Choosing γ so that $c\gamma^{n+1} \leqslant \frac{1}{2}$, letting $N \to \infty$ and $\varepsilon \to 0$ completes the proof of Theorem III.

By standard arguments Theorem III implies the following important corollary.

(4.21) Corollary. *For* $(a, f) \in L^{(p,p_0)}$, $\sum_{j=0}^{n} (1/p_j) = 1/q < n+1$, $1 < p_0 < \infty$, $1 < p_j \leqslant \infty$, *and* $T_\varepsilon^m(a, f)$ *and* $T_*^m(a, f)$ *the operators defined in Theorem* III, *the following properties are satisfied:*

(1) $\|T_*^m(a, f)\|_q \leqslant c\|(a, f)\|_{(p,p_0)}$.

(2) $\lim_{\varepsilon \to 0} T_\varepsilon^m(a, f)(x)$ *exists almost everywhere.*

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An analog of the Marcinkiewicz integral in ergodic theory

by

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Abstract. Let T be an invertible measure preserving point transformation from a space X onto itself. Define $\tau_B(x) = \inf\{n > 0 \mid T^n x \in B\}$. The analog of the classical Marcinkiewicz integral $I(f)(x)$, is defined by

$$I(f)(x) = \sum_{k=1}^{\infty} \frac{\tau_B(T^k x) f(T^k x)}{k^2}.$$

If f is the characteristic function of a set B, then this integral, like its classical analog, gives a measure of the distance from a point x to the set B. Intuitively it is the average amount of time the point spends outside the set B during its orbit. It is used to give a direct proof that the ergodic Hilbert transform is weak type (1, 1).

Theorems. Let (X, Σ, m) denote a complete nonatomic probability space, and T an ergodic measure preserving invertible point transformation from X onto itself. For $B \in \Sigma$, with $0 < m(B) < 1$ and a point x, consider the orbit, $x, Tx, T^2x, \ldots$ Following this orbit we will enter and leave the set B infinitely often. In the following we will be interested in various measures of the distance from the point x to the set B.

A natural measure is the recurrence time, defined by

$$\nu_B(x) = \begin{cases} \inf\{n > 0 \mid T^n x \in B\}, & x \in B, \\ 0, & x \notin B. \end{cases}$$

This function has been previously studied by Kac [6] and Blum and Rosenblatt [1]. Kac has shown that $\|\nu_B\|_1 = 1/m(B)$, and Blum and Rosenblatt have studied the higher moments.

A second measure, related to the recurrence time, is defined by

$$\tau(x) = \inf\{n \geqslant 0 \mid T^{-n}x \in B\}.$$

It is not hard to see that $\tau(x)$ may fail to be in $L^1(X)$. In fact $\tau(x) \in L^1(X)$ if and only if $\nu_B(x)$ has a finite second moment.

Both of the above measurements are local in the sense that after a return to B, they fail to observe the remainder of the orbit. However, to study certain operators, such as the ergodic Hilbert transform [see Section 3], a measure of distance which looks at the entire orbit must be used. With this in mind we define a distance function $I(x)$ by

$$I(x) = \sum_{k=1}^{\infty} \frac{\tau(T^k x)}{k^2}.$$

Note that if τ is bounded, then $I(x)$ is bounded, and if τ grows large early in the orbit, then $I(x)$ is large. The function $I(x)$ also has the property that it looks at the entire (positive) orbit. A very long excursion from the set B late in the orbit would be detected by a large $I(x)$. Intuitively $I(x)$ is a measure of the average length of time spent outside the set B.

This function can be studied by replacing $I(x)$ by the operator $I(f)(x)$ defined by

$$I(f)(x) = \sum_{k=1}^{\infty} \frac{\tau(T^k x) f(T^k x)}{k^2}.$$

The original problem is then the study of $I(\chi_{B^c})(x)$ since $\chi_{B^c}(T^k x) = 0$ if and only if $\tau(T^k x) = 0$, where $\chi_A(x)$ is the characteristic function of A, taking the value 1 for $x \in A$ and 0 for $x \notin A$.

