$;

(ii) the linear functional $\Phi_u\colon R(E') \to C,$

$$\begin{array}{c} R(E') \\ \cap \\ H' \end{array}$$

$$\Phi_u(f) \underset{\mathrm{df}}{=} a'(y, u) \qquad \text{for any } f \in E'Y,\ f = E'y$$

(Φ_u is well defined because of (i)) is bounded on $R(E') \subset H'$ for the topology of H'.

Observe that the space U always exists, however it may reduce to $\{0\} \subset X''$. In this case the only Φ_0 is 0 which obviously is bounded.

At any case the functional Φ_u defined in (ii) can be extended by the Hahn–Banach Theorem over the whole space H'.

We shall denote by the same symbol Φ_u some fixed extension of this kind.

Since $\Phi_u \in H''$, for any $u \in U$, we can write

(1) $$\Phi_u(f) = \langle f, Au\rangle_{H'},$$

for any $f \in H'$, where the operator

$$A\colon \begin{array}{c} U \to H'' \\ \cap \\ X'' \end{array}$$

is not necessarily linear, nor bounded. However, for any $f = E'y$, formula (1) can be replaced by

(2) $$a'(y, u) = \langle E'y, Au\rangle_{H'},$$

that is, because of linearity of the form a', for these particular values of f, A is still linear as a function of $u \in U \subset X''$.

Observe that A is in general non-uniquely determined (as the extension of Φ_u is in general non-unique), except for the case $\overline{E'Y} = H'$.

THEOREM 1 (Existence). *Let X, Y, H be linear normed spaces, $a\colon X \times Y \to C$ an X-bounded form such that $a'\colon Y \times Z \to C$ is coercive. Let $E'\colon Y \to H'$ be linear (not necessarily bounded) operator, and $g \in D(E'') \subset H''$ be arbitrary, where E'' is the dual of E'.*

Consider the following variational equation

(3) $$a'(y, w) = \langle E'y, g\rangle_{H'} \quad \text{for any } y \in Y.$$

Then the set of all solutions w of equation (3) *is non-void and is contained in $U \subset Z \subset X''$, where U is the subspace of $Z = D(F')$ defined by* (i) *and* (ii).

If A is the operator defined by (1), *then any solution of the equation*

(4) $$Aw = g, \quad w \in U$$

is a solution of (3).

If in addition $\overline{E'Y} = H'$, then (3) *and* (4) *are equivalent.*

Proof. By X-boundedness of a,

$$a(x, y) = \langle x, Fy\rangle_X$$

and

$$a'(y, z) = \langle y, F'z\rangle_Y$$

for any $x \in X, y \in Y, z \in Z = D(F')$ where

$$F\colon Y \to X'$$

$$F'\colon \begin{array}{c} Z \to Y'. \\ \cap \\ X'' \end{array}$$

Using equation (3), we can write

$$\langle y, F'w\rangle_Y \equiv a'(y, w) = \langle E'y, g\rangle_{H'} \equiv \langle y, E''g\rangle_Y,$$

where $E''\colon D(E'') \to Y'$ is the dual of E'. Hence (3) is equivalent to the following equation

(5) $$F'w = E''g.$$

By coerciveness of a' and the Lemma 4 we have that

$$F'^{-1}\colon R(F') \to Y$$

is bounded, and hence by the Banach Theorem (see [1], p. 63, Theorem 11.4.4 (i))

$$R(F') = Y'$$

which means that (5) has always a solution $w \in Z$ (for any fixed E'). Let w be such a solution; then

$$a'(y, w) = \langle E'y, g\rangle_{H'} \quad \text{for any } y \in Y,$$

and if $E'y = 0$, then $a'(y, w) = 0$.

Hence the linear functional

$$E'y \mapsto a'(y, w) = \langle E'y, g\rangle_{H'}$$

is well defined over $R(E') \subset H'$ and bounded there for the topology of H' (its norm is equal to $\|g\|$). This means that (i) and (ii) are satisfied and $w \in U$; hence any solution of (3) is in U. By (2) we can write the following variational equation

(6) $$a'(y, w) = \langle E'y, Aw\rangle_{H'} = \langle E'y, g\rangle_{H'} \quad \text{for any } y \in Y$$

which is equivalent to equation (3).

