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Multilinear singular integrals

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Abstract. This paper uses Fourier Transform and Mellin Transform analysis to obtain L^p estimates for certain multilinear singular integrals. The results obtained here extend estimates by Calderón, Coifman and Meyer on commutators of singular integrals to a wider class of multilinear singular integrals.

§ 1. Introduction. In this paper sharp estimates are obtained for operators of the type:

$$\text{(1.1)} \qquad p.v. \int \prod_{j=1}^{n} \left\{ \frac{r_{m_j}(A_j; x, y)}{(x-y)^{m_j}} \right\} \frac{f(y)}{x-y}\, dy$$

where

$$r_{m_j}(A_j;\ x,\ y) = A_j(x) - \sum_{k=1}^{m_j-1} \frac{A_j^{(k)}(y)(x-y)^k}{k!}.$$

These operators are related to those introduced by Calderón in [2] and [3] and studied by Coifman and Meyer in [5], [6] and [7]. We will sometimes denote these operators by $D^N H\{\prod_{j=1}^{n} r_{m_j}(A_j;\ x,\ \cdot) f(\cdot)\}$ where $D^N H$ is the Hilbert transform followed by the nth derivative. (The reason for this notation will become apparent in §3.)

The operators studied in this paper arise naturally from the study of higher commutators of differential and pseudo-differential operators. The simplest case is the commutator $[A, D^N H]$ where A is pointwise multiplication by the function $A(x)$. It has been shown by Calderón [3] that this commutator can be written as the sum of pseudo-differential operators of degree less than or equal to $N-1$ plus an operator of the type studied in this paper.

One can show that higher commutators of the form $[A_1, \ldots, [A_n, D^N H] \ldots]$ can be written as the sum of pseudo-differential operators of degree less than or equal to $N-n$ plus the sum of operators of the type considered in this paper. One example is the second commutator $[A, [B, D^N H]]$

which can be written

$$[A,[B,D^NH]]f = D^NH[r_m(A;\,x,\cdot)r_n(B;\,x,\cdot)f(\cdot)] + \sum_{k=1}^{n-1}(-1)^k\binom{N}{k}D^{N-k}H[r_{N-k}(A;\,x,\cdot)B^{(k)}(\cdot)f(\cdot)] + \sum_{j=1}^{m-1}(-1)^j\binom{N}{j}D^{N-j}H[r_{N-j}(B;\,x,\cdot)A^{(j)}(\cdot)f(\cdot)] + \sum_{t=2}^{N-1}\sum_{j+k=t}(-1)^{j+k}\binom{N}{j,k}D^{N-j-k}H[A^{(j)}B^{(k)}f]$$

where $m+n=N$, $j\geqslant 1$, $k\geqslant 1$ and $\binom{N}{j,k} = \dfrac{N!}{j!\,k!\,(N-j-k)!}$.

The results and methods used in this paper closely follow those developed by Coifman and Meyer in [6] and [7]. It appears as a natural follow up to [7] and the notation and organization have been chosen to be as consistent as possible with it.

Let $a_j\in\mathcal{S}$, $j=1,\ldots,n$, $f\in\mathcal{S}$ ($\mathcal{S}$ is the space of functions which are C^∞ and rapidly decreasing). Let $\sigma\colon R^{n+1}\to C$ be bounded and measurable. Assume $\alpha=(\alpha_1,\ldots,\alpha_n)\in R^n$, $s\in R$ and $(\alpha,s)=(\alpha_1,\ldots,\alpha_n,s)\in R^{n+1}$. We adopt the notation $\sigma(\alpha,s)=\sigma(\alpha_1,\ldots,\alpha_n,s)$, $\hat a(\alpha)=\prod_{j=1}^n \hat a_j(\alpha_j)$ and $d\alpha=\prod_{j=1}^n d\alpha_j$.

Let $P_n=\{(p_1,\ldots,p_n,p_0)\colon 1<p_j<\infty;\ \sum_{j=1}^n(1/p_j)=1/q<1\}$. For $(p,p_0)=(p_1,\ldots,p_n,p_0)$, we define

$$L^{(p,p_0)}=\{(a,f)\colon a_j\in L^{p_j},\ j=1,2,\ldots,n,\ f\in L^{p_0}\},$$

and

$$\|(a,f)\|_{(p,p_0)}=\Big\{\prod_{j=1}^n\|a_j\|_{p_j}\Big\}\|f\|_{p_0}.$$

Using the above notation and assumptions on the functions a_j,f and σ we make the following definitions.

(1.3) DEFINITION.

