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Hölder continuous functions on compact sets and function spaces

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Abstract. Let F be a closed set, and let $\mathrm{Lip}(\beta, F)$ denote the space of all functions which are Hölder continuous on F with exponent β. Necessary and sufficient conditions on F are given, which guarantee that every function in $\mathrm{Lip}(\beta, F)$ may be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n)$ belonging to a Besov space or a Bessel potential space. Applications to the theories of harmonic functions and multiple Fourier series are given.

0. Introduction. Let F be a closed set and $0 < \beta \leqslant 1$. It is well known from the Whitney extension theorem that every function in $\mathrm{Lip}(\beta, F)$ may be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n)$. (For the definition of the Lipschitz space $\mathrm{Lip}(\beta, F)$, the Besov space $B_\alpha^{p,q}(\boldsymbol{R}^n)$ and the space $L_\alpha^p(\boldsymbol{R}^n)$ of Bessel potentials, which for integer α coincides with a Sobolev space, we refer to Section 1.) In this paper we consider the following question. For which sets F is it true that every function in $\mathrm{Lip}(\beta, F)$ may be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n) \cap B_\alpha^{p,q}(\boldsymbol{R}^n)$ or $\mathrm{Lip}(\beta, \boldsymbol{R}^n) \cap L_\alpha^p(\boldsymbol{R}^n)$? The answer, given in Theorem 1 and Theorem 2, is definitive. The extension operator used in the theorems is the same as in the Whitney extension theorem. Applications of the result are given in Theorem 3 and Theorem 4.

A corresponding question for continuous functions was solved by H. Wallin in [11] and, more generally, by T. Sjölin in [7] and [8]. They proved that every function in $C(F)$, the space of continuous functions on F, may be extended to a function in $C(\boldsymbol{R}^n) \cap L_\alpha^p(\boldsymbol{R}^n)$ or $C(\boldsymbol{R}^n) \cap B_\alpha^{p,p}(\boldsymbol{R}^n)$ if and only if a certain capacity is zero. Our case with Lipschitz continuous functions is of different nature than the continuous case, and the methods used in this paper are entirely different than those in [7], [8] and [11]. It should also be mentioned that the extension parts of Theorem 1 and Theorem 2 were proved, in a less general form, in [2].

Our result is also related to the imbedding theory for functions of several variables (see e.g. [6]), where e.g. the trace of functions in $B_\alpha^{p,q}(\boldsymbol{R}^n)$ or $L_\alpha^p(\boldsymbol{R}^n)$ to sufficiently smooth manifolds is characterized.

1. The spaces $\mathrm{Lip}(\beta, F)$, $L_\alpha^p(\boldsymbol{R}^n)$, and $B_\alpha^{p,q}(\boldsymbol{R}^n)$. Notation.

1.1. General references for the spaces defined in this section are [6] and [9]. The spaces $L_\alpha^p(\boldsymbol{R}^n)$ and $B_\alpha^{p,q}(\boldsymbol{R}^n)$ may be defined in many equivalent ways, and the definitions given here are chosen to suit our need in the proof of the theorems.

For any closed set F in $\mathbf{R}^n$ and $0 < \beta \leqslant 1$, the *Lipschitz space* $\mathrm{Lip}(\beta, F)$ consists of all functions f for which there exists a constant M such that $|f(x)| \leqslant M$ and $|f(x)-f(y)| \leqslant M|x-y|^\beta$, $x, y \in F$. The smallest M possible in this definition is taken as the norm of f.

The following definition of the *space* $L^p_\alpha(\mathbf{R}^n)$ *of Bessel potentials*, $\alpha > 0$, $1 < p < \infty$, is taken from [5]. Let l be an integer, $\alpha < l$. Then $L^p_\alpha(\mathbf{R}^n)$ is the space of all functions f with finite norm

$$\|f\|_p + \Big\|\Big\{\int_0^\infty \Big(\int_{t<|h|<2t} |\Delta_h^l f(x)|\,|h|^{-n-\alpha}dh\Big)^2 t^{-1}dt\Big\}^{1/2}\Big\|_p. \tag{1.1}$$

Here, $\Delta_h^l f(x)$ denotes the difference of order l with step h at the point x, i.e. $\Delta_h^1 f(x) = f(x+h)-f(x)$ and, for $l > 1$, $\Delta_h^l f(x) = \Delta_h^1(\Delta_h^{l-1}f)(x)$.

The *Besov space* $B^{p,q}_\alpha(\mathbf{R}^n)$, $\alpha > 0$, $1 \leqslant p, q < \infty$ consists of all functions with finite norm

$$\|f\|_{B^{p,q}_\alpha(\mathbf{R}^n)} = \|f\|_p + \Big(\int_{|h|<a} \|\Delta_h^l f(x)\|_p^q |h|^{-n-\alpha q}dh\Big)^{1/q}, \tag{1.2}$$

where $l > \alpha$ and $0 < a \leqslant \infty$. (Different values of l and a give raise to equivalent norms.) For $1 \leqslant p \leqslant \infty$, $q = \infty$, the norm is given by

$$\|f\|_{B^{p,\infty}_\alpha(\mathbf{R}^n)} = \|f\|_p + \sup_{0<|h|<a} \|\Delta_h^l f(x)\|_p |h|^{-\alpha}. \tag{1.3}$$

1.2. *Notation.* We will use the notation

$$\mathrm{Lip}(\beta, F) \to S(\mathbf{R}^n)$$

signifying that every function in $\mathrm{Lip}(\beta, F)$ has an extension Ef, defined on $\mathbf{R}^n$ and belonging to the normed function space $S(\mathbf{R}^n)$, and that the extension operator is continuous, i.e. we have $\|Ef\|_{S(\mathbf{R}^n)} \leqslant c\|f\|_{\mathrm{Lip}(\beta,F)}$. If S_1 and S_2 are two normed spaces, $S_1 \cap S_2$ will as usual denote $S_1 \cap S_2$ equipped with the norm $\|\cdot\|_{S_1\cap S_2} = \max(\|\cdot\|_{S_1}, \|\cdot\|_{S_2})$.