We can also consider

$$I_\lambda(f)(x) = \sum_{k=1}^{\infty} \frac{\tau^\lambda(T^k x) f(T^k x)}{k^{1+\lambda}}.$$

This operator should be compared with the classical Marcinkiewicz Integral defined by

$$J_\lambda(f)(x) = \int_{-\infty}^{\infty} \frac{\delta^\lambda(y) f(y)}{|x-y|^{1+\lambda}}\, dy$$

where $\delta(y)$ is the distance from y to a fixed set B.

Following the idea used by Zygmund [8], we modify $I_\lambda(f)$ and consider

$$I_\lambda^*(f)(x) = \sum_{k=1}^{\infty} \frac{\tau^\lambda(T^k x) f(T^k x)}{k^{1+\lambda} + \tau^{1+\lambda}(x)},$$

which coincides with $I_\lambda(f)(x)$ for all $x \in B$. The advantage of $I_\lambda^*(f)$ over $I_\lambda(f)$ is that $I_\lambda^*(f)$ will be in $L^1(X)$ for $f \in L^1(X)$ while $I_\lambda(f)$ may fail to be in $L^1(X)$ even for $f \in L^\infty(X)$. In fact we can prove the following theorem:

THEOREM 1.1. *If $f \in L^p(X)$, $1 \leqslant p < \infty$, then $I_\lambda^*(f) \in L^p$ and $\|I_\lambda^*(f)\|_p \leqslant c_{\lambda,p}\|f\|_p$.*

In reference to the original problem, this says that

$$\int_X \Big| \sum_{k=1}^{\infty} \frac{\tau^\lambda(T^k x)\chi_{B^c}(T^k x)}{k^{1+\lambda} + \tau(x)} \Big|^p dx \leqslant c_{\lambda,p} \int_X |\chi_{B^c}(x)|^p\, dx$$

or if we integrate only over the set B, then we have

$$(*) \qquad \int_B \Big| \sum_{k=1}^{\infty} \frac{\tau^\lambda(T^k x)}{k^{1+\lambda}} \Big|^p dx \leqslant c_p m(B^c).$$

This estimate is good if B is large, and in which case it says that most of the points in the large set B are close to it with the distance function $I(x)$.

However, if B has small measure, the estimate $(*)$ seems rather large, since we are integrating only over a small set. We can improve the situation with the following result.

THEOREM 1.2. *For all $\lambda > 0$, there exists a constant c_λ such that*

$$\int_B \sum_{k=1}^{\infty} \frac{\tau^\lambda(T^k x)}{k^{1+\lambda}}\, dx \leqslant c_\lambda m(B)\big(1 - \log m(B)\big).$$

This result is the best possible in the sense that as a function of B, the term $1-\log m(B)$ cannot be replaced by a more slowly growing function.

Proofs. To prove the above theorems we begin with two simple but important lemmas. In this section, c and c_λ denote constants, not necessarily the same from line to line.

LEMMA 1.3. *For any positive integer δ, we have*

$$\sum_{k=1}^{\infty} \frac{\delta^\lambda f(T^k x)}{k^{1+\lambda} + \delta^{1+\lambda}} \leqslant c f^*(x),$$

where

$$(**) \qquad f^*(x) = \sup_{n>0} \frac{1}{n} \sum_{k=1}^{\infty} |f(T^k x)|.$$

Proof. We split the sum into two pieces, getting separate estimates for each piece. For the first piece we have

$$\sum_{k=1}^{\delta} \frac{\delta^\lambda f(T^k x)}{k^{1+\lambda} + \delta^{1+\lambda}} \leqslant \sum_{k=1}^{\delta} \frac{f(T^k x)}{\delta} \leqslant f^*(x).$$

For the second piece we sum by parts, yielding

$$\sum_{k=\delta}^{\infty}\frac{\delta^{\lambda}f(T^kx)}{k^{1+\lambda}+\delta^{1+\lambda}} \leqslant \sum_{k=\delta}^{\infty}\frac{\delta^{\lambda}f(T^kx)}{k^{1+\lambda}} \leqslant \sum_{k=\delta}^{\infty}\delta^{\lambda}f(T^kx)\sum_{n=k}^{\infty}\left(\frac{1}{n^{1+\lambda}}-\frac{1}{(n+1)^{1+\lambda}}\right)$$

$$\leqslant (1+\lambda)\sum_{n=\delta}^{\infty}\frac{\delta^{\lambda}}{n^{2+\lambda}}\sum_{k=\delta}^{n}f(T^kx)$$

$$\leqslant c_{\lambda}f^*(x)\sum_{n=\delta}^{\infty}\frac{n\delta^{\lambda}}{n^{2+\lambda}} \leqslant c_{\lambda}f^*(x).$$