(1) $T_\sigma(a,f)(x)=\displaystyle\int_{R^{n+1}} e^{isx}\sigma(\alpha,s)\hat a(\alpha)\hat f\Big(s-\sum_{j=1}^n\alpha_j\Big)\,d\alpha\,ds.$

(2) For $(p,p_0)\in P_n$, $\displaystyle\sum_{j=0}^n(1/p_j)=\frac{1}{q}<1$,

$$M_n(p,p_0)=\{\sigma\colon \|T_\sigma(a,f)\|_q\leqslant c\,\|(a,f)\|_{(p,p_0)}\}$$

and

$$M_n=\bigcap_{(p,p_0)\in P_n}M_n(p,p_0).$$

The function σ will be called the *symbol of the operator* T_σ.

For $g(x)$ a function ($x\in R^1$) with $m\geqslant 1$ derivatives, we define the Taylor series remainder operator

$$R^m_{-\alpha}g(x)=g(s-\alpha)-\sum_{k=0}^{m-1}\frac{g^{(k)}(s)(-\alpha)^k}{k!}.$$

We let $R^0_{-\alpha}g(s)=g(s-\alpha)$.

If $m=(m_1,\ldots,m_n)\in Z^n$, $0\leqslant m_j$, $\alpha\in R^n$ and g has $|m|=\sum_{j=1}^n m_j$ derivatives, we denote the n-fold composition of Taylor series of g by

$$R^{(m)}_{(-\alpha)}g(s)=R^{m_1}_{-\alpha_1}\ldots R^{m_n}_{-\alpha_n}g(s).$$

Let

$$\omega_m(\alpha,s)=\frac{R^{(m)}_{(-\alpha)}s^{|m|}\operatorname{sgn}s}{\prod_{j=1}^n\alpha_j^{m_j}}.$$

The operator T_{ω_m} will be called a *commutator of order* $|m|$. We note that if $m_j=1$ for $j=1,2,\ldots,n$, then the operator T_{ω_m} is the nth commutator treated in the papers of Coifman and Meyer [6] and [7]. (The order of the operator T_ω does not have the same meaning as the order of a pseudo-differential operator.)

The main results of this paper are the following theorems.

THEOREM I. *If* $a_1,\ldots,a_n\in\mathcal{S}$, *then*

$$\|T_{\omega_m}(a,f)\|_q\leqslant c\,\|f\|_{p_0}\prod_{j=1}^n\|a_j\|_{p_j}$$

where $1>1/q=\sum_{j=0}^n(1/p_j)$, $1<p_j<\infty$, $j=0,1,\ldots,n$.

THEOREM II. *For* $a_j\in\mathcal{D}$ (C^∞ *functions with compact support*), $a_j=A_j^{(m_j)}=\left(\dfrac{d}{dx}\right)^{m_j}A_j$, *and*

$$r_{m_j}(A_j;\,x,y)=A_j(x)-\sum_{k=0}^{m_j-1}\frac{A_j^{(k)}(y)(x-y)^k}{k!},$$

(1.4)
$$\mathrm{p.v.}\int\prod_{j=1}^n\left\{\frac{r_{m_j}(A_j;\,x,y)}{(x-y)^{m_j}}\right\}\frac{f(y)}{(x-y)}\,dy = c_m\int e^{isx}\omega_m(\alpha,s)\hat a(\alpha)\hat f\Big(s-\sum_{j=1}^n\alpha_j\Big)\,d\alpha\,ds$$

where $c_m=\dfrac{(-i\pi)(-1)^{|m|}}{|m|!\,(2\pi)^{n+1}}$.

THEOREM III. *We define* $T_\varepsilon^m(a,f) = T_\varepsilon^{m_1,\ldots,m_n}(a_1,\ldots,a_n)$ *by*

$$T_\varepsilon^m(a,f)(x) = \int\limits_{|x-y|>\varepsilon} \prod_{j=1}^{n} \left\{\frac{r_{m_j}(A_j;\, x,\, y)}{(x-y)^{m_j}}\right\} \frac{f(y)}{x-y}\, dy$$

and let

$$T_*^m(a,f)(x) = \sup_{\varepsilon>0} |T_\varepsilon^m(a,f)(x)|.$$

Then, for $0<q<\infty$

$$\int |T_*^m(a,f)(x)|^q dx \leqslant c\int \Big\{\Big[\prod_{j=1}^{n} a_j^*(x)\Big] f^*(x)\Big\}^q dx$$

where a_j^* *is the Hardy–Littlewood maximal function of* a.

§ 2. An L^p estimate for smooth functions. We defined the order of the symbol ω_m to be $|m| = \sum_{j=1}^{n} m_j$ where $m = (m_1,\ldots,m_n)\in Z^n$ and $0\leqslant m_j$. The proof of Theorem 1 is by induction on the order of ω_m.

To clarify the inductive hypothesis we make the following observations.