Let F be a closed set in $\mathbf{R}^n$. Then F_h will denote the set of points with distance less than or equal to h from F, and $|F_h|$ the n-dimensional Lebesgue measure of F_h. To simplify the notation, we shall often write h_i for 2^{-i}, F_i for $F_{2^{-i}}$, and ΔF_i for $F_i \setminus F_{i+1}$, $i \in \mathbf{Z}$. (In this way we have two definitions of F_i when i is an integer, but this will not cause any confusion.) In the proofs, c will denote a constant, in general not the same every time it appears.

2. Statement of theorems.

THEOREM 1. *Let $1 < p < \infty$, $0 < \beta < 1$, $\beta < \alpha < \beta + n/p$, and let F be a compact set in $\mathbf{R}^n$. Then*

$$\mathrm{Lip}(\beta, F) \to \mathrm{Lip}(\beta, \mathbf{R}^n) \cap L^p_\alpha(\mathbf{R}^n)$$

if and only if

$$\int_0^1 |F_h|\, h^{(\beta-\alpha)p}\frac{dh}{h} < \infty. \tag{2.1}$$

THEOREM 2. *Let $1 \leqslant p, q < \infty$, $0 < \beta < 1$, $0 < \alpha < \beta + n/p$, and let F be a compact set in $\mathbf{R}^n$. Then*

$$\mathrm{Lip}(\beta, F) \to \mathrm{Lip}(\beta, \mathbf{R}^n) \cap B^{p,q}_\alpha(\mathbf{R}^n)$$

if and only if

$$\int_0^1 |F_h|^{q/p} h^{(\beta-\alpha)q}\frac{dh}{h} < \infty. \tag{2.2}$$

For $1 \leqslant p < \infty$, $q = \infty$ the result holds if (2.2) is replaced by

$$\sup_{0<h<1} |F_h|^{1/p} h^{(\beta-\alpha)} < \infty. \tag{2.3}$$

Several remarks are in order in connection with these theorems.

Remark 1. The extension in Theorem 1 and Theorem 2 is given by an operator E which is linear, and has the property that Ef is infinitely differentiable outside F if $f \in \mathrm{Lip}(\beta, F)$. On the other hand, the conditions (2.1), (2.2) and (2.3) are necessary even if we do not require the corresponding extension operators to be continuous.

Remark 2. The assumption that F is compact is not a limitation, since if E is an arbitrary set, then every $f \in \mathrm{Lip}(\beta, E)$ has a unique extension to a function in $\mathrm{Lip}(\beta, \bar{E})$, and if F is closed but not compact, then the conditions (2.1)–(2.3) are not fulfilled, and there is of course a function in $\mathrm{Lip}(\beta, F)$ which cannot be extended even to a function in $\mathrm{Lip}(\beta, F) \cap L^p(\mathbf{R}^n)$.

Remark 3. If $\alpha \geqslant \beta + n/p$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, then the imbeddings in Theorem 1 and Theorem 2 are clearly possible if the compact set F consists of a finite number of points. This condition is also necessary. To see this, we recall the imbeddings (cf. [6], p. 236)

$$B^{p,q}_\alpha(\mathbf{R}^n) \to B^{\infty,\infty}_\gamma(\mathbf{R}^n) \quad \text{if} \quad \gamma = \alpha - n/p > 0, \tag{2.4}$$

and

$$L^p_\alpha(\mathbf{R}^n) \to B^{p,p}_\alpha(\mathbf{R}^n),\ 2 \leqslant p < \infty \quad \text{and} \quad L^p_\alpha(\mathbf{R}^n) \to B^{p,2}_\alpha(\mathbf{R}^n),\ 1 < p \leqslant 2.$$

From these, we see that

$$|f(x+h)-f(x)| = o(|h|^\beta), \quad h\to 0$$

if f is continuous and

$$f\in L_\alpha^p(\mathbf{R}^n) \text{ or } B_\alpha^{p,p}(\mathbf{R}^n), \quad \alpha-n/p>\beta,$$

and we may obtain this even for $\alpha-n/p=\beta$, using that $C_0^\infty(\mathbf{R}^n)$ is dense in $B_\alpha^{p,q}(\mathbf{R}^n)$, $q<\infty$, and the continuity of imbedding (2.4). Thus, if F has a cluster point, it is easy to construct a function in $\mathrm{Lip}(\beta, F)$ which can not be extended to $L_\alpha^p(\mathbf{R}^n)$ or $B_\alpha^{p,q}(\mathbf{R}^n)$, $1\leqslant p, q<\infty$, $\alpha\geqslant\beta+n/p$.

Our first application is to the theory of harmonic functions. The result we have in mind should be compared to the famous example by Hadamard, which shows the existence of a function f, harmonic in the open unit disc U of $\mathbf{R}^2$, continuous on $\overline{U}$ with infinite Dirichlet integral, thereby showing that, in general, the Dirichlet problem can not be solved directly using Dirichlet's principle. Actually, it is known that for $\beta\leqslant 1/2$, the function f can be taken in $\mathrm{Lip}(\beta, \overline{U})$; this is also a consequence of the theorem below.

A corresponding result for continuous functions is given in [11].

THEOREM 3. *Let $0<\beta<1$, let F be a closed subset of the boundary δU of U, where U is the unit sphere in $\mathbf{R}^n$. Then every $f\in\mathrm{Lip}(\beta, F)$ can be extended to a function in $\overline{U}$, harmonic in U, belonging to $\mathrm{Lip}(\beta,\overline{U})$, with finite Dirichlet integral $\int_U \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 dx$ if and only if*

