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Fredholm Toeplitz operators on strongly pseudoconvex domains

by

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Abstract. Venugopalkrishna in [15] investigated conditions which ensure that a Toeplitz operator acting on a Hardy space on a strongly pseudoconvex domain $D \subseteq C^n$ $(n \geqslant 1)$ is Fredholm. In [11] McDonald proved that when $D = B^n$, the open unit ball in C^n, then the Toeplitz operator T_φ, for $\varphi \in H^\infty + C$, is Fredholm if and only if φ is bounded away from zero in a neighbourhood of ∂B^n. We extend this result to a general strongly pseudoconvex domain, D, with smooth boundary in C^n with $n > 2$, and give a similar result for Toeplitz operators acting on a Hardy space on ∂D. We also note that the property of a Toeplitz operator, T_φ, being Fredholm depends only on the local properties of the symbol φ on ∂D.

1. Introduction. Let D be a strongly pseudoconvex domain with smooth boundary in C^n, i.e., D is a bounded domain in C^n and there exists a real-valued function ϱ such that

(1) $D = \{z : \varrho(z) < 0\}$,

(2) $\operatorname{grad}\varrho \neq 0$ on ∂D,

(3) ϱ is strictly plurisubharmonic in a neighbourhood of ∂D,

(4) ϱ is of class C^∞ in a neighbourhood of $\overline{D}$.

Denote by L^2 the space of functions $f\colon D \to C$ which are square integrable with respect to Lebesgue measure, dV, in C^n. Write L^∞ for the essentially bounded measurable functions on D. H^2 is the space of all functions $f \in L^2$ which are holomorphic in D, with norm $\|f\|_2 = \left(\int_D |f(z)|^2 dV(z)\right)^{1/2}$. C is the space of all continuous functions on $\overline{D}$, and $A(\overline{D})$ is the space of all holomorphic functions in D which extend continuously to $\overline{D}$. H^∞ is the space of all bounded holomorphic functions on D.

Let σ denote the surface area measure on ∂D. We write $L^\infty(\partial D)$ for $L^\infty(d\sigma)$, $L^2(\partial D)$ for $L^2(d\sigma)$. $H^2(\partial D)$ denotes the closure in L^2 of the boundary values of holomorphic functions which extend smoothly to $\overline{D}$. Since the boundary of D is smooth, this definition is equivalent to requiring that $\sup_{\varepsilon>0}\left[\int_{\partial D_\varepsilon} |f(z)|^2 d\sigma_\varepsilon(z)\right]^{1/2} < \infty$ where $D_\varepsilon = \{z \in D\colon \varrho(z) < -\varepsilon\}$, $d\sigma_\varepsilon$ is surface area measure on ∂D_ε, and $f(z)$ is the Poisson integral extension of f into D. The norm for $H^2(\partial D)$ is given by

$$\|f\|_2 = \left(\int_{\partial D} |f(z)|^2 d\sigma(z)\right)^{1/2}.$$

For f in H^∞ and $z \in \partial D$ let v_z be the outward normal to ∂D at z and set $f^*(z) = \lim_{\varepsilon\to 0} f(z-\varepsilon v_z)$. Such a limit exists almost everywhere on ∂D with respect to $d\sigma$ (see [14]). The map $f \to f^*$ gives an isometric isomorphism of H^∞ onto a closed subalgebra, $H^\infty(\partial D)$, of $L^\infty(\partial D)$ ([14], or see [1], Prop. 10). f can be regained from its boundary values by using the Szegö kernel, (see [14], p. 18), i.e., $f(z) = \int_{\partial D} S(z, w) f^*(w)\, d\sigma(w)$ where $S(w, z)$ represents the Szegö kernel. Let $C(\partial D)$ denote the space of continuous functions on ∂D. It is shown in [1] that $H^\infty + C$ is a closed subalgebra of L^∞ and that $H^\infty(\partial D) + C(\partial D)$ is a closed subalgebra of $L^\infty(\partial D)$. A shorter proof is given in [7]. Lemma 8 of [10] can be easily adapted to the case of strongly pseudoconvex domains with smooth boundary to show that the map $f \to f^*$ is an algebra homomorphism from $H^\infty + C$ onto $H^\infty(\partial D) + C(\partial D)$ with kernel $\{f \in C\colon f^* = 0\}$.