LEMMA 1.4. *Let $f\in L^p(x)$ and $g\in L^q(x)$, with $1/p+1/q=1$, and $1\leqslant p<\infty$, then*

$$\int_X\left(\sum_{k=1}^{\infty}\frac{\tau^{\lambda}(T^kx)f(T^kx)}{k^{1+\lambda}+\tau^{1+\lambda}(T^kx)}\right)g(x)\,dx \leqslant c_{\lambda}\|f\|_p\|g\|_q.$$

Proof. For f and g as above, and using Lemma 1.4, we have

$$\int_X\sum_{k=1}^{\infty}\frac{\tau^{\lambda}(T^kx)f(T^kx)}{k^{1+\lambda}+\tau^{1+\lambda}(T^kx)}g(x)\,dx \leqslant \sum_{k=1}^{\infty}\int_X\frac{\tau^{\lambda}(T^kx)f(T^kx)}{k^{1+\lambda}+\tau^{1+\lambda}(T^kx)}g(x)\,dx$$

$$\leqslant \sum_{k=1}^{\infty}\int_X\frac{\tau^{\lambda}(x)f(x)}{k^{1+\lambda}+\tau^{1+\lambda}(x)}g(T^{-k}x)\,dx$$

$$\leqslant \int_X f(x)\sum_{k=1}^{\infty}\frac{\tau^{\lambda}(x)g(T^{-k}x)}{k^{1+\lambda}+\tau^{1+\lambda}(x)}\,dx$$

$$\leqslant \int_X f(x)g^*(x)\,dx \leqslant \|f\|_p\|g^*\|_q \leqslant c\|f\|_p\|g\|_q.$$

The last step can be made because the maximal function (**) is a bounded operation from $L^q(X)$ to $L^q(X)$, $1<q\leqslant\infty$.

Proof of Theorem 1.1. By the above lemma, it follows that the operator

$$H_{\lambda}(f)(x) = \sum_{k=1}^{\infty}\frac{\tau^{\lambda}(T^kx)f(T^kx)}{k^{1+\lambda}+\tau^{1+\lambda}(T^kx)}$$

is bounded on L^p, $1\leqslant p<\infty$. Since $\tau(T^kx)\leqslant k+\tau(x)$ and consequently $\tau^{1+\lambda}(T^kx)\leqslant c_{\lambda}[k^{1+\lambda}+\tau^{1+\lambda}(x)]$, there exist constants c_1 and c_2 such that $c_1I_{\lambda}^*(f)(x)\leqslant H_{\lambda}(f)(x)\leqslant c_2I_{\lambda}^*(f)(x)$. The fact that H_{λ} is L^p bounded, and the above inequality implies I_{λ}^* is L^p bounded, $1\leqslant p<\infty$.

Proof of Theorem 1.2. Proceeding as in the proof of Lemma 1.4, we have:

$$\int_B I_{\lambda}(x)\,dx = \int_X \chi_B(x)I_{\lambda}(x)\,dx = \int_X\chi_B(x)\sum_{k=1}^{\infty}\frac{\tau^{\lambda}(T^kx)}{k^{1+\lambda}}\,dx$$

$$= \sum_{k=1}^{\infty}\int_X\chi_B(x)\frac{\tau^{\lambda}(T^kx)}{k^{1+\lambda}}\,dx = \sum_{k=1}^{\infty}\int_X\frac{\chi_B(T^{-k}x)\tau^{\lambda}(x)}{k^{1+\lambda}}\,dx$$

$$= \int_X\sum_{k=1}^{\infty}\frac{\chi_B(T^{-k}x)\tau^{\lambda}(x)}{k^{1+\lambda}}\,dx.$$