(1) Let $\sigma(a_1,\ldots,a_n,s)\in M_n$. *Then if* $\tau(a_1,\ldots,a_n,\beta,s) = \sigma(a_1,\ldots,a_n,s)$, $\tau\in M_{n+1}$.

(2) *If* $\sigma\in M_n$, *then* $R_{-\beta}^0\sigma(a_1,\ldots,a_n,s) = \sigma(a_1,\ldots,a_n,s-\beta)\in M_{n+1}$.

(3) *The zero order symbols are those* ω*'s of the form* $R_{-a_1}^0\ldots R_{-a_n}^0\operatorname{sgn} s = \operatorname{sgn}(s-a_1-\ldots-a_n)$.

Applying observations (1)–(3) and the fact that $\widehat{Hf}(s) = (-i\pi)\operatorname{sgn} s\hat f(s)$ where $Hf(x) = p.v.\int \frac{f(y)}{x-y}\,dy$, we see that the zero commutators are of the form $[\prod_{j=1}^{n-k} a_j][H(f\prod_{j=k+1}^{n} a_j)]\in M_n$.

The inductive hypothesis is the following: for $\overline{m} = (\overline{m}_1,\ldots,\overline{m}_{\overline{n}})\in Z^{\overline{n}}$, $0\leqslant \overline{m}_g$ and $|\overline{m}|\leqslant |m|-1$, the symbol $\omega_{\overline{m}}\in M_{\overline{n}}$. From observations (1)–(3) it is clear that we need only show that the inductive hypothesis implies $\omega_m\in M_n$ when $m = (m_1,\ldots,m_n)\in Z^n$, $\sum_{j=1}^{n} m_j = |m|$ and $m_j\geqslant 1$.

The symbol

$$\overline{\omega}_m = \frac{sR_{-a_1}^{m_1}\ldots R_{-a_n}^{m_n} s^{|m|-1}\operatorname{sgn} s}{\prod_{j=1}^{n} a_j^{m_j}}$$

is introduced and it is shown that $\omega_m - \overline{\omega}_m$ is a sum of lower order commutators. It is enough to show that $\overline{\omega}_m\in M_n$.

(2.1) DEFINITION. $\mathscr{S}^x = \{\varphi\colon \varphi(e^x)\in\mathscr{S}\}$.

(2.2) LEMMA. *For all* $\varphi\in\mathscr{S}^x$, $\sigma\in M_n$, *and all integers* j, $1\leqslant j\leqslant n$, $\varphi(|s/a_j|)\sigma(a,s)\in M_n$.

Proof. For $\varphi\in\mathscr{S}^x$,

$$\varphi(x) = \int_{-\infty}^{\infty} \hat\psi(\gamma)x^{i\gamma}\,d\gamma$$

where $\psi(x) = \varphi(e^x)\in\mathscr{S}$. Hence

$$\varphi(|s/a_j|)\sigma(a,s) = \int_{-\infty}^{\infty} |s|^{i\gamma}|a_j|^{-i\gamma}\sigma(a,s)\hat\psi(\gamma)\,d\gamma.$$

But $|s|^{i\gamma}$ is a bounded multiplier with norm not exceeding $(1+|\gamma|)$ so that the operator $T_{\varphi(|s/a_j|)\sigma}$ has norm less than or equal to $\int_{-\infty}^{\infty}(1+|\gamma|)^2\|\sigma\|\hat\psi(\gamma)\,d\gamma$ where $\|\sigma\|$ is the operator norm of T_σ.

Since $\mathscr{S}$ is closed under the Fourier Transform, $\hat\psi\in\mathscr{S}$ and we have $\int_{-\infty}^{\infty}|\hat\psi(\gamma)|(1+|\gamma|)^N d\gamma<\infty$ for any positive integer N.

(2.3) LEMMA. *Let* $\tau\colon R^{n+1}\to C$ *be a measurable function. Let* $\mathscr{C}$ *be the set of* $\gamma = (t_1,\ldots,t_n)$ *such that* $|a_1|^{t_1}\ldots|a_n|^{t_n}\tau(a,s)\in M_n$. *Then* $\mathscr{C}$ *is a convex set.*

Proof. The proof is given in proposition 3 of [6]. Its importance to this paper is the following corollary.