For φ in L^∞ (resp. $L^\infty(\partial D)$) the Toeplitz operator with symbol φ, T_φ, is defined by $T_\varphi f = P(\varphi f)$ for $f \in H^2$ (resp. $H^2(\partial D)$) where P denotes the orthogonal projection of L^2 (resp. $L^2(\partial D)$) onto H^2 (resp. $H^2(\partial D)$). The Toeplitz operators, T_φ, have integral representations:

$$T_\varphi f(z) = \int_D K(z, w)\varphi(w) f(w)\, dV(w) \qquad (f \in H^2)$$

where $K(z, w)$ is the Bergman kernel, and

$$T_\varphi f(z) = \Big[\int_{\partial D} S(w, \zeta)\varphi(\zeta) f(\zeta)\, d\sigma(\zeta)\Big]^*(z) \qquad (f \in H^2(\partial D))$$

where $S(w, \zeta)$ is the Szegö kernel (see [14]).

From this point on we will assume that $n > 2$. The results for Toeplitz operators acting on H^2 (in particular, Theorem 1 below) in Sections 3, 4 and 6 all carry through in the case $n = 2$ with identical proofs. The difficulty for the case $n = 2$ when we consider Toeplitz operators on $H^2(\partial D)$ is discussed briefly at the end of Section 5.

The main aim of this paper is to prove the following theorems:

THEOREM 1. *If $\varphi \in H^\infty + C$, then T_φ is Fredholm if and only if φ is bounded away from zero in a neighbourhood of ∂D.*

THEOREM 2. *If $\varphi \in H^\infty(\partial D) + C(\partial D)$, then T_φ is Fredholm if and only if the extension of φ to D (via the Poisson–Szegö kernel) is bounded away from zero in a neighbourhood of ∂D.*

For $\varphi \in C$ Venugopalkrishna [15] proved that T_φ is Fredholm acting on H^2 if φ is non-zero on ∂D. For the particular case when $D = B^n$, the open unit ball in C^n, McDonald proved Theorem 1. We shall adapt his methods to prove our theorems.

We note here some elementary propositions which we shall need. For any Hilbert space, H, let $\mathrm{BL}(H)$ denote the algebra of all bounded linear operators on H, and $\mathscr{K}(H)$ the ideal of all compact operators on H. The identity operator on H is denoted by I.

PROPOSITION 3. *If $\varphi \in L^\infty$ (resp. $L^\infty(\partial D)$) and $\psi \in C$ (resp. $C(\partial D)$), then $T_\varphi T_\psi - T_{\varphi\psi} \in \mathscr{K}(H^2)$ (resp. $\mathscr{K}(H^2(\partial D))$).*

Proof. A simple extension of [15], Theorem 2.1 shows that when $\psi \in C$, the operator $(I-P)M_\psi$ is compact when restricted to H^2 (where M_ψ denotes the usual multiplication operator on L^2). [12], Theorem 1.2 shows that when $\psi \in C(\partial D)$, the operator $(I-P)M_\psi$ is compact when restricted to $H^2(\partial D)$. In either case, $T_\varphi T_\psi - T_{\varphi\psi} = PM_\varphi(P-I)M_\psi$ and so $T_\varphi T_\psi - T_{\varphi\psi}$ is compact.

PROPOSITION 4. *If $\psi \in H^\infty + C$ (resp. $H^\infty(\partial D) + C(\partial D)$) and $\varphi \in L^\infty$ (resp. $L^\infty(\partial D)$), then $T_\varphi T_\psi - T_{\varphi\psi} \in \mathscr{K}(H^2)$ (resp. $\mathscr{K}(H^2(\partial D))$).*

Proof. This follows from Proposition 3 by verifying that for $\psi \in H^\infty$ (resp. $H^\infty(\partial D)$) and $\varphi \in L^\infty$ (resp. $L^\infty(\partial D)$) $T_\varphi T_\psi = T_{\varphi\psi}$.

PROPOSITION 5. *If $h \in C$ and $h|_{\partial D} = 0$, then T_h is compact.*

Proof. This is just Theorem 2.3 of [15].

It is appropriate to comment here that in [2] an index formula is described for Fredholm Toeplitz operators on strongly pseudoconvex domains with smooth boundary.

2. Results on maximal ideal spaces. In this section we state some results on the maximal ideal spaces of the algebras $H^\infty + C$ and $H^\infty(\partial D) + C(\partial D)$ which were proved for the case $D = B^n$ by McDonald [10]. The proofs for the general strongly pseudoconvex domain with smooth boundary are almost identical and we leave it to the reader to make the necessary changes in the proofs of [10].