However, the sum does not start until $\chi_B(T^{-k}x)\neq 0$; i.e., until $T^{-k}x\in B$. The first k for which this is true is $\tau(x)$. Consequently the integrand becomes

$$\sum_{k=\tau(x)}^{\infty}\frac{\chi_B(T^{-k}x)\tau^{\lambda}(x)}{k^{1+\lambda}},$$

but by the proof of Lemma 1.4 this is less than $\chi_B^*(x)$, the maximal function of $\chi_B(x)$. We now have

$$\int_B I(x)\,dx \leqslant c\int_X\chi_B^*(x)\,dx \leqslant c\int_0^{\infty}m\{x\,|\,\chi_B^*(x)>\lambda\}\,d\lambda$$

$$\leqslant c\int_0^{m(B)}m\{x\,|\,\chi_B^*(x)>\lambda\}\,d\lambda + c\int_{m(B)}^{1}m\{x\,|\,\chi_B^*(x)>\lambda\}\,d\lambda$$

$$\leqslant cm(B)+c\int_{m(B)}^{1}\frac{1}{\lambda}\|\chi_B\|_1\,d\lambda$$

$$\leqslant cm(B)+cm(B)[\log 1-\log m(B)]\leqslant cm(B)[1-\log m(B)].$$

To see that the estimate in Theorem 1.2 cannot be improved, consider a very tall Rokhlin tower of height N. Let B be the base of the tower, then for $x\in B$, $I(x)\geqslant\sum_{k=1}^{N}\frac{k}{(k+1)^2}$. Thus $I(x)\lesssim\log m(B)$ for $x\in B$. Integrating over B, we get $\int_B I(x)\,dx\geqslant -m(B)\log m(B)$ since $N=1/m(B)$. A similar argument also shows that if the function

$$I(x) = \sum_{k=1}^{\infty}\frac{\tau(T^kx)}{k^2}$$

is replaced by

$$\sum_{k=1}^{\infty} \frac{\tau(T^k x)}{k^{2-\varepsilon}},$$

this function fails to satisfy any inequality of the above type; in fact, the integral of this function over the set B, can be $+\infty$, for any $\varepsilon > 0$.

Higher dimensional results. Theorem 1.1 can be extended to higher dimensions. In particular, let S and T be two non-commuting, measure preserving, point transformations mapping the space X onto itself. For a given set B, define

$$\tau(x) = \inf\{(m^2+n^2)^{1/2} \,|T^m S^n x \in B\}.$$

The analog of the operator $I_\lambda^*(f)(x)$ is the operator

$$H_\lambda^*(f)(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\tau^\lambda(T^k S^j x) f(T^k S^j x)}{(k^2+j^2)^{3/2} + \tau^{2+\lambda}(T^k S^j x)}.$$

THEOREM 2.1. *For $f \in L^p(X)$, $1 \leqslant p < \infty$; the operator $H_\lambda^*(f) \in L^p$ and* $\|H^*(f)\|_p \leqslant c_p \|f\|_p$.

Proof. The proof is essentially the same as the proof of Theorem 1.1. The required maximal function is defined by

$$f^*(x) = \sup_{m>0, n>0} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} |f(S^j T^k x)|.$$

This maximal function has been shown (by Zygmund [7]) to be bounded on $L^p(X)$, $1 < p \leqslant \infty$. Integrating with respect to $g \in L^q(X)$, we have

$$\int H_\lambda^*(f)(x) g(x)\,dx \leqslant c_\lambda \int f(x) g^*(x)\,dx \leqslant c_\lambda \|f\|_p \|g^*\|_q \leqslant c_{\lambda,p} \|f\|_p \|g\|_q.$$

Taking the sup over all $g \in L^p(X)$ with $\|g\|_q = 1$ completes the proof.

The ergodic Hilbert transform. As an application of this distance function, consider the ergodic Hilbert transform defined by

$$\tilde{f}(x) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{f(T^k x)}{k}.$$

This transform was introduced by Cotlar [4] in 1955, and has since been studied by Calderón [2], Coifman and Weiss [3], and others. The first question is, does $f(x)$ exist for $\tilde{f} \in L^p(X)$? The usual proofs use the fact

that the classical Hilbert transform, defined by

$$f(x) = \lim_{\varepsilon\to 0} \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \frac{f(x-y)}{y}\,dy,$$

is weak type $(1, 1)$ and strong type (p, p). The classical results and a transference argument imply the same results in the ergodic theory setting. However, the results of Theorem 1.1 can be used to give a direct proof that the ergodic Hilbert transform is weak type $(1, 1)$.