(2.4) COROLLARY. *Assume that* $\omega_{m'}\in M_n$ *for* $|m'|<|m|$. *Then for* $0\leqslant\theta_j\leqslant 1$ *and* $\sum_{j=1}^{n}\theta_j = 1$, *we have*,

$$|a_1|^{\theta_1}\ldots|a_n|^{\theta_n}\frac{R_{-a_1}^{m_1}\ldots R_{-a_n}^{m_n}s^{|m|-1}\operatorname{sgn} s}{\prod_{j=1}^{n}a_j^{m_j}}\in M_n.$$

Proof. The proof only involves the case $\theta_1 = 1$, $\theta_j = 0$ for $j = 2,\ldots,n$. The cases $\theta_j = 1$, $\theta_k = 0$ for $k\neq j$ are taken care of by a symmetric argument and the rest of the cases follow by convexity from Lemma (2.3). So one must show

$$|a_j|\frac{R_{-a_1}^{m_1}\ldots R_{-a_n}^{m_n}s^{|m|-1}\operatorname{sgn} s}{\prod_{j=1}^{n}a_j^{m_j}}\in M_n. \tag{2.5}$$

A simple argument shows that if $\sigma(a,s)\in M_n$, then $\operatorname{sgn} a_j\sigma(a,s)\in M_n$. Hence we may replace $|a_j|$ by a_j in (2.5).

If $m_1 = 1$, let $m' = (m_2, m_3, \ldots, m_n)$. Then

$$a_1 \frac{R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^{n} a_j^{m_j}} = \frac{R^{m_2}_{-a_2} \ldots R^{m_n}_{-a_n} (s-a_1)^{|m|-1} \operatorname{sgn}(s-a_1)}{\prod_{j=2}^{n} a_j^{m_j}} - \frac{R^{m_2}_{-a_2} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=2}^{n} a_j^{m_j}}.$$

Calling the first term σ_1 and the second term σ_2 we see that $T_{\sigma_1}(a, f) = a_1 T_{\omega_{m'}}(a, f)$ and $T_{\sigma_2}(a, f) = T_{\omega_{m'}}(a, a_1 f)$. Both T_{σ_1} and T_{σ_2} satisfy Definition (1.3) and so they are both in M_n.

If $m_1 > 1$, let $m' = (m_2, \ldots, m_n)$ as above and let $m'' = (m_1 - 1, m_2, \ldots, m_n)$. Then once again,

$$a_1 \frac{R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^{n} a_j^{m_j}} = \frac{R^{m_1}_{-a_1} R^{m_2}_{-a_2} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{a_1^{m_1-1} \prod_{j=2}^{n} a_j^{m_j}} - (-1)^{m_1-1} \frac{(|m|-1)!}{(m_1-1)!\,|m'|!} \frac{R^{m_2}_{-a_2} \ldots R^{m_n}_{-a_n} s^{|m'|} \operatorname{sgn} s}{\prod_{j=2}^{n} a_j^{m_j}}.$$

The first term is the symbol of the operator $T_{\omega_{m''}}(a, f)$ and the second is a constant times the symbol of $T_{\omega_{m'}}(a, a_1 f)$. Both of these are commutators of order lower than $|m|$ and so satisfy Definition (1.3) by induction. This proves the corollary.

(2.6) Lemma. *$\overline{\omega}_m - \omega_m \in M_n$ where*

$$\overline{\omega}_m = \frac{s R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{j=1}^{n} a_j^{m_j}}.$$

Proof. Lemma (2.6) follows from the following identity. For $\varphi(s)$ a function with $|m|$ derivatives,

(2.7) $$R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s\varphi(s) = s R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} \varphi(s) - \sum_{j=1}^{n} a_j R^{m_1}_{-a_1} \ldots R^{m_j-1}_{-a_j} \ldots R^{m_n}_{-a_n} \varphi(s).$$

Hence,

$$\overline{\omega}_m - \omega_m = \sum_{j=1}^{n} \frac{R^{m_1}_{-a_1} \ldots R^{m_j-1}_{-a_j} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{a_j^{m_j-1} \prod_{\substack{k \neq j \\ k=1}}^{n} a_k^{m_k}}$$

and all of the terms in this sum are in M_n since they are the symbols of commutators of order less than $|m|$.

(2.8) Lemma. *Let $\eta \in C_0^\infty$ where η is even, non-negative, at most one and satisfies $\eta(t) \equiv 1$ for $t \in [-n, n]$ and $\eta(t) \equiv 0$ for $t \notin [-2n, 2n]$. Then*
$$\overline{\omega}_m(a, s) \prod_{j=1}^{n} \big(1 - \eta(s/a_j)\big) \equiv 0.$$

Proof. If $1 - \eta(s/a_j) \neq 0$ for all j, then $|s| > n|a_j|$ for all j and so $\operatorname{sgn}(s - \sum_{j \in J} a_j) = \operatorname{sgn} s$ for any subset J of $\{1, 2, \ldots, n\}$. Thus $\prod_{j=1}^{n} \big(1 - \eta(s/a_j)\big) \neq 0$ implies that

$$\overline{\omega}_m(a, s) = s \operatorname{sgn} s \frac{R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s^{|m|-1}}{\prod_{j=1}^{n} a_j^{m_j}} \equiv 0.$$