First we establish some notation so that we can state the results. For any Banach algebra A let $M(A)$ denote its maximal ideal space. $M(A(\bar D))$ consists of just evaluation functionals, e_w, where $w \in \bar D$ and $e_w(f) = f(w)$ $(f \in A(\bar D))$. The map $w \to e_w$ is a homeomorphism from $\bar D$ onto $M(A(\bar D))$. If $w \in D$, we can consider e_w as an element of both $M(H^\infty)$ and $M(H^\infty + C)$. We put $D_\infty = \{m \in M(H^\infty)\colon m = e_w,\ w \in D\}$. The map $f \to f^*$ is an isometric isomorphism from H^∞ onto $H^\infty(\partial D)$ and hence there is an induced homeomorphism τ from $M(H^\infty)$ onto $M(H^\infty(\partial D))$ defined by $\tau(m)(f^*) = m(f)$ $(m \in M(H^\infty), f \in H^\infty)$. Let $D_{\infty,\partial D} = \tau(D_\infty)$. It is clear that $D_{\infty,\partial D} = \{m \in M(H^\infty)\colon m(f^*) = f(w)$ for some $w \in D$ and all $f^* \in H^\infty(\partial D)\}$.

THEOREM 6. *The map $p\colon M(H^\infty) \to \bar D$ defined by $p(m) = m|_{A(\bar D)}$ is continuous and onto. The map p^{-1} is well-defined on D and is a homeomorphism onto an open subset of $M(H^\infty)$. This subset is precisely D_∞.*

For $\lambda \in \partial D$, let $F_\lambda = \{p^{-1}(\lambda)\}$. It follows from Theorem 6 that $M(H^\infty) \setminus D_\infty$ is closed and that $M(H^\infty)\setminus D_\infty = \bigcup_{\lambda\in\partial D} F_\lambda$. Define $p_D\colon M(H^\infty + C)$

$\to M(H^\infty)$ by $p_D(m) = m|_{H^\infty}$ and $p_{\partial D}$: $M(H^\infty(\partial D)+C(\partial D))\to M(H^\infty(\partial D))$ by $p_{\partial D}(m) = m|_{H^\infty(\partial D)}$.

THEOREM 7. *The map p_D is a homeomorphism from $M(H^\infty+C)$ onto $M(H^\infty)$.*

THEOREM 8. *The map $p_{\partial D}$ is a homeomorphism from $M(H^\infty(\partial D)+C(\partial D))$ onto $M(H^\infty(\partial D))\setminus D_{\infty,\partial D}$.*

3. The necessity part of Theorem 1. We follow the proof of McDonald [11] for the case $D = B^n$ but we will have to do a little work to show that the modified ideas of [11] will go through for general D.

First we need some results on the Bergman kernel function, $K(w, z)$. In [6] Hörmander showed that if $\lambda \in \partial D$, then $K(z, z)\to\infty$ as $z\to\lambda$. Also Kerzman [9] showed that $K(w, z)$ is C^∞ up to the boundary of D simultaneously in z and w, except at points where $z = w\in\partial D$, i.e., $K\in C^\infty(\bar D\times\bar D\setminus F)$ where $F = \{(w, z)\in\partial D\times\partial D: z = w\}$.

Now given $\lambda\in\partial D$ and $\lambda_m\in D$ converging to λ as $m\to\infty$ we let $f_m(z) = K(z, \lambda_m)/K(\lambda_m, \lambda_m)^{1/2}$ for $z\in D$. Then $f_m\in H^2$ and $\|f_m\|_2 = 1$. It is clear that f_m converges pointwise to zero on D a.e. by the above remarks. Also, by Kerzman's theorem, $K(\lambda_m, z)$ is uniformly bounded on the complement of any open neighbourhood of λ for m large enough whereas $K(\lambda_m, \lambda_m)\to\infty$ as $m\to\infty$. It follows that f_m converges uniformly to zero in the complement of any open neighbourhood of λ in D.

LEMMA 9. *Let $\varphi\in C$, $\lambda\in\partial D$, $\varphi(\lambda) = \alpha$. For any sequence $\{\lambda_m\}$ in D converging to λ we have $\|(T_\varphi-\alpha)f_m\|_2\to 0$.*

Proof.

$$\|T_\varphi f_m-\alpha f_m\|_2^2 = \|\varphi f_m-\varphi(\lambda)f_m\|_2^2 = \int_D \big|(\varphi-\varphi(\lambda))f_m(z)\big|^2 dV(z)$$

$$= \int_N \big|(\varphi-\varphi(\lambda))f_m(z)\big|^2 dV(z)+\int_{D\setminus N}\big|(\varphi-\varphi(\lambda))f_m(z)\big|^2 dV(z)$$

for any neighbourhood N of λ. The right-hand side can be made as small as we like by choosing N to be a neighbourhood of λ so that $|\varphi-\varphi(\lambda)|$ is sufficiently small in N and by choosing m large enough so that $|f_m(z)|$ is uniformly small on $D\setminus N$.