$$\int_0^1 |F_h|\, h^{(\beta-1)2}\frac{dh}{h}<\infty. \tag{2.5}$$

Proof. Assuming that (2.5) is satisfied, we obtain from Theorem 1, since $L_1^2(\mathbf{R}^n)$ is equal to the Sobolev space $W_1^2(\mathbf{R}^n)$, that there exists a function Ef in $\mathrm{Lip}(\beta, \mathbf{R}^n)$, equal to f on F, infinitely differentiable outside F with finite Dirichlet integral. Applying Dirichlet's principle, minimizing the Dirichlet integral over all functions in $\overline{U}$ which are equal to Ef on δU, we get an extension with the desired properties, since a harmonic function in U with boundary values in $\mathrm{Lip}(\beta, \delta U)$ is actually in $\mathrm{Lip}(\beta,\overline{U})$. To prove the converse, let u be a harmonic function in U with finite Dirichlet integral belonging to $\mathrm{Lip}(\beta,\overline{U})$. Then, defining u outside U by $u(x) = u(x/|x|^2)$, and multiplying this extended function with an infinitely differentiable function with compact support equal to 1 on $\overline{U}$, we obtain a function in $\mathrm{Lip}(\beta,\mathbf{R}^n)\cap W_1^2(\mathbf{R}^n) = \mathrm{Lip}(\beta,\mathbf{R}^n)\cap L_1^2(\mathbf{R}^n)$ (cf. the proof of Theorem 5 in [11]). From this the converse follows, in view of Theorem 1.

Our last theorem is an immediate consequence of Theorem 2 and a criterion for absolute convergence of multiple Fourier series. It generalizes the corresponding one-dimensional result in [4]. It should be noted that in [4] it is also proved that the theorem below admits a partial converse when $n=1$.

THEOREM 4. *Let $0<\beta<1$ and let F be a closed subset of the cube $C=\{x=(x_1, x_2, \ldots, x_n)|\ |x_i|<\pi,\ i=1,2,\ldots,n\}$ such that*

$$\int_0^1 |F_h|^{1/2} h^{\beta-n/2-1}\,dh<\infty.$$

Then every $f\in\mathrm{Lip}(\beta, F)$ has an extension to a function with period 2π, which has absolute convergent Fourier series.

Proof. By Theorem 2, f has an extension $\tilde f$ in $B_{2,1}^{n/2}(\mathbf{R}^n)$. Multiplying this function $\tilde f$ by an infinitely differentiable function φ equal to one on F and equal to zero outside the cube $\{x|\ |x_i|<\pi-\delta/2\}$, where δ is the distance from F to the complement of C, we obtain a function $\varphi\tilde f$ in $B_{2,1}^{n/2}(\mathbf{R}^n)$. Extending the restriction of this function to C periodically with period 2π to $\mathbf{R}^n$, we obtain a function which is easily seen to belong to the periodical Besov space $B_{2,1}^{n/2}(T^n)$ (see e.g. [10] for the definition of this space). But a function in $B_{2,1}^{n/2}(T^n)$ has absolute convergent Fourier series, see e.g. [10], so we are done.

3. Proof of the sufficiency of the conditions on F in Theorem 1 and Theorem 2.

3.1. In the proofs, the conditions (2.1) and (2.2) occur in an equivalent form involving sums. We take care of this once and for all in a simple lemma, whose proof is an immediate consequence of the monotonicity of $|F_h|$.

LEMMA 3.1. *Let $F\subset\mathbf{R}^n$ be compact, $s>0$, and $-\infty<\gamma<\infty$. Then the sum $\sum_{i=1}^\infty 2^{i\gamma}|F_i|^s$ and the integral $\int_0^1 |F_h|^s h^{-\gamma-1}\,dh$ converge simultanously.*

In the proofs in this section, the main tool is the Whitney extension theorem. A version of this theorem suitable for our purposes may be found in [9] and reads as follows.

THEOREM (Whitney). *Let F be a closed set in $\mathbf{R}^n$. Then there exists a continuous, linear extension operator E mapping $\mathrm{Lip}(\beta, F)$ into $\mathrm{Lip}(\beta,\mathbf{R}^n)$. Furthermore, Ef is infinitely differentiable outside F for $f\in\mathrm{Lip}(\beta, F)$.*

We shall need the following estimate on the derivatives of Ef. Let $d(x, F)$ denote the distance from x to F, and let s be a multiindex, not equal to zero. Then, for $f\in\mathrm{Lip}(\beta, F)$,

$$\left|\frac{\partial^s(Ef)}{\partial x^s}(x)\right|\leqslant c\,d(x,F)^{\beta-|s|}. \tag{3.1}$$

For $|s|=1$, this estimate may be found in [9], p. 174, and the general case follows in the same way.

In the proof of the extension parts of Theorem 1 and Theorem 2, we shall take $\varphi\circ(Ef)$ as our extension of $f\in \mathrm{Lip}(\beta, F)$, where φ is a fix, infinitely differentiable function with compact support, equal to one for all x with $d(x, F)\leqslant 1$. The estimate (3.1) clearly holds with Ef replaced by $\varphi\circ(Ef)$, and using the mean value theorem repeatedly (write $\Delta_h^l f(x) = \Delta_h^1 g(x)$, where $g(x) = \Delta_h^{l-1} f(x)$), we obtain from this the following fact. Suppose that the line segment L between x and $x+lh$ does not intersect F. Then

$$|\Delta_h^l \varphi(Ef)(x)| \leqslant c|h|^l \sup_{x'\in L}\big(d(x', F)\big)^{\beta-l}. \tag{3.2}$$

3.2. We assume first that F satisfies (2.1), and prove that the extension $f = \varphi E f_F$ given above belongs to L_α^p if $f_F\in \mathrm{Lip}(\beta, F)$. Since f is bounded and has compact support we have $\|f\|_p<\infty$, so using norm (1.1) for L_α^p, it remains to show that the second term in (1.1) is finite. This term is less than a constant times $(\sum_{i=0}^{\infty} A_i + \sum_{i=0}^{\infty} B_i)^{1/p}$, where

$$A_i = \int_{x\in\Delta F_i}\bigg(\int_0^{h_i/8l}\Big(\int_{t<|h|<2t}\frac{|\Delta_h^l f(x)|}{|h|^{n+\alpha}}\,dh\Big)^2\frac{dt}{t}\bigg)^{p/2}dx,$$

and B_i the same expression but with the t-integration from $h_i/8l$ to infinity. Here we have put $\Delta F_i = F_i\setminus F_{i+1}$, $i\geqslant 1$, and $\Delta F_0 = \{x|\ d(x, F) > \frac{1}{2}\}$.