Proof of Theorem I. To prove Theorem I it suffices to show that $\overline{\omega}_m \in M_n$ since by Lemma (2.6) $\overline{\omega}_m - \omega_m \in M_n$. From Lemma (2.8),

$$\overline{\omega}_m(a, s) = -\sum_{J \in \mathscr{J}_0} (-1)^{|J|} \prod_{j \in J} \eta(s/a_j) \overline{\omega}_m(a, s)$$

where $\mathscr{J}_0$ is the set of non-empty subsets of $\{1, 2, \ldots, n\}$ and $|J|$ is the number of elements in J. If we define

$$\omega_J(a, s) = \prod_{j \in J} \eta(s/a_j) \overline{\omega}_m(a, s),$$

then

$$\omega_J = \operatorname{sgn} s \Big\{\prod_{j \in J} \eta(|s/a_j|)|s/a_j|^{1/|J|}\Big\} \Big\{\Big[\prod_{j \in J} |a_j|^{1/|J|}\Big] \frac{R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s^{|m|-1} \operatorname{sgn} s}{\prod_{k=1}^{n} a_k^{m_k+1}}\Big\}.$$

Let

$$\tilde{\omega}_J = \Big\{\Big[\prod_{j \in J} |a_j|^{1/|J|}\Big] \frac{R^{m_1}_{-a_1} \ldots R^{m_n}_{-a_n} s^{|m|-1} s \operatorname{sgn} s}{\prod_{k=1}^{n} a_k^{m_k+1}}\Big\}.$$

Then $\tilde{\omega}_J \in M_n$ by Corollary (2.4). $\tilde{\omega}_J \prod_{j \in J} \eta(|s/a_j|)|s/a_j|^{1/|J|} \in M_n$ by Lemma (2.2) and the fact that $t^{1/|J|}\eta(t) \in \mathscr{S}^\infty$. Finally, summing up the J's we get that $\overline{\omega}_m \in M_n$.

§ 3. The fundamental identity. The purpose of this section is to show that the operators considered in §2 can be realized as singular integrals if enough regularity is assumed on the a_j's and f. Specifically, for $a_j \in \mathscr{D}$, $j = 1, 2, \ldots, n$, $f \in \mathscr{D}$, $m = (m_1, \ldots, m_n)$ and $N = \sum_{j=1}^{n} m_j$, T_{ω_m}

$= D^N H\big[\prod_{j=1}^{n} r_{m_j}(A_j;\, x,\, \cdot) f(\cdot)\big]$ and this operator is given by

$$\frac{1}{c_m}\, p.v. \int \prod_{j=1}^{n} \left\{\frac{r_{m_j}(A_j;\, x,\, y)}{(x-y)^{m_j}}\right\} \frac{f(y)}{x-y}\, dy \quad \text{where} \quad c_m = \frac{(-i\pi)(-1)^{|m|}}{|m|!\,(2\pi)^{n+1}}.$$

(3.1) Lemma. *Assume $g \in \mathscr{D}$ and $x \in R$.*

(1) *If N is a positive integer and $g(x) = g^{(1)}(x) = \ldots = g^{(N-1)}(x) = 0$, then*

$$\left(\frac{d}{dx}\right)^N \left\{p.v. \int \frac{g(y)}{x-y}\, dy\right\} = (-1)^N N!\, p.v. \int \frac{g(y)}{(x-y)^{N+1}}\, dy. \tag{3.2}$$

(2) *If k, N are positive integers with $k \leqslant N$, then*

(3.3)

$$\left(\frac{d}{dx}\right)^N \left\{p.v. \int \frac{g(y)(x-y)^k}{(x-y)}\, dy\right\} = \frac{(-1)^k N!}{(N-k)!} \left(\frac{d}{dx}\right)^{N-k} \left\{p.v. \int \frac{g(y)}{x-y}\, dy\right\}.$$

Proof. Choose $M > 0$ large enough so that the interval $[-M, M]$ contains the support of g. Let $\eta \in \mathscr{D}$ be an even function where $0 \leqslant \eta(t) \leqslant 1$, $\eta(t) \equiv 1$ for $t \in [-4M, 4M]$ and $\eta(t) \equiv 0$ for $t \notin [-8M, 8M]$. Let $\eta_k(t) = \eta(t/k)$. For $k \geqslant 1$ we clearly have for $x \in [-2M, 2M]$

$$p.v. \int \frac{g(y)}{x-y}\, dy = \int \frac{g(y)-g(x)}{x-y}\, \eta_k(x-y)\, dy.$$

By differentiating N times and letting $k \to \infty$ we get the formula

$$\left(\frac{d}{dx}\right)^N \left\{p.v. \int \frac{g(y)}{(x-y)}\, dy\right\} = (-1)^N N! \lim_{\delta\to\infty} \int_{|x-y|\leqslant\delta} \frac{g(y) - \sum_{j=1}^{N} \frac{g^{(j)}(x)(y-x)^j}{j!}}{(x-y)^{N+1}}\, dy.$$

If $g(x) = g^{(1)}(x) = \ldots = g^{(N-1)}(x) = 0$, formula (3.4) yields

$$\lim_{\delta\to\infty} \int_{|x-y|\leqslant\delta} \frac{g(y) - \frac{g^{(N)}(x)(y-x)^N}{n!}}{(x-y)^{N+1}}\, dy = p.v. \int \frac{g(y)}{(x-y)^{N+1}}\, dy$$

which establishes (3.2).