LEMMA 10. *If $\varphi\in H^\infty+C$ and φ is not bounded away from α on some neighbourhood of ∂D, then there exists $\lambda\in\partial D$ and a sequence $\{\lambda_m\}$ in D such that $\lambda_m\to\lambda$, $\varphi(\lambda_m)\to\alpha$ and $\|(T_{\bar\varphi}-\bar\alpha)f_m\|_2\to 0$ as $m\to\infty$.*

Proof. The same proof as given for Lemma 2.3 in [11] will work here.

THEOREM 11. *For $\varphi\in H^\infty+C$, T_φ is right Fredholm only if φ is bounded away from zero in a neighbourhood of ∂D.*

Proof. Suppose φ is not bounded away from zero in a neighbourhood of ∂D. Then, by Lemma 10, there exists a sequence of unit vectors $\{f_m\}$ in H^2 with $\|T_{\bar\varphi}f_m\|_2\to 0$. Choose $g\in H^2$ and a subsequence $\{f_{m_k}\}$ with $f_{m_k}\to g$ weakly. If T_φ were right Fredholm, then there would exist $S\in \mathrm{BL}(H^2)$ and $K\in\mathscr{K}(H^2)$ such that $ST_{\bar\varphi}+K = I$. We would then have

$$\|f_{m_k}-Kg\|_2 = \|(ST_{\bar\varphi}+K)f_{m_k}-Kg\|_2\leqslant\|S\|\,\|T_{\bar\varphi}f_{m_k}\|_2+\|Kf_{m_k}-Kg\|_2.$$

Since K is compact, the right-hand side converges to zero as $k\to\infty$ and so f_{m_k} converges to Kg in norm. But $\{f_{m_k}\}$ converges pointwise to zero a.e. and so the only function it can converge to in H^2 in norm is 0. However $\{f_{m_k}\}$ is a sequence of unit vectors and so would have to converge in norm to a unit vector. Thus the sequence $\{f_{m_k}\}$ cannot converge in norm and this contradiction completes the proof.

4. The sufficiency part of Theorem 1. Our initial aim in this section is to show, using results of Section 2, that if $\varphi\in H^\infty+C$ and is bounded away from zero in a neighbourhood of ∂D, then there exists ψ in $H^\infty+C$ and h in C with $h|_{\partial D}\equiv 0$ such that $\varphi\psi+h\equiv 1$.

We consider φ^* which belongs to $H^\infty(\partial D)+C(\partial D)$ and we want to prove that φ^* is invertible in $H^\infty(\partial D)+C(\partial D)$ when φ is bounded away from zero in a neighbourhood of ∂D. By Theorem 8, $M(H^\infty(\partial D)+C(\partial D)) = M(H^\infty(\partial D))\setminus D_{\infty,\partial D}$ where $D_{\infty,\partial D} = \{m\in M(H^\infty(\partial D)):\ m(f^*) = f(w)$ for some $w\in D$ and all $f^*\in H^\infty(\partial D)\}$. Then let m be any element of $M(H^\infty(\partial D))\setminus D_{\infty,\partial D}$. Then m belongs to the fibre above a point $\lambda\in\partial D$, i.e., in the notation of Section 2, $m\in\tau(F_\lambda)$ for some $\lambda\in\partial D$.

Write $\varphi^* = f^*+g^*$ where $f^*\in H^\infty(\partial D)$ and $g^*\in C(\partial D)$. We consider the function $k^* = f^*+g^*(\lambda)$ which belongs to $H^\infty(\partial D)$. Now $\varphi^*-k^* = g^*-g^*(\lambda)$. By our assumption k is bounded away from zero in a neighbourhood of λ (since $\varphi-k = g-g^*(\lambda)$ and so $|\varphi|\leqslant|g-g^*(\lambda)|+|k|$ and so we choose our neighbourhood so that φ is bounded away from zero and $|g-g^*(\lambda)|$ is sufficiently small). We now state a crucial lemma.

LEMMA 12. *If $H\in H^\infty$ is bounded away from zero in a neighbourhood of $\lambda\in\partial D$, then $\varphi(H)\neq 0$ for $\varphi\in F_\lambda$.*

Proof. See Theorem 2.3 of [13].

Using this lemma it follows that $\tau^{-1}(m)(k)\neq 0$, i.e., $m(k^*)\neq 0$. But $m(\varphi^*) = m(k^*)+m(g^*)-g^*(\lambda) = m(k^*)+g^*(\lambda)-g^*(\lambda)\neq 0$. Since m was any element of $M(H^\infty(\partial D)+C(\partial D))$, it follows that φ^* is invertible in $H^\infty(\partial D)+C(\partial D)$. Thus we have the following theorem.