Using (3.2) we easily get

$$A_i \leqslant c\int_{x\in\Delta F_i}\bigg(\int_0^{h_i/8l}\Big(\int_{t<|h|<2t}|h|^{-n-\alpha+l}h_i^{\beta-l}\,dh\Big)^2\frac{dt}{t}\bigg)^{p/2}dx \leqslant c|\Delta F_i|\,h_i^{(\beta-\alpha)p},$$

and using just that $f\in \mathrm{Lip}(\beta, \boldsymbol{R}^n)$, we obtain

$$B_i \leqslant c\int_{x\in\Delta F_i}\bigg(\int_{h_i/8l}^{\infty}\Big(\int_{t<|h|<2t}|h|^{-n-\alpha+\beta}\,dh\Big)^2\frac{dt}{t}\bigg)^{p/2}dx \leqslant c|\Delta F_i|\,h_i^{(\beta-\alpha)p}.$$

Altogether, as $|\Delta F_i|\leqslant|F_i|$, this shows together with Lemma 3.1 that f belongs to L_α^p, and as the constants in all estimates are independent of f_F as long as $\|f_F\|_{\mathrm{Lip}(\beta,R^n)} = 1$, we actually obtain that $\mathrm{Lip}(\beta, F)\to L_\alpha^p(\boldsymbol{R}^n)$.

3.3. Next we show that $f\in B_\alpha^{p,q}$ under (2.2). For $|h|<h_i/(4l)$, $l>\alpha$, we have, since (3.2) holds and $f\in \mathrm{Lip}(\beta, \boldsymbol{R}^n)$,

$$\int|\Delta_h^l f(x)|^p\,dx = \Big(\sum_{\nu=0}^{i}+\sum_{\nu=i+1}^{\infty}\Big)\int_{\Delta F_\nu}|\Delta_h^l f(x)|^p\,dx$$
$$\leqslant c\sum_{\nu=0}^{i}|\Delta F_\nu|\,h_i^{lp}h_\nu^{\beta p-lp} + c\sum_{\nu=i+1}^{\infty}|\Delta F_\nu|\,h_i^{\beta p},$$

where the last term is less than $c|F_i|\,h_i^{\beta p}$.

We shall estimate the second term in (1.2), which raised to the power q is less than a constant times

$$\sum_{i=0}^{\infty} h_i^{-\alpha q-n}\int_{h_{i+1}/(4l)\leqslant|h|<h_i/(4l)}\Big(\int|\Delta_h^l f(x)|^p\,dx\Big)^{q/p}dh.$$

Inserting the estimate above for the x-integration, and using that $(\Sigma a_i)^{1/p}\leqslant \Sigma a_i^{1/p}$, we see that this is less than a constant times

$$\sum_{i=0}^{\infty} h_i^{(l-\alpha)q}\Big(\sum_{\nu=0}^{i}|\Delta F_\nu|^{1/p}h_\nu^{\beta-l}\Big)^q + \sum_{i=0}^{\infty} h_i^{(\beta-\alpha)q}|F_i|^{q/p}.$$

Using Hardy's inequality for sums, we get that the left member of this expression is less than a constant times $\sum_{i=0}^{\infty}|\Delta F_i|^{q/p}h_i^{(\beta-\alpha)q}$, and we are finished in view of Lemma 3.1. The continuity of the extension operator follows in the same way as in Section 3.2. Under (2.3) we have

$$\sum_{\nu=0}^{i}|\Delta F_\nu|\,h_\nu^{\beta p-lp}h_i^{lp}h_\nu^{-\alpha p+\alpha p}\leqslant c\sum_{\nu=0}^{i}h_\nu^{(\alpha-l)p}h_i^{lp}\leqslant c\,h_i^{\alpha p},$$

which combined with the first formula in this section easily settles the case $1\leqslant p<\infty$, $q=\infty$.

4. Proof of the necessity of the conditions on F in Theorem 1 and Theorem 2.

4.1. Our problem is to construct a function $f\in \mathrm{Lip}(\beta, F)$, which cannot be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n)$ and $L_\alpha^p(\boldsymbol{R}^n)$, $B_\alpha^{p,q}(\boldsymbol{R}^n)$, or $B_\alpha^{p,\infty}(\boldsymbol{R}^n)$, if the conditions (2.1), (2.2) or (2.3), respectively, are not fulfilled. The construction of f is given in Lemma 4.1, and in Section 4.4 we prove that f has the desired properties.

LEMMA 4.1. *Let $F\subset \boldsymbol{R}^n$ be a compact set, $0<\beta<1$, and let k_0 be a positive integer. Let furthermore the integer N be big enough, depending on k_0 and β, and let $0\leqslant N_0<N$. Then there exists a function $f\in \mathrm{Lip}(\beta, F)$, and points $p_{ki}\in F$, k integer, $k\geqslant 1$, $i = 1, 2, \ldots, i_k$, with the following properties:*

(i) *$|p_{ki}-p_{kj}|\geqslant h_k$, $i\neq j$, and to every point p_{ki} there exists a point p'_{ki}, equal to some p_{kj}, $j\neq i$, with*

$$h_k\leqslant|p_{ki}-p'_{ki}|\leqslant 2^{k_0}h_k.$$

(ii) *Put $G_k = \{x|\ |x-p_{ki}|\leqslant\frac{1}{2}h_k$ for some $i\}$. Then, for every $s>0$, we* *have the estimate*

$$|G_k|^s \geqslant c_1(|F_k|^s - c_2 2^{-k_0 ns}|F_{k-k_0}|^s). \tag{4.1}$$

Here, the positive constants c_1 and c_2 depend only on s and n.