Applying (3.4) to the function $\psi(y) = g(y)(x-y)^k$ (where k, N are positive integers with $k \leqslant N$) establishes (3.3).

(3.5) Remark. Assume $A_1, \ldots, A_n \in \mathscr{S}$. Then

$$\left(\frac{d}{dy}\right)^N \prod_{j=1}^{n} r_{m_j}(A_j;\, x,\, y)\big|_{y=x} = 0$$

for $N = 0, 1, \ldots, |m|-1$ where $|m| = \sum_{j=1}^{n} m_j$.

Remark (3.5) and Lemma (3.1) show that

$$D^{|m|} H\Big[\prod_{j=1}^{n} r_{m_j}(A_j;\, x,\, \cdot) f(\cdot)\Big]$$

$$= (-1)^{|m|}\, |m|!\; p.v. \int \prod_{j=1}^{n} \left\{\frac{r_{m_j}(A_j;\, x,\, y)}{(x-y)^{m_j}}\right\} \frac{f(y)}{x-y}\, dy.$$

(3.6) Lemma. *Assume $f \in \mathscr{D}$, $A_1, \ldots, A_n \in \mathscr{S}$. Then*

$$i^{|m|} C_m \int_{R^{n+1}} R^{(m)}_{(-\alpha)} s^{|m|} \operatorname{sgn} s\, \hat{A}(\alpha) \hat{f}\Big(s - \sum_{j=1}^{n} \alpha_j\Big) e^{isx}\, d\alpha\, ds \tag{3.7}$$

$$= p.v. \int \prod_{j=1}^{n} \left\{\frac{r_{m_j}(A_j;\, x,\, y)}{(x-y)^{m_j+1}}\right\} \frac{f(y)}{x-y}\, dy$$

where C_m is as in Theorem II.

Proof. To show this we introduce some multiple index notation:

$\mathscr{J}$ = set of subsets of $\{1, 2, \ldots, n\}$,
J' = complement of J in $\{1, 2, \ldots, n\}$,
$|J|$ = number of elements in J,
$k_J = (k_{j_1}, \ldots, k_{j_l})$ where $J = \{j_1, \ldots, j_l\}$ and $j_1 < j_2 < \ldots < j_l$,
$K_J = \{k_J\colon\ 0 \leqslant k_{j_1} \leqslant m_{j_1}-1;\ \ldots;\ 0 \leqslant k_{j_l} \leqslant m_{j_l}-1\}$,
$k_J! = k_{j_1}!\ k_{j_2}! \ldots k_{j_l}!$,
$|k_J| = k_{j_1} + k_{j_2} + \ldots + k_{j_l}$,
$g^{(k_J)}(x) = g^{(k_{j_1}+\ldots+k_{j_l})}(x)$.

Using this notation we have the expansion for any $g \in \mathscr{S}$

$$R^{(m)}_{(-\alpha)} g(s) = \sum_{J\in\mathscr{J}} \sum_{k_J\in K_J} \frac{(-1)^{|J|} g^{(k_J)}\big(s - \sum_{j\in J'} \alpha_j\big)}{k_J!} \prod_{j\in J} (-\alpha_j)^{k_j}. \tag{3.8}$$

For $\alpha \in R^n$, $x \in R$, we use the following notation:

$$\hat{A}_J(\alpha) = \prod_{j\in J} \hat{A}_j(\alpha_j),$$

$$A_J^{(k_J)}(x) = \prod_{j\in J} A_j^{(k_j)}(x)$$

and

$$d\alpha = \prod_{j\in J} d\alpha_j.$$

With this additional notation, setting $g(s) = s^{|m|}\operatorname{sgn} s$ and using (3.8)

$$(3.9)\quad \int_{R^{n+1}} R^{(m)}_{(-\alpha)} s^{|m|} \operatorname{sgn} s \hat{A}(\alpha)\hat{f}\Big(s-\sum_{j\in 1}^{n}\alpha_j\Big)e^{isx}\,d\alpha\,ds$$

$$= \sum_{J\in\mathscr{J}}\sum_{k_J\in K_J}\frac{(-1)^{|m|-|k_J|+|J|}}{(-i\pi)}\cdot\frac{i^{|m|}|m|!}{(|m|-|k_J|)!}\int_R e^{isx}\times$$

$$\times\int_{R^{|J'|}}\Big[i\Big(s-\sum_{j\in J}\alpha_j\Big)\Big]^{|m|-|k_J|}(-i\pi)\operatorname{sgn}\Big(s-\sum_{j\in J}\alpha_j\Big)\hat{A}_{J'}(\alpha)\,d\alpha_{J'}\times$$

$$\times\int_{R^{|J|}}\frac{A_J^{(k_J)\wedge}(\alpha)}{k_J!}\hat{f}\Big(s-\sum_{j=1}^{n}\alpha_j\Big)d\alpha_J\,ds.$$