THEOREM 13. *Let $\varphi\in H^\infty+C$. The following are equivalent:*

(1) *φ is bounded away from zero in a neighbourhood of ∂D;*

(2) *φ^* is invertible in $H^\infty(\partial D)+C(\partial D)$.*

Proof. The remarks above show that (1) $\Rightarrow$ (2). Standard Poisson integral calculations show (2) $\Rightarrow$ (1).

So for any $\varphi\in H^\infty+C$ which is bounded away from zero in a neighbourhood of ∂D there is $\psi^*\in H^\infty(\partial D)+C(\partial D)$ with $\varphi^*\psi^*\equiv 1$ on ∂D, i.e., there exists $h\in C$ with $h|_{\partial D}\equiv 0$ such that $\varphi\psi+h\equiv 1$ on D.

THEOREM 14. *If $\varphi \in H^\infty + C$ is bounded away from zero in a neighbourhood of ∂D, then T_φ is Fredholm.*

Proof. By above $\varphi\psi + h \equiv 1$ on D for some $\psi \in H^\infty + C$ and $h \in C$ with $h|_{\partial D} \equiv 0$. Then we have

$$T_{\varphi\psi+h} = I \Rightarrow T_{\varphi\psi} + T_h = I \Rightarrow T_\varphi T_\psi + T_h = I + K_1 \quad \text{(by Prop. 4)}$$
$$\Rightarrow T_\varphi T_\psi = I + K_2 \quad \text{(by Prop. 5)}.$$

Similarly $T_\psi T_\varphi = I + K_3$. ($K_i$, $1 \leqslant i \leqslant 3$, are compact operators.) Hence T_φ is Fredholm.

Theorems 11 and 14 give Theorem 1.

5. Toeplitz operators with symbol in $H^\infty(\partial D) + C(\partial D)$. The proof of Theorem 2 follows the same pattern as that of Theorem 1 and so we will only point out the essential differences.

The necessity part of Theorem 2. Here we will need to use the Szegö kernel rather than the Bergman kernel. We refer to [14], p. 17, 18 for a description of the basic properties of the Szegö kernel. We need the following result concerning the singularities of the Szegö kernel.

THEOREM 15 [3]. *Let D be a strongly pseudoconvex domain with smooth boundary and let ϱ be a defining function for D. Then*

(a) *the Szegö kernel $S(\zeta, w)$ is C^∞ on $\partial D \times \partial D \setminus \{(w, w) \colon w \in \partial D\}$;*

(b) *there exists $\psi(\zeta, w) \in C^\infty(C^n \times C^n)$ such that*

(1) $\psi(x, x) = -i\varrho(x)$,

(2) $\bar\partial_\zeta$ *and* $\bar\partial_w$ *vanish to infinite order at* $\zeta = w$,

(3) $\psi(\zeta, w) = -\overline{\psi(w, \zeta)}$,

(4) *there exist $F; G \in C^\infty(\partial D \times \partial D)$ such that* $S(\zeta, w) = F.(-i\psi)^{-n} + G.\log(-i\psi)$.

This theorem tells us that if $S(\zeta, w)$ becomes infinitely large as $w \to \zeta \in \partial D$, then it does so in a predictable fashion. It is C^∞ everywhere else.

The Szegö kernel can be defined in the following way (see [14]): let $\{\psi_j(\zeta)\}_{j=1}^\infty$ be an orthonormal basis for $H^2(\partial D)$ where we consider ψ_j as defined on D; then $S(\zeta, w) = \sum_{j=1}^\infty \psi_j(\zeta)\bar\psi_j(w)$ where the series converges uniformly for ζ, w restricted to any compact subset of $D \times D$. Therefore, for $\zeta \in D$,

$$(1) \qquad S(\zeta, \zeta) = \sum_{j=1}^\infty |\psi_j(\zeta)|^2 = \sup_{f_j \in l^2} \frac{\left|\sum_{j=1}^\infty f_j \psi_j(\zeta)\right|}{\sum_{j=1}^\infty |f_j|^2} = \sup_{f \in H^2(\partial D)} \frac{|f(\zeta)|^2}{\|f\|_2^2}.$$

Let $\zeta_0 \in \partial D$. Now, since D is strictly pseudoconvex with smooth boundary, there is a holomorphic peaking function, g, at ζ_0; i.e., g is holomorphic in a neighbourhood of $\bar D$, $g(\zeta_0) = 1$ and $|g(\zeta)| < 1$ for all $\zeta \in \bar D \setminus \{\zeta_0\}$, [5], p. 275. By taking a high enough power of g, say g^N, we may suppose that $\|g^N\|_2$ is very small $(g \in H^2(\partial D))$. However, by choosing ζ close enough to ζ_0, it is clear that $|g^N(\zeta)|$ is as near to 1 as we please. It follows from (1) that $S(\zeta, \zeta)$ becomes arbitrarily large if ζ is near enough to ζ_0, i.e., $|S(\zeta, \zeta)| \to \infty$ as $\zeta \to \zeta_0$. ... (2)