(iii) *We have*

$$|f(p_{ki}) - f(p'_{ki})| \geqslant h_k^\beta$$

for all p_{ki} with $k = N\nu + N_0$, $\nu = 1, 2, \ldots$

Before proving the lemma, we remark that if $0 \leqslant \gamma < ns$, it follows from (4.1) that $\sum_1^\infty |G_k|^s h_k^{-\gamma}$ diverges if $\sum_1^\infty |F_k|^s h_k^{-\gamma}$ diverges, if k_0 is big enough. Indeed, we have

$$\sum_{k_0}^{M} |G_k|^s h_k^{-\gamma} \geqslant c_1\Big(\sum_{k_0}^{M} |F_k|^s h_k^{-\gamma} - \sum_{k_0}^{M} c_2 2^{-k_0 ns + k_0\gamma}|F_{k-k_0}|^s h_{k-k_0}^{-\gamma}\Big)$$

and if k_0 is big enough, the constant in the last sum is less than 1/2, so then the right member of the inequality is greater than

$$c_1\Big(\sum_{k_0}^{M-k_0} \tfrac{1}{2}|F_k|^s h_k^{-\gamma} + \sum_{M-k_0+1}^{M} |F_k|^s h_k^{-\gamma} - \tfrac{1}{2}\sum_{0}^{k_0-1} |F_k|^s h_k^{-\gamma}\Big),$$

from which the assertion follows.

This also means that given $N > 0$ there exists an N_0 with $0 \leqslant N_0 < N$ such that $\sum_{\nu=1}^{\infty} |G_{N\nu+N_0}|^s h_{N\nu+N_0}^{-\gamma}$ diverges if $\sum_{k=1}^{\infty} |F_k|^s h_k^{-\gamma}$ diverges (which by Lemma 3.1 is the case if and only if $\int_0^1 |F_h|^s h^{-\gamma-1}\,dh$ diverges).

Choosing a sequence k_n such that $|F_{k_n}|^s h_{k_n}^{-\gamma} = \max_{1\leqslant i\leqslant k_n} |F_i|^s h_i^{-\gamma}$, we easily get from (4.1) that $\sup_k |G_k|^s h_k^{-\gamma} = +\infty$ if $\sup_{0<h<1} |F_h|^s h^{-\gamma} = +\infty$, k_0 big enough, and consequently then also $\sup_\nu |G_{N\nu+N_0}|^s h_{N\nu+N_0}^{-\gamma} = +\infty$ for some N_0.

4.2. *Proof of the existence of points $p_{ki} \in F$ satisfying* (i) *and* (ii) *in Lemma* 4.1. Divide $\boldsymbol{R}^n$ into a net of closed cubes with sides of length h_k. Choose points $b_i \in F$ and cubes B_i from the net in the following way. Let b_1 be an arbitrary point from F, and B_1 a cube containing b_1, and having chosen $b_1, b_2, \ldots, b_{i-1}$, choose, if possible, $b_i \in F$ and B_i with $b_i \in B_i$ so that $B_i \cap (\bigcup_{\nu=1}^{i-1} B_\nu) = \emptyset$. This gives a finite number of cubes B_i, with the property that the cubes $5B_i$ contain all cubes from the net intersecting cubes from the net intersecting F, whence

$$\sum |5B_i| = 5^n \sum |B_i| \geqslant |F_k|. \tag{4.2}$$

Here, $5B_i$ denotes the cube obtained from B_i, by expanding B_i with the factor 5, keeping the center of B_i fixed. Next put $M = 2^{k_0}/\sqrt{n}$, and define the index set I by the condition that $i \in I$ if and only if $MB_i \cap MB_j \neq \emptyset$ for some $i \neq j$. Since $MB_i \subset F_{k-k_0}$ we get $\sum_{i\notin I} |MB_i| \leqslant |F_{k-k_0}|$ or

$$\sum_{i\notin I} |B_i| \leqslant M^{-n}|F_{k-k_0}|. \tag{4.3}$$

Now we let the points b_i, $i \in I$, be the points p_{ki}, $i = 1, 2, \ldots, i_k$ in the lemma. By construction, condition (i) of the lemma is fulfilled. From (4.2), (4.3) and the equality $|B_i| = \omega^{-1}(1/2)^{-n}|\{x|\ |x-b_i| \leqslant h_k/2\}|$, where ω is the volume of the unit sphere in $\boldsymbol{R}^n$, we get

$$|F_k| \leqslant 5^n \sum |B_i| = 5^n\Big(\sum_{i\notin I} |B_i| + \sum_{i\in I} |B_i|\Big) \leqslant 5^n\big(M^{-n}|F_{k-k_0}| + (1/2)^{-n}\omega^{-1}|G_k|\big)$$

and thus, using $(a+b)^s \leqslant 2^s(a^s+b^s)$,

$$|F_k|^s \leqslant 2^s 5^{ns}\big(M^{-ns}|F_{k-k_0}|^s + (1/2)^{-ns}\omega^{-s}|G_k|^s\big).$$

This gives (4.1) (with $c_1 = 2^{-s}\omega^s 5^{-ns} 2^{-ns}$ and $c_2 = (5\sqrt{n})^{ns} 2^s$).

4.3. *Construction of the function f in Lemma* 4.1. Define a function f_k on $\boldsymbol{R}^n$ in the following way. On the points p_{ki}, $i = 1, 2, \ldots, i_k$, define f_k by $f_k(p_{k1}) = h_k^\beta$, and if f_k is defined on $p_{k\nu}$, $\nu = 1, 2, \ldots, i-1$, put $f_k(p_{ki}) = h_k^\beta$ if f is not previously defined on p'_{ki}, otherwise put $f_k(p_{ki}) = -f_k(p'_{ki})$. Outside the points p_{ki}, put $f_k(x) = f_k(p_{ki})(1 - 2|x - p_{ki}|/h_k)$ if the distance from x to p_{ki} is less than $h_k/2$, and $f_k(x) = 0$ otherwise. Then we have

$$|f_k(x) - f_k(y)| \leqslant 2|x-y|h_k^{\beta-1} \leqslant 2|x-y|^\beta \quad \text{if} \quad |x-y| \leqslant h_k \tag{4.4}$$

and since trivially $|f_k(x) - f_k(y)| \leqslant 2h_k^\beta \leqslant 2|x-y|^\beta$ if $|x-y| \geqslant h_k$, we see that for any x and y holds $|f_k(x) - f_k(y)| \leqslant 2|x-y|^\beta$. Now, put $f = \sum_{\nu=1}^{\infty} f_{\bar\nu}$, where $\bar\nu = N\nu + N_0$. Then for any $m > 1$ we have if $|x-y| \geqslant h_{\bar m}$ ($\bar m = Nm + N_0$)