Integrating in $\alpha_{J'}$ the inner integral equals

$$(2\pi)^{|J|}\Big\{\frac{fA_J^{(k_J)}}{k_J!}\Big\}^{\wedge}\Big(s-\sum_{j\in J}\alpha_j\Big).$$

Integrating next in $\alpha_{J'}$ we get the middle integral is equal to

$$(2\pi)^n\Big\{A_{J'}D^{|m|-|k_J|}H\Big(\frac{fA_J^{(k_J)}}{k_J!}\Big)\Big\}^{\wedge}(s).$$

Integrating in s and using Lemma (3.1) and Remark (3.5), the right-hand side of (3.9) becomes

$$\sum_{J\in\mathscr{J}}\sum_{k_J\in K_J}\frac{(-i)^{|m|}(2\pi)^{n+1}}{-i\pi}(-1)^{|J|}\prod_{j\in J}A_j(x)D^{|m|}H\Big\{f(\cdot)\prod_{j\in J}\frac{A_j^{(k_j)}(\cdot)}{k_j!}(x-\cdot)^{k_j}\Big\}(x)$$

$$= i^{|m|}C_m\,p.v.\int\frac{\prod_{j=1}^{n}r_{m_j}(A_j;\,x,y)}{(x-y)^{|m|+1}}f(y)\,dy.$$

Proof of Theorem II. Let

$$I(\varepsilon) = i^{|m|}C_m\int_{R^{n+1}}e^{isx}\frac{R^{(m)}_{(-\alpha)}s^{|m|}\operatorname{sgn} s}{\prod_{j=1}^{n}(i\alpha_j+\varepsilon)^{m_j}}\hat{a}(\alpha)\hat{f}\Big(s-\sum_{j=1}^{n}\alpha_j\Big)d\alpha\,ds.$$

If $A_j^{\varepsilon}(x) = e^{-\varepsilon x}\int_{-\infty}^{x}\int_{-\infty}^{t_1}\dots\int_{-\infty}^{t_{m_j-1}}a_j(s)e^{\varepsilon s}ds$, then $A_j^{\varepsilon}\in\mathscr{S}$ and $A_j^{\varepsilon}(\alpha_j)$ $= \dfrac{\hat{a}_j(\alpha_j)}{(i\alpha_j+\varepsilon)^{m_j}}$. Applying Lemma (3.6)

$$I(\varepsilon) = p.v.\int\prod_{j=1}^{n}\Big\{\frac{r_{m_j}(A_j^{\varepsilon};\,x,y)}{(x-y)^{m_j}}\Big\}\frac{f(y)}{x-y}dy.$$

Since

$$R^{(m)}_{(-\alpha)}g(s) = \frac{1}{(m-1)!}\int_0^{-\alpha_1}\dots\int_0^{-\alpha_n}g^{(|m|)}\Big(s-\sum_{i=1}^{n}u_i\Big)\prod_{j=1}^{n}u_j^{m_j-1}du_j$$

(where $(m-1)! = (m_1-1)!\dots(m_n-1)!$), for any g with $|m|$ derivatives, and since $\left(\dfrac{d}{ds}\right)^{|m|}s^{|m|}\operatorname{sgn} s = |m|!\operatorname{sgn} s$ we have the estimate

$$|R^{(m)}_{(-\alpha)}s^{|m|}\operatorname{sgn} s|\leqslant\frac{|m|!}{(m-1)!}\prod_{j=1}^{n}|\alpha_j|^{m_j}.$$

Applying the Lebesgue dominated convergence theorem to $I(\varepsilon)$ we see

$$\lim_{\varepsilon\to 0}I(\varepsilon) = C_m\int_{R^{n+1}}e^{isx}\frac{R^{(m)}_{(-\alpha)}s^{|m|}\operatorname{sgn} s}{\prod_{j=1}^{n}\alpha_j^{m_j}}\hat{a}(\alpha)\hat{f}\Big(s-\sum_{j=1}^{n}\alpha_j\Big)d\alpha\,ds.$$