From (6) it follows that any solution of equation (4) is a solution of (3). Moreover, if $R(E') \subset H'$ is dense in H', then obviously (3) and (4) are equivalent. ■

THEOREM 2. (Local uniqueness). *Let as before $K\colon X \to X''$ be the canonical embedding, and consider the form $a\colon X \times Y \to C$ of Theorem* 1. *Assume that $R(K) \subset Z = D(F')$ and that a is X-bounded and coercive.*

Then equation (3) *has at most one solution in the set*

$$M = U \cap R(K) \subset X''.$$

Proof. If $w = Kx$ is a solution of (3), then

$$F'Kx = E''g.$$

The local uniqueness of the solution follows from Lemma 3, because, if w is the solution of (3), then

(7) $$x = (F'K)^{-1}E''g,$$

and hence $w = K(F'K)^{-1}E''g$ is unique in M. ■

COROLLARY. (1) *Observe that if $E'\colon Y \to H'$ is bounded, then $D(E'') = H''$ and Theorem* 1 *holds for any $g \in H''$.*

Moreover, in this case E'' is bounded as well and (7) *implies that under assumptions of Theorem* 2 *the unique solution of* (3) *in the set M* (*if it exists!*) *depends continously on g* (see Lemma 3).

(2) *If $K: X \to X''$ maps onto the whole space X'' (X is reflexive!), then under the assumptions of Theorems 1 and 2 there exists exactly one solution of equation* (3).

Consider now the special case where $Y = X'$ and the form $a: X \times X' \to C$ is simply

$$a(x, f) = \langle x, f\rangle_X = f(x) \tag{8}$$

for any $x \in X$ and $f \in X'$. Clearly the dual $a': X' \times X'' \to C$ is of the form

$$a'(f, z) = \langle f, z\rangle_{X'} = z(f) \tag{9}$$

for any $f \in X'$ and $z \in X''$. Observe that both forms a and a' are bounded and coercive.

Take now two normed spaces X and H, the bounded linear operator $E': X' \to H'$ and consider in this case the operator

$$\begin{array}{l} A\colon\ U \to H'' \\ \cap \\ X'' \end{array}$$

defined by (i) and (ii) for the form a. Then we have the following

THEOREM 3. *If a is defined by* (8) *and $E': X' \to H'$ is bounded, then*

$$E''A = I_{X''}.$$

Proof. We have $a'(f, u) = \langle E'f, Au\rangle_{H'}$ for any $f \in X'$ and $u \in U \subset X''$. By (9)

$$\langle f, u\rangle_{X'} = \langle E'f, Au\rangle_{H'} = \langle f, E''Au\rangle_{X'}$$

for any $f \in X'$ and hence

$$E''Au = u$$

for any $u \in U \subset X''$. ■

LEMMA 6. *Let $E: H \to X$ be a linear and bounded operator, and take as E' the dual of E. Then for the form* (8)

$$KEH \subset U \subset X'',$$

where $K: X \to X''$ is the canonical embedding.

Proof. It is enough to verify that for $u = KEh$, where $h \in H$ is arbitrary, (i) and (ii) hold. Indeed, $E'f = 0$ implies that

$$a'(f, KEh) = \langle f, KEh\rangle_{X'} = \langle Eh, f\rangle_X = \langle h, E'f\rangle_H = 0$$

hence (i) holds.

Now if $L: H \to H''$ is the canonical embedding, then

$$E'f \mapsto a'(f, KEh) = \langle h, E'f\rangle_H = \langle E'f, Lh\rangle_{H'}$$

and this means that $\Phi_u: H' \to C$ is bounded in H' (its norm is $\|Lh\| = \|h\|$). Hence (ii) is fulfilled. ■

THEOREM 4. *Let assumptions of Lemma 6 be satisfied, and let $\overline{E'X'} = H'$. Then, for the operator defined by the form* (8) *we have*

$$L^{-1}AKE = I_H, \tag{11}$$

where $K: X \to X''$ and $L: H \to H''$ are canonical embeddings.

Proof. If $u \in U$, then for any $f \in X'$

$$a'(f, u) = \langle E'f, Au\rangle_{H'}.$$

Put $u = KEh$, $h \in H$; then, according to Lemma 6, $u \in U$ and

$$a'(f, KEh) = \langle E'f, AKEh\rangle_{H'}.$$

But from (8)

$$\begin{aligned} a'(f, KEh) &= f(Eh) = \langle Eh, f\rangle_X = \langle h, E'f\rangle_H \\ &= \langle E'f, Lh\rangle_{H'} = \langle E'f, AKEh\rangle_{H'} \end{aligned}$$

and by the density of $E'X'$ we have

$$AKE = L. \ ■$$

Comment. In the special case of the form a given by (8) Theorem 3 shows that A is simply the right inverse to E''. If we assume that the map E' is the dual of some map $E: H \to X$ with $R(E')$ dense in H', then by Theorem 4 A can be considered as an extension of E^{-1} over $U \subset X''$.