Now, given $\zeta_0 \in \partial D$ and a sequence $\zeta_m \to \zeta_0$ as $m \to \infty$, we let $f_m(\zeta) = S(\zeta, \zeta_m)/S(\zeta_m, \zeta_m)^{1/2}$ where we are considering $S(\zeta, w)$ as being defined on $\bar D \times \bar D$. $f_m \in H^2(\partial D)$ and $\|f_m\|_2 = 1$ for each m. For $\zeta \neq \zeta_0$ $S(\zeta, w)$ is uniformly bounded in $w \in \partial D$ in a neighbourhood of ζ_0 by Theorem 15 (a). This extends to $w \in D$ by the definition of the Szegö kernel. It then follows from (2) that f_m converges pointwise to zero a.e. on ∂D. It also follows from the above remarks that f_m converges uniformly to zero in the complement of any neighbourhood of ζ_0 in ∂D. Hence $\{f_m\}$ cannot have a convergent subsequence.

Using the above information we can then prove the appropriate Lemmas 9′ and 10′. We then have

LEMMA 9′. *Let $\varphi \in C(\partial D)$, $\lambda \in \partial D$, $\varphi(\lambda) = a$. For any sequence $\{\lambda_m\}$ in D converging to λ we have $\|(T_\varphi - a)f_m\|_2 \to 0$.*

Proof. Similar to the proof of Lemma 9.

LEMMA 10′. *If $\varphi \in H^\infty(\partial D) + C(\partial D)$ and φ (extended to D through the Poisson-Szegö integral) is not bounded away from a on some neighbourhood of ∂D, then there exists $\lambda \in \partial D$ and a sequence $\{\lambda_m\}$ in D such that $\lambda_m \to \lambda$, $\varphi(\lambda_m) \to a$ and $\|(T_{\bar\varphi} - \bar a)f_m\|_2 \to 0$ as $m \to \infty$.*

Proof. Since $\bar D$ is compact, we can find $\{\lambda_m\} \subseteq D$ converging to some $\lambda \in \partial D$ with $\varphi(\lambda_m) \to a$. By the remarks in the introduction we have $\varphi = (f+g)^* + k^*$ where $f \in H^\infty$, $g \in C$, and $k \in C$ with $k|_{\partial D} \equiv 0$. Then

$$\|(T_{\bar\varphi} - \bar a)f_m\| \leqslant \|(T_{\bar f^*} - \bar\beta)f_m\|_2 + \left\|\left(T_{\overline{(g+k)^*}} - (\bar a - \bar\beta)\right)f_m\right\|_2$$

where $\beta = \lim f(\lambda_m)$. The second term on the left tends to zero by Lemma 9′. It is not hard to see from the reproducing properties of the Szegö kernel that $\|T_{\bar f^*} f_m\|_2 = |\bar f(\lambda_m)|$. The result follows.

Theorem 11′ follows in the same way as Theorem 11.

The sufficiency part of Theorem 2. It follows from Section 4 that if $\varphi \in H^\infty(\partial D) + C(\partial D)$ is bounded away from zero in a neighbourhood of ∂D, then φ is invertible in $H^\infty(\partial D) + C(\partial D)$, i.e., there exists $\psi \in H^\infty(\partial D) + C(\partial D)$ with $\varphi\psi \equiv 1$ on ∂D. Then

$$T_{\varphi\psi} = I \Rightarrow T_\varphi T_\psi + K_1 = T_\psi T_\varphi + K_2 = I \quad \text{(by Prop. 4)}$$
$$\Rightarrow T_\varphi \text{ is Fredholm.}$$

We have now proved Theorem 2.