$$\sum_{\nu>m} |f_{\bar\nu}(x) - f_{\bar\nu}(y)| \leqslant 2\sum_{\nu>m} h_{\bar\nu}^\beta = 2h_{\bar m}^\beta/(h_N^{-\beta} - 1) \leqslant |x-y|^\beta A(N),$$

where $A(N) = 2/(h_N^{-\beta} - 1)$. If $|x-y| \leqslant Mh_{\bar m}$, $M > 0$, we instead have, by (4.4),

$$\sum_{\nu=1}^{m-1} |f_{\bar\nu}(x) - f_{\bar\nu}(y)| \leqslant 2|x-y|\sum_{\nu=1}^{m-1} h_{\bar\nu}^{\beta-1}$$

$$= 2|x-y|(h_{\bar m}^{\beta-1} - h_{N+N_0}^{\beta-1})/(h_N^{\beta-1} - 1) \leqslant 2|x-y|^\beta B(N, M),$$

where $B(N, M) = M^{1-\beta}/(h_N^{\beta-1}-1)$. Here, in the last inequality we replaced $|x-y|$ by $|x-y|^\beta M^{1-\beta} h_{\overline{m}}^{1-\beta}$ and dropped a negative term. From these estimates we easily obtain the following facts. Firstly, if x and y are given with $|x-y| < h_{N+N_0}$ and we choose m so that $h_{\overline{m}} \leqslant |x-y| < h_{\overline{m-1}} = h_{\overline{m}} 2^N$, we obtain that $f \in \mathrm{Lip}(\beta, \boldsymbol{R}^n)$, since f is obviously bounded. Secondly, for $k = Nm+N_0$ and $M = 2^{k_0}$, we have

$$(4.5)\qquad |f(p_{ki})-f(p'_{ki})| = \Big|2h_k^\beta - \sum_{\nu\neq m}\big(f_{\bar\nu}(p_{ki})-f_{\bar\nu}(p'_{ki})\big)\Big|$$
$$\geqslant 2h_k^\beta - \big(A(N)+B(N, 2^{k_0})\big)|p_{ki}-p'_{ki}|^\beta$$
$$\geqslant h_k^\beta\Big(2-2^{k_0\beta}\big(A(N)+B(N, 2^{k_0})\big)\Big) \geqslant h_k^\beta$$

if N is big enough, which shows that f satisfies (iii) of Lemma 4.1. Thus the restriction to F of the function f constructed here satisfies the conditions of Lemma 4.1.

4.4. *Proof of the necessity of the conditions on F in the case $a < 1$.* The case $a \geqslant 1$ will be reduced to this case in Section 4.5. We begin with some preparatory estimates.

Let F be a compact set, and let $f \in \mathrm{Lip}(\beta, F)$ be the function given in Lemma 4.1. Assume that f is extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n)$, and denote this function also by f. Since $f \in \mathrm{Lip}(\beta, \boldsymbol{R}^n)$, there exists a constant $a > 0$ such that $|f(x+h)-f(x)| \leqslant h_k^\beta/4$ if $|h| \leqslant a h_k$. If $x \in S_{ki} = \{t|\ |t-p_{ki}| \leqslant a h_k\}$, then it follows from (iii) in Lemma 4.1 that $|f(x+h)-f(x)| \geqslant h_k^\beta/2$ if $x+h$ belongs to the sphere S'_{ki} of radius $a h_k$ around p'_{ki}, if k is of the form $k = N\nu+N_0$.

Assume as we may that $a < 1/8$, and set for a moment $d = |p_{ki}-p'_{ki}|$. Then, if $x \in S_{ki}$ and $t \in S'_{ki}$, we have $d-2ah_k \leqslant |x-t| \leqslant d+2ah_k$, and hence

$$(4.6)\qquad S'_{ki} \subset \{t|\ d-h_k/4 \leqslant |x-t| \leqslant d+h_k/4\} \subset \{t|\ |x-t| \leqslant 2^{k_0+1}h_k\}.$$

This gives the estimate

(4.7)

$$\int_0^\infty \Big(\int_{t<|h|<2t} |f(x+h)-f(x)|\,|h|^{-n-a}\,dh\Big)^2 \frac{dt}{t} \geqslant c h_k^{(\beta-a)2}, \qquad x \in S_{ki},\ k = N\nu+N_0,$$

where c is independent of x and k, since the left member of (4.7) is by the left inclusion in (4.6) greater than

$$\int_{d-3h_k/8}^{d-h_k/4} \Big(\int_{x+h\in S'_{ki}} |f(x+h)-f(x)|\ \ |h|^{-n-a}\,dh\Big)^2 \frac{dt}{t},$$

from which (4.7) immediately follows. Similarly, using the right inclusion in (4.6), we obtain for $b = 2^{k_0+1}$

$$(4.8)\qquad \int_{|x-y|<bh_k} |f(x)-f(y)|^p\,dy \geqslant c h_k^{\beta p+n}, \qquad x \in S_{ki},\ k = N\nu+N_0,$$

where c is independent of x and k.

Now we assume that $\int_0^1 |F_h|\, h^{(\beta-a)p-1}\,dh = +\infty$, and show that the function f above does not belong to L_a^p. Put $G_k^a = \bigcup_{i=1}^{i_k} S_{ki}$ (then $G_k^{1/2}$ is equal to G_k as defined in Lemma 4.1). We write $\bigcup_{\nu=1}^{K} G^a_{N\nu+N_0} = \bigcup_{i=1}^{K} A_i$, where $A_K = G^a_{NK+N_0}$ and $A_i = G^a_{Ni+N_0} \setminus \bigcup_{i+1}^{K} G^a_{Ni+N_0}$, $1 \leqslant i < k$. Then, using (1.1), (4.7), and that

$$\sum_{\nu=1}^{i} b^\nu = \frac{b^{i+1}-b}{b-1} \leqslant b^i\Big(\frac{b}{b-1}\Big) = cb^i, \qquad b > 1,$$