Let now $\{V, (\ )_V\}$, $\{H, (\ )_H\}$ be two real Hilbert spaces such that $J: V \subset H$, J being the linear, bounded embedding. We assume that $\overline{JV} = H$

$$\begin{tikzcd} V \arrow[r, hook, shift left, "J"] \arrow[dd, bend right=60, "K"'] & H \arrow[l, shift left, "E"] \arrow[dd, bend left=60, "L"] \\ V' \arrow[u, "S"] \arrow[r, "E'"'] & H' \\ V'' & H'' \arrow[l, "E''"] \end{tikzcd}$$

Let $S: V' \to V$ be the Riesz representation operator (defined by $S(\cdot, g)_V = g$) and let K, L be the canonical embeddings of V and H, respectively. Because of reflexiveness we put $K = I_V$. Define

$$E: H \to V,$$

$$Eh = S(h, J\cdot)_H \tag{12}$$

for any $h \in H$.

LEMMA 7. *We have*

$$E = J^*, \tag{13}$$

$$E'v' = (\cdot, JSv')_H \tag{14}$$

for any $v' \in V'$. *Moreover,* $E'V'$ *is dense in* H'.

Proof. For any $h \in H$ and $v \in V$

$$(h, Jv)_H = (J^*h, v)_V$$

hence

$$(h, J\cdot)_H = (\cdot, J^*h)_V$$

and

$$Eh = S(h, J\cdot)_H = J^*h$$

i.e. (13) holds. For any $h \in H$ and $v' \in V'$

$$\begin{aligned}\langle h, E'v'\rangle_H &= \langle Eh, v'\rangle_V = (J^*h, Sv')_V \\ &= (h, JSv')_H,\end{aligned}$$

hence (14) holds.

The density of $E'V'$ in H' follows by the Riesz Representation Theorem, because $R(J)$ is dense in H. ∎

Consider now the bounded bilinear form

$$a\colon V \times V \to \boldsymbol{R},$$

where

$$a(u, v) = \langle u, Fv\rangle_V = \langle u, FSv'\rangle_V \underset{\mathrm{df}}{=} b(u, v')$$

and $F\colon V \to V'$, $v = Sv'$, $v' \in V'$. For the form b we have:

$$b\colon V \times V' \to \boldsymbol{R}.$$

LEMMA 8. *For any* $v, z \in V$, $v = Sv'$:

$$a'(v, z) = a(z, v) \underset{\mathrm{df}}{=} b(z, v') = b'(v', z).$$

Proof. Because V is reflexive ($K = I$), by the definition of a' we have

$$\begin{aligned}a'(v, z) &= \langle v, F'z\rangle_V = \langle Fv, z\rangle_{V'} = \langle z, Fv\rangle_V \\ &= a(z, v) = b(z, v') = b'(v', z). \ \blacksquare\end{aligned}$$

COROLLARY. *a' is coercive iff a is coercive; b' is coercive iff b is coercive.*

Observe that in this case Theorems 1 and 2 both hold under the assumption that a is bounded and coercive. We get in this way the existence and uniqueness of the solution to the variational equation

$$a(u, v) = (g, Jv)_H \quad \forall v \in V \tag{15}$$

when $g \in H$ is arbitrary. In the discussed case (15) is equivalent to the equation

$$Au = g, \tag{16}$$

where

$$\begin{array}{c} A\colon U \to H'' \\ \cap \\ V \end{array}$$

is the unique operator defined by (i) and (ii) for the form a and the operator E. This is just the version of the Lax–Milgram Lemma discussed in [2].

EXAMPLE 2. Take the Sobolev spaces:

$$H = H^0(0, 1) = L^2(0, 1), \qquad V = H_0^1(0, 1);$$

then

$$(h_1, h_2)_H = \int_0^1 h_1 h_2\, dt,$$

$$(v_1, v_2)_V = \int_0^1 Dv_1 Dv_2\, dt,$$

D being the derivative operator. In this case $J\colon V \subset H$, $\overline{JV} = H$ hold.

Let E be defined by (12) and put as $a\colon V \times V \to \boldsymbol{R}$ the following bounded and coercive form:

$$a(u, v) = \int_0^1 Du\, Dv\, dt = v'(u),$$

where $v = Sv'$, $v' \in V'$. Hence Theorem 4 can be applied and we see that the operator $A\colon U \to H''$ defined by (i) and (ii) is simply an extension of $(J^*)^{-1}$ over its domain U in V.