Remarks. (1) In the remarks before Lemma 9′ we noted that $|S(\zeta, \zeta)| \to \infty$ as $\zeta \in D$ tends to $\zeta_0 \in \partial D$. Now S. Krantz has pointed out that, for $\zeta, w \in \partial D$, $S(\zeta, w)$ *must* grow arbitrarily large near (ζ_0, ζ_0) where $\zeta_0 \in \partial D$ for the following reason. Suppose $S(\zeta, w)$ were bounded in a neighbourhood of (ζ_0, ζ_0) in ∂D. Then for ζ near ζ_0, $\zeta \in \partial D$, $S(\zeta, w)$ is uniformly bounded in w by Theorem 15 (a) $(w \in \partial D)$. This extends to $\zeta \in D$, ζ near ζ_0 by the definition of the Szegö kernel. Now let $g \in C(\bar{D})$ be a holomorphic peaking function which peaks at ζ_0. Let $\zeta_m \in D$ converge to ζ_0 non-tangentially. Then $[g(\zeta_m)]^k = \int_{\partial D} S(\zeta, \zeta_m)[g(\zeta)]^k\, d\sigma(\zeta)$ for each m. Letting $k, m \to \infty$ appropriately the left-hand side of the equation tends to 1, while the right-hand side tends to zero. (Recall that, if we choose k large enough, $\int_{\partial D} |g(\zeta)|^{2k} d\sigma(\zeta)$ can be made arbitrarily small.) This contradiction shows that $S(\zeta, w)$ is unbounded on $\partial D \times \partial D$ near (ζ_0, ζ_0) for $\zeta_0 \in \partial D$. By Theorem 15 (b) (4), $S(\zeta, w)$ is unbounded as $(\zeta, w) \to (\zeta_0, \zeta_0)$ $((\zeta, w) \in \partial D)$. It follows from the definition of the Szegö kernel that $|S(\zeta, w)| \to \infty$ as $(\zeta, w) \in \bar{D} \times \bar{D}$ tends to (ζ_0, ζ_0) where $\zeta, w \to \zeta_0$ non-tangentially.

It would be of interest to settle the following question: If $(\zeta_0, \zeta_0) \in \partial D \times \partial D$ and $(\zeta, w) \in D \times D$, is $\lim_{(\zeta,w)\to(\zeta_0,\zeta_0)} |S(\zeta, w)| = \infty$? According to the above remarks, the only case unresolved is when $\zeta \neq w$ and either ζ or w (or both) converge to ζ_0 tangentially. It may be possible to prove the conjecture for this case also by extending the methods of [3].

(2) In proving the sufficiency part of Theorem 2 we used Proposition 4. In order to extend Theorem 2 to the case $n = 2$ we would need the following result:

$\psi \in C(\partial D) \Rightarrow$

the restriction of $(I-P)M_\psi$ to $H^2(\partial D)$ is a compact operator.

As noted in the proof of Proposition 3 this was proved by Raeburn [12], Theorem 1.2 for the case $n > 2$. The methods of Raeburn do not carry through since the Neumann theory for the $\bar{\partial}_b$ complex does not work right for $n = 2$. It is possible that there may be a way to get around this.

6. Local Toeplitz operators. In [4] Douglas shows that, when $D = B^1$, the open unit disc in $\boldsymbol{C}$, then the Fredholmness of T_φ for $\varphi \in L^\infty(\partial D)$ depends only on the local properties of the symbol φ. We want to remark here that an analogous result holds for a strongly pseudoconvex domain with smooth boundary. We will only give an indication of the necessary changes needed in Douglas' proof and we leave it to the reader to fill in the details.

For a family of functions $\mathscr{F}$ in L^∞, let $\mathfrak{T}(\mathscr{F})$ be the C^*-algebra generated by the Toeplitz operators on H^2 with symbol in $\mathscr{F}$. Note that by

Proposition 3 the quotient algebra $\mathfrak{T}(L^\infty)/\mathscr{K}(H^2)$ has a non-trivial centre which contains $\mathfrak{T}(C)/\mathscr{K}(H^2)$ and this last algebra is equal to $C(\partial D)$ by [12], Theorem 3.1.

Recall from Section 2 that $M(H^\infty)\setminus D$ is fibred by ∂D. The main tool we need in making Douglas' proofs go through is the fact that the cluster set of $f \in H^\infty$ at $\lambda \in \partial D$ is equal to the range of the Gelfand transform $\hat{f}$ on the fibre in $M(H^\infty)$ over λ. This is proved in [13], by using a well-known result of Kerzman [8] concerning the existence of a smooth solution, u, to the equation $\bar{\partial} u = \varphi$ where φ is a smooth, bounded, closed $(0, 1)$-form on D.

For $\lambda \in \partial D$ let $\mathfrak{T}_\lambda$ be the closed ideal in $\mathfrak{T}(L^\infty)$ generated by $\{T_\varphi\colon \varphi \in C,\ \varphi(\lambda) = 0\}$ and let $G_\lambda = \mathfrak{T}(L^\infty)/\mathfrak{T}_\lambda$ and let Φ_λ be the canonical map from $\mathfrak{T}_\lambda(L^\infty) \to G_\lambda$. Then it is possible to prove:

THEOREM 16. *If Φ is the *-homomorphism defined by $\Phi = \sum_{\lambda\in\partial D} \oplus \Phi_\lambda$ from $\mathfrak{T}(L^\infty) \to \sum_{\lambda\in\partial D} \oplus G_\lambda$, then the sequence*

$$\{0\} \to \mathscr{K}(H^2) \to \mathfrak{T}(L^\infty) \xrightarrow{\Phi} \sum_{\lambda\in\partial D} \oplus G_\lambda$$

is exact at $\mathfrak{T}(L^\infty)$.