we get (since $a > \beta$)

$$\|f\|^p_{L^p_a(R^n)} \geqslant \int_{\bigcup_{i=1}^{K} A_i} \bigg(\int_0^\infty \Big(\int_{t<|h|<2t} |f(x+h)-f(x)|\,|h|^{-n-a}\,dh\Big)^2 \frac{dt}{t}\bigg)^{p/2} dx$$
$$\geqslant c\sum_{i=1}^{K} |A_i|\, h^{(\beta-a)p}_{Ni+N_0} \geqslant c\sum_{i=1}^{K} |A_i| \sum_{\nu=1}^{i} h^{(\beta-a)p}_{N\nu+N_0}$$
$$= c\sum_{\nu=1}^{K}\sum_{i=\nu}^{K} |A_i|\, h^{(\beta-a)p}_{N\nu+N_0} \geqslant c\sum_{\nu=1}^{K} |G^a_{N\nu+N_0}|\, h^{(\beta-a)p}_{N\nu+N_0}.$$

If now k_0, N and N_0 on beforehand are chosen so that $\sum_{\nu=1}^{\infty} |G^a_{N\nu+N_0}|\, h^{(\beta-a)p}_{N\nu+N_0}$ diverges (this happens simultanously for all $0 < a \leqslant 1/2$, since $a^{-n}|G_k^a| = (1/2)^{-n}|G_k^{1/2}|$) which is possible due to the discussion after Lemma 4.1, we may conclude that f is not in $L^p_a(\boldsymbol{R}^n)$.

Next assume that $\int_0^1 |F_h|^{q/p} h^{(\beta-a)q}\,dh/h < \infty$. In [3] we proved that an equivalent norm for $B_a^{p,q}(\boldsymbol{R}^n)$ for $a < 1$ is given by

$$\|f\|_p + \Big(\sum_{\nu=0}^{\infty} h_\nu^{-aq}\Big(h_\nu^{-n} \iint_{|x-y|<bh_\nu} |f(x)-f(y)|^p\,dx\,dy\Big)^{q/p}\Big)^{1/q},$$

where $b > 0$. Using this and (4.8), we obtain

$$\|f\|^q_{B^\alpha_{p,q}(\mathbf{R}^n)} \geqslant c \sum_{\nu=1}^{\infty} h^{-\alpha q}_{N\nu+N_0} (|G^\alpha_{N\nu+N_0}| h^{\beta p}_{N\nu+N_0})^{q/p},$$

where the sum as before may be taken divergent.

Finally, suppose that $\sup\limits_{0<h<1} |F_h|^{1/p} h^{\beta-\alpha} = +\infty$. If $f \in B^{p,\infty}_\alpha$, then $\int |f(x+h)-f(x)|^p dx \leqslant c|h|^{\alpha p}$, and consequently $h_k^{-n} \iint\limits_{|x-y|<bh_k} |f(x)-f(y)|^p dx\,dy \leqslant c h_k^{\alpha p}$. By (4.8), the left side of this inequality is larger than $c|G^\alpha_k| h^{\beta p}_k$, $k = N\nu+N_0$. From this it follows that $f \notin B^{p,\infty}_\alpha$, again because of the comments after Lemma 4.1.

4.5. *The case* $\alpha \geqslant 1$. Let $\alpha_0 = \alpha\theta+\beta(1-\theta)$, $1/p_0 = \theta/p$, and $1/q_0 = \theta/q$, where $0 < \theta < 1$. It is then well known that $\mathrm{Lip}(\beta, \mathbf{R}^n) \cap B^{p,q}_\alpha(\mathbf{R}^n) \subset B^{p_0,q_0}_{\alpha_0}(\mathbf{R}^n)$, and we obviously have

$$\int_0^1 |F_h|^{q/p} h^{(\beta-\alpha)p} dh/h = \int_0^1 |F_h|^{q_0/p_0} h^{(\beta-\alpha_0)p_0} dh/h.$$

Thus, to show that (2.2) is necessary in Theorem 2, just choose θ so close to zero that $\alpha_0 < 1$ and use the case $\alpha < 1$. The necessity of (2.3) is obtained analogously.

In order to show that (2.1) is necessary in Theorem 1, we use a similar argument. Put

$$N^\alpha_{p,s,l}(f) = \Big\| \Big\{ \int_0^\infty \Big(\int\limits_{t<|h|<2t} |\Delta^l_h f(x)|\,|h|^{-n-\alpha} dh\Big)^s dt/t \Big\}^{1/s} \Big\|_p.$$

From the lemmas below, and the fact that the norms (1.1) are equivalent for $l > \alpha$, we deduce that $N^{\alpha_0}_{p_0,s_0,1}(f) < \infty$ if $f \in \mathrm{Lip}(\beta, \mathbf{R}^n) \cap L^p_\alpha(\mathbf{R}^n)$, where $\alpha_0 = \alpha\theta+\beta(1-\theta)$, $1/p_0 = \theta/p$, $1/s_0 = \theta/2$, $0 < \theta < 1$. Choose θ so close to zero that $\alpha_0 < 1$. In Section 4.7 we constructed a function in $\mathrm{Lip}(\beta, F)$ such that for any extension f of it, holds

$$N^{\alpha_0}_{p_0,2,1}(f) = +\infty \quad \text{if} \quad \int_0^1 |F_h| h^{(\beta-\alpha)p} dh/h = \int_0^1 |F_h| h^{(\beta-\alpha_0)p_0} dh/h = +\infty,$$

but an inspection of the proof shows that actually also $N^{\alpha_0}_{p_0,s_0,1}(f) = +\infty$. Thus we are finished.