To see what really A is, let's calculate J^* in this case. To this end take $v \in V$ arbitrary, and $h \in H$ such that $D^2 J^* h \in H$. Then

$$\begin{aligned}(Jv, h)_H &= (v, J^*h)_V = \int_0^1 Dv\, DJ^*h\, dt \\ &= v\cdot DJ^*h\Big|_0^1 - \int_0^1 v D^2 J^* h\, dt \\ &= -\int_0^1 v D^2 J^* h\, dt \\ &= -(Jv, D^2 J^* h)_H,\end{aligned}$$

i.e.

$$h = -D^2 J^* h$$

for all such $h \in H$. Taking into account (11) we see that A is an extension of $-D^2$. Finally observe that in this case

$$Eh = J^* h = t\int_0^1\int_0^s h(\tau)\,d\tau\,ds - \int_0^t\int_0^s h(\tau)\,d\tau\,ds. \tag{17}$$

EXAMPLE 3. Take $X = C^1([0,1])$, $Y = X'$, $H = C([0,1])$, and $E\colon H \to V$ defined by

$$Eh(t) = \varphi(h) + \int_0^t h(s)\,ds,$$

where $\varphi \in H'$ is an arbitrary, fixed functional.

If we put $a(x, f) = f(x)$ for any $x \in X$, $f \in X' = Y$, then $a'(f, z) = z(f)$ for any $f \in X' = Y$ and $z \in X''$, and both forms a and a' are bounded and coercive; hence Theorems 1 and 2 apply.

Observe that $DE = I_H$, where $D\colon V \to H$ is the bounded derivative operator; hence $R(E') = H'$, and Theorem 4 applies as well. This means that A is the extension of E^{-1} over $U \subset X''$.

We have $E^{-1} = D|R(E)$.

It is easy to see that $R(E) \subset V$ is the set of all $v \in V$ such that the following boundary condition is satisfied:

$$v(0) = \varphi(Dv). \tag{18}$$

Indeed, if $v \in R(E)$, then $v(t) = \varphi(h) + \int_0^t h(s)\,ds$ and $v(0) = \varphi(h)$, and since $D^0 v = h$, (18) holds.

If, on the other hand, (18) holds for some $v \in V$, then $h = Dv$, $v \in V$ and

$$v(t) = v(0) + \int_0^t h(s)\,ds = \varphi(h) + \int_0^1 h(s)\,ds = Eh.$$

We can consider E^{-1} as the operator D over $R(E) \subset V$ with the boundary condition (18).

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Maximal operators defined by Fourier multipliers

by

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Abstract. The authors develop a linearization for maximal operators defined through Fourier multipliers, and establish for such operators transplantation and restriction theorems. Applications are discussed.

Introduction. Let λ be an $L^\infty(\boldsymbol{R}^n)$ function; define for each real number $R > 0$ an operator T_R on $L^2(\boldsymbol{R}^n)$ by $\widehat{T_R f}(\xi) = \lambda(\xi/R)\hat{f}(\xi)$ and $\tilde{T}_R$ on $L^2(\boldsymbol{T}^m)$ by $\widehat{\tilde{T}_R f}(n) = \lambda(n/R)\hat{f}(n)$. We say λ is *p-maximal* on $\boldsymbol{R}^n$ (or *weak p-maximal* on $\boldsymbol{R}^n$) if the operator T^* defined by $T^* f(x) = \sup_{R>0} |T_R f(x)|$ is bounded (or weakly bounded) on $L^p(\boldsymbol{R}^n)$; similarly for $\tilde{T}^*$ on $L^p(\boldsymbol{T}^m)$. The purpose of this note is to establish for p-maximal operators results on transplantation between T^* on $\boldsymbol{R}^n$ and $\tilde{T}^*$ on $\boldsymbol{T}^m$, and restriction of T^* and $\tilde{T}^*$ to subspaces. These results are similar to those of de Leeuw [3] for Fourier multipliers. The study of such transplantations was initiated by A. P. Calderòn [1] and by Coifman and Weiss [2].

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1. A linearization.

LEMMA. *Fix p, $1 < p < \infty$. The function λ is p-maximal if and only if*

$$\Big\|\sum_k T_{R_k} f_k\Big\|_{p'} \leqslant c\Big\|\sum_k |f_k|\Big\|_{p'}$$

uniformly in all sequences of positive reals $\{R_k\}$.

Proof. Define the Banach space $L^p(G, l^\infty(\boldsymbol{Z}^+))$ for $G = \boldsymbol{R}^n$ or $G = \boldsymbol{T}^m$, as the collection of all sequences of $L^p(G)$ functions $\{f_k\}$ such that the norm $\|\sup_k |f_k|\|_p$ is finite. It is clear that λ is p-maximal if and only if the linear

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