As usual, the first step is a decomposition of $f \in L^1(X)$ into a sum:

$$f(x) = g(x) + b(x)$$

where g is in $L^2(X)$ and b is supported on a small set. This is just the ergodic analog of the Calderón–Zygmund decomposition. This analog is discussed in [5]. In this section we need to use a two-sided maximal function defined by

$$f^*(x) = \sup_n \frac{1}{|n|+1} \sum_{k=0}^{n} |f(T^k x)|.$$

Using the notation of [5], we let $\{x|\ f^*(x) \leqslant \lambda\}$ be the base of the Kakutani construction. The function b_j is supported on the column C_j and is obtained from f by subtracting off the mean value. More precisely we define

$$b_j(x) = \begin{cases} f(x) - \dfrac{1}{j}\displaystyle\sum_{k=1}^{j-1} f(T^k x^*), & x \in C_j, \\ 0, & \text{elsewhere} \end{cases}$$

where x^* is the first point in the sequence $T^{-1}x$, $T^{-2}x, \ldots$ which lies in the base. The function b is defined by

$$b(x) = \sum_j b_j(x)$$

and g is defined by

$$g(x) = f(x) - b(x).$$

As in the classical case $g(x)$ is in $L^2(X)$ and $L^\infty(X)$, with $\|g\|_\infty \leqslant C\lambda$. For this piece we need to know that $\tilde{f}$ is strong type $(2, 2)$. This is Cotlar's result [4] using his theory of quasiorthogonal operators. The proof is exactly the same in the ergodic theory setting as in the classical setting.

For the functions b_j we need to work harder. In the classical case the function b_j is supported on an interval I_j. Estimates are needed for the expression

$$\int_{-\infty}^{\infty} \frac{b_j(y)}{x-y}\,dy.$$

As y varies from $-\infty$ to ∞, we pass through I_j exactly once. In our case we study

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{b_j(T^k x)}{k}.$$

As k varies from $-\infty$ to ∞ we pass through the support of b_j not once but infinitely often. As usual, we need an estimate of $m\{x|\ \tilde{b}(x)>\lambda\}$. Denote by C_j^* the column C_j expanded 3 times; i.e., $C_j^* = T^{-(j-1)}C_j \cup C_j \cup T^{j-1}C_j$. Since $m\{\bigcup_j C_j^*\} \leqslant \frac{C}{\lambda}\|f\|_1$, it is enough to study $m\{x \in (\bigcup_j C_j^*)^c|\ \tilde{b}(x)>\lambda\}$. By Chebyshev we have

$$(3.1)\qquad m\{x\in(\bigcup_j C_j^*)^c|\ \tilde{b}(x)>\lambda\} \leqslant \frac{1}{\lambda}\int_{(\cup C_j^*)^c} |\tilde{b}(x)|\,dx$$

$$\leqslant \frac{1}{\lambda}\int_{(\cup C_j^*)^c}\Big|\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{b(T^k x)}{k}\Big|\,dx \leqslant \sum_j \frac{1}{\lambda}\int_{(\cup C_j^*)^c}\Big|\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{\chi_{C_j}(T^k x)\,b(T^k x)}{k}\Big|\,dx.$$

Using the mean value property of b_j we can subtract off an appropriate constant each time the sequence $\{T^k x\}$ passes through the column C_j. Messy but straightforward arguments show that

$$\Big|\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{\chi_{C_j}(T^k x)\,b(T^k x)}{k}\Big| \leqslant \lambda \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{\tau_j(T^k x)\,\chi_{C_j}(T^k x)}{k^2},$$

where $\tau_j(x)$ is the distance from x to the set $(C_j^*)^c$. If we split the sum into 2 pieces, a sum with $k>0$ and a sum with $k<0$, then we can use the estimates obtained in Theorem 1.1 on each piece. Consequently

$$\int_{(C_j^*)^c}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{\tau_j(T^k x)\,\chi_{C_j}(T^k x)}{k^2}\,dx \leqslant 2cm(C_j).$$

From the above we get that

$$m\{x\in(\bigcup_j C_j^*)^c|\,\tilde{b}>\lambda\} \leqslant \sum_j 2cm(C_j) \leqslant cm(\bigcup_j C_j) \leqslant \frac{c}{\lambda}\|f\|_1.$$

The last step follows from Theorem (2.1) of [5]. Combining the results on $g+b$ we get that $\tilde{f}$ is weak type $(1,1)$.

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