Lemma 4.2. *Let* $0 < \theta < 1$, $1 < p, s < \infty$, $0 < \beta < 1$, $0 < \alpha < l$, $\alpha_0 = \alpha\theta+\beta(1-\theta)$, $1/p_0 = \theta/p$, *and* $1/s_0 = \theta/s$. *Then*

$$N^{\alpha_0}_{p_0,s_0,l}(f) \leqslant c\big(N^\alpha_{p,s,l}(f)\big)^\theta (\|f\|_{\mathrm{Lip}(\beta,\mathbf{R}^n)})^{1-\theta}.$$

Proof. Using Hölder's inequality, we get

$$\Big(\int\limits_{t<|h|<2t} |\Delta^l_h f(x)|^{\theta+1-\theta} |h|^{-n-\alpha_0} dh\Big)^{s_0}$$
$$\leqslant \Big(\int\limits_{t<|h|<2t} |\Delta^l_h f(x)|\,|h|^{(-n-\alpha_0)/\theta} dh\Big)^{s_0\theta} \Big(\int\limits_{t<|h|<2t} |\Delta^l_h f(x)|\, dh\Big)^{s_0(1-\theta)}$$
$$\leqslant c\Big(\int\limits_{t<|h|<2t} |\Delta^l_h f(x)|\,|h|^{-n-\alpha} dh\Big)^{s} \|f\|^{s_0(1-\theta)}_{\mathrm{Lip}(\beta,\mathbf{R}^n)},$$

from which the result follows.

Lemma 4.3. *Let* $0 < \alpha < 1$, $1 < p, s < \infty$ *and* $l_0 \geqslant 1$, *and assume that* $f \in L^p(\mathbf{R}^n)$ *and* $N^\alpha_{p,s,l}(f) < \infty$ *for* $l \geqslant l_0$. *Then* $N^\alpha_{p,s,l}(f) < \infty$ *for* $l \geqslant 1$.

Proof. We use the identity (see e.g. [1]. p. 228)

$$\Delta^l_h f(x) - 2^{-l} \Delta^l_{2h} f(x) = -\tfrac{1}{2} \sum_{k=0}^{l-1} \sum_{s=0}^{k} 2^{-k} \binom{k}{s} \Delta^{l+1}_h f(x+sh) \tag{4.9}$$

and the obvious identity

$$\Delta^{l+1}_h f(x+sh) = -\Delta^{l+2}_h f(x+(s-1)h) + \Delta^{l+1}_h f(x+(s-1)h).$$

Repeated use of the latter gives $|\Delta^{l+1}_h f(x+sh)| \leqslant c \sum\limits_{\nu=l+1}^{\nu=l+s+1} |\Delta^\nu_h f(x)|$, and in view of this we obtain from (4.9)

$$|\Delta^l_h f(x)| - 2^{-l} |\Delta^l_{2h} f(x)| \leqslant c \sum_{\nu=l+1}^{2l} |\Delta^\nu_h f(x)|.$$

After integrating, one obtains

$$(1-2^{-l+\alpha}) N^\alpha_{p,s,l} \leqslant c \sum_{\nu=l+1}^{2l} N^\alpha_{p,s,\nu}. \tag{4.10}$$

(Put $A_\varepsilon = \Big(\int_\varepsilon^\infty \Big(\int\limits_{t<|h|<2t} |\Delta^l_h f(x)|\,|h|^{-n-\alpha}\Big)^s \frac{dt}{t}\Big)^{1/s}$ and let A'_ε be the same expression with Δ^l_h replaced by Δ^l_{2h}; then straightforward substitutions show that $A'_\varepsilon = 2^\alpha A_{2\varepsilon} \leqslant 2^\alpha A_\varepsilon$, and since $f \in L^p$, we have that $A_\varepsilon < \infty$ a.e. From this (4.10) is easily deduced.) The result now follows from (4.10) by induction.

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Sur le minimum des fonctionnelles dans les espaces de Banach

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Résumé. Soit Φ une fonctionnelle continue sur l'espace de Banach, remplissant l'inégalité; $\Phi(tx+(1-t)y) \leqslant t\Phi(x)+(1-t)\Phi(y)-\Gamma(t, \|x-y\|)$ où $\Gamma(t, s)$ satisfait aux certaines conditions spéciales. On montre l'existence d'un seul point minimal de Φ: $\Phi(\bar{x}) = \inf \Phi(x)$: $x \in X$ et on établie une limitation pour la distance $\|x-\bar{x}\|$ pour x arbitraire.

Soit Ψ une autre fonctionnelle, continue, convexe et non-négative; à tout $\lambda \geqslant 0$ on fait correspondre un élément x_λ tel que $\Phi(x_\lambda)+\lambda(x_\lambda) = \inf(\Phi(x)+\lambda\Psi(x))$: $x \in X$. La fonction x_λ possède la limite $\bar{z}$ (si $\lambda \to \infty$) et on a: $\Phi(\bar{z}) = \inf \Phi(x)$: $\Psi(x) = 0$.

Enfin, on présente une application de la théorie exposée ci-dessus à la calculation numérique d'un point minimal absolu et relatif, en se servant d'ainsi dits espaces des suite concordantes [2].

Ce travail comprend trois parties; dans la premiére, nous considérons le problème du minimum "absolu"

$$(*) \qquad \inf \Phi(x): \ x \in X,$$

X étant un espace de Banach (non réflexif), Φ une fonctionnelle définie sur X. Nous montrons l'existence d'une solution $\bar{x}$ unique de $(*)$ et déduisons quelques limitations pour $\|x-\bar{x}\|$, $x \in X$ — arbitraire. La deuxième partie contient la solution du problème du minimum "relatif", c.-à-d. du problème consistant à trouver $\bar{z} \in Z$ tel que

$$(**) \qquad \inf \Phi(z): \ z \in Z = \Phi(\bar{z}),$$

Z étant un sous-ensemble fermé convexe de X. Enfin, dans la troisième partie, nous appliquons les résultats des parties 1° et 2° aux problèmes $(*)$ et $(**)$ dans des espaces spéciaux, appelés "espaces de suites concordantes" ou bien espaces du type $\{X_n\}$. Expliquons-le plus en détail: soit $\{X_n\}$ une suite d'espaces de Banach et considérons les suites $\{x_n\}$: $x_n \in X_n$. Certaines d'elles sont distinguées comme "concordantes" (corrélatif des suites de Cauchy); les suites concordantes forment un espace de Banach, appelé espace du type $\{X_n\}$. Voici l'ordre des idées de la troisiéme partie: on représente l'espace donné X comme un espace du type $\{X_n\}$ (ce qui