For each $\lambda \in \partial D$ denote $F_\lambda \cap M(L^\infty)$ by ∂F_λ. We have the following propositions:

PROPOSITION 17. *If $\varphi, \psi \in L^\infty$ with Gelfand transforms $\hat{\varphi}, \hat{\psi}$ on $M(L^\infty)$ and λ is in ∂D, then $\Phi_\lambda(T_\varphi) = \Phi_\lambda(T_\psi)$ if $\hat{\varphi}|_{\partial F_\lambda} \equiv \hat{\psi}|_{\partial F_\lambda}$.*

COROLLARY 18. *If $\varphi \in L^\infty$, then T_φ is Fredholm if and only if for each $\lambda \in \partial D$ there exists $\psi \in L^\infty$ such that T_ψ is Fredholm and $\varphi = \psi$ on ∂F_λ.*

COROLLARY 19. *If $\varphi, \psi \in L^\infty$ such that for each $\lambda \in \partial D$ either $\hat{\varphi} - \hat{f}|_{\partial F_\lambda} \equiv 0$ for some f with $\bar{f}$ in H^∞ or $\hat{\psi} - \hat{g}|_{\partial F_\lambda} \equiv 0$ for some g in H^∞, then $T_\varphi T_\psi - T_{\varphi\psi}$ is compact.*

Remark. These results extend results on p. 198, 199 of [4].

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Hölder continuous functions on compact sets and function spaces

by

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Abstract. Let F be a closed set, and let $\mathrm{Lip}(\beta, F)$ denote the space of all functions which are Hölder continuous on F with exponent β. Necessary and sufficient conditions on F are given, which guarantee that every function in $\mathrm{Lip}(\beta, F)$ may be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n)$ belonging to a Besov space or a Bessel potential space. Applications to the theories of harmonic functions and multiple Fourier series are given.

0. Introduction. Let F be a closed set and $0 < \beta \leqslant 1$. It is well known from the Whitney extension theorem that every function in $\mathrm{Lip}(\beta, F)$ may be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n)$. (For the definition of the Lipschitz space $\mathrm{Lip}(\beta, F)$, the Besov space $B_\alpha^{p,q}(\boldsymbol{R}^n)$ and the space $L_\alpha^p(\boldsymbol{R}^n)$ of Bessel potentials, which for integer α coincides with a Sobolev space, we refer to Section 1.) In this paper we consider the following question. For which sets F is it true that every function in $\mathrm{Lip}(\beta, F)$ may be extended to a function in $\mathrm{Lip}(\beta, \boldsymbol{R}^n) \cap B_\alpha^{p,q}(\boldsymbol{R}^n)$ or $\mathrm{Lip}(\beta, \boldsymbol{R}^n) \cap L_\alpha^p(\boldsymbol{R}^n)$? The answer, given in Theorem 1 and Theorem 2, is definitive. The extension operator used in the theorems is the same as in the Whitney extension theorem. Applications of the result are given in Theorem 3 and Theorem 4.

A corresponding question for continuous functions was solved by H. Wallin in [11] and, more generally, by T. Sjölin in [7] and [8]. They proved that every function in $C(F)$, the space of continuous functions on F, may be extended to a function in $C(\boldsymbol{R}^n) \cap L_\alpha^p(\boldsymbol{R}^n)$ or $C(\boldsymbol{R}^n) \cap B_\alpha^{p,p}(\boldsymbol{R}^n)$ if and only if a certain capacity is zero. Our case with Lipschitz continuous functions is of different nature than the continuous case, and the methods used in this paper are entirely different than those in [7], [8] and [11]. It should also be mentioned that the extension parts of Theorem 1 and Theorem 2 were proved, in a less general form, in [2].

Our result is also related to the imbedding theory for functions of several variables (see e.g. [6]), where e.g. the trace of functions in $B_\alpha^{p,q}(\boldsymbol{R}^n)$ or $L_\alpha^p(\boldsymbol{R}^n)$ to sufficiently smooth manifolds is characterized.

1. The spaces $\mathrm{Lip}(\beta, F)$, $L_\alpha^p(\boldsymbol{R}^n)$, and $B_\alpha^{p,q}(\boldsymbol{R}^n)$. Notation.

1.1. General references for the spaces defined in this section are [6] and [9]. The spaces $L_\alpha^p(\boldsymbol{R}^n)$ and $B_\alpha^{p,q}(\boldsymbol{R}^n)$ may be defined in many equiv-