Finally, as $\varepsilon\to 0$, $A_j^{\varepsilon}(x)\to A_j(x)$, $1\leqslant j\leqslant n$, uniformly on all compact sets, and the same is true of the derivatives of $A_j^{\varepsilon}(x)$. It follows that

$$\lim_{\varepsilon\to 0}p.v.\int\prod_{j=1}^{n}\Big\{\frac{r_{m_j}(A_j^{\varepsilon};\,x,y)}{(x-y)^{m_j}}\Big\}\frac{f(y)}{x-y}dy$$

$$= p.v.\int\prod_{j=1}^{n}\Big\{\frac{r_{m_j}(A_j;\,x,y)}{(x-y)^{m_j}}\Big\}\frac{f(y)}{x-y}dy.$$

§ 4. Real variable methods. In this section real variable methods are introduced to extend the estimates of §2 to a full range of L^p functions. The truncated operator $T_\varepsilon^m(a,f)$ is introduced along with its associated maximal function T_*^m. A good λ inequality is proved showing that T_*^m is bounded in L^q by the L^q norm of an appropriate product of Hardy–Littlewood maximal functions.

Some more notation and definitions must be introduced. Adopting the notation of §1, for $p = (p_1,\dots,p_n)$ and $(p,p_0) = (p_1,\dots,p_n,p_0)$. $L^{(p,p_0)}$ and $\|(a,f)\|_{(p,p_0)}$ are defined as before. In addition we define:

$$\|a\|_{(p)} = \prod_{j=1}^{n}\|a_j\|_{p_j},$$

$$a_j^*(x) = \sup_{I\ni x}\frac{1}{|I|}\int_I|a_j(t)|\,dt$$

where I is an interval and $|I|$ is its length. We introduce the corresponding multi-index maximal functions,

$$a^{(*)}(x) = \prod_{j=1}^{n}a_j^*(x)$$

and

$$(a,f)^{(*)}(x) = \Big\{\prod_{j=1}^{n} a_j^*(x)\Big\} f^*(x).$$

Next define the truncated operator

$$T_\varepsilon^m(a,f)(x) = \int\limits_{|x-y|>\varepsilon} t_m(x,y)f(y)\,dy$$

where

$$t_m(x,y) = \frac{1}{x-y}\prod_{j=1}^{n}\frac{r_{m_j}(A_j;\,x,y)}{(x-y)^{m_j}} \quad\text{and}\quad T_*^m(a,f)(x) = \sup_{\varepsilon>0}|T_\varepsilon^m(a,f)(x)|.$$

We are now ready to state the principal result of this section.

(4.1) Proposition. *If $a_j \in L^1$ for $j = 1,\ldots,n$, $f \in L^1$, then there exists γ_0 so that for $\gamma < \gamma_0$,*

$$|x\colon\ T_*^m(a,f)(x) > 2\lambda,\ (a,f)^{(*)}(x) \leqslant \gamma\lambda| \leqslant c\gamma^{n+1}|x\colon\ T_*^m(a,f)(x) > \lambda|.$$

(The notation $|x\colon$ "..."$|$ means the measure of the set x such that "...".)

Theorem 1 will follow as an immediate consequence of Proposition (4.1). The proof will be given in a series of lemmas, following closely the real variable methods used by Coifman and Meyer in [5].

(4.2) Lemma. *If $|x-y| > \varepsilon$, $|x-x_1| < \varepsilon/4$, then for $x_2 \in (x-\varepsilon/4,\ x+\varepsilon/4)$ we have the following estimates.*

(a) $$|t_m(x,y) - t_m(x_1,y)| < c\,\frac{a^{(*)}(x_2)\,|x-x_1|}{|x-y|^2},$$

(b) $$\Big|\int\limits_{|x-y|>\varepsilon}[t_m(x,y) - t_m(x_1,y)]f(y)\,dy \leqslant c(a,f)^{(*)}(x_2)\Big|$$

and for $|y-y_1| < \frac{1}{2}|x-y|$, $|y-y_2| < \frac{1}{2}|x-y|$,

(c) $$|t_m(x,y) - t_m(x,y_1)| \leqslant ca^{(*)}(y_2)\,\frac{|y-y_1|}{|x-y|^2}.$$

Proof.

$$t_m(x,y) - t_m(x_1,y) = \prod_{j=1}^{n} r_{m_j}(A_j;\,x,y)\Big[\frac{1}{(x-y)^{|m|+1}} - \frac{1}{(x_1-y)^{|m|+1}}\Big] +$$
$$+\frac{1}{(x_1-y)^{|m|+1}}\sum_{j=1}^{n}[r_{m_j}(A_j;\,x,y) - r_{m_j}(A_j;\,x_1,y)]\prod_{k<j} r_{m_k}(A_k;\,x,y)\times$$
$$\times\prod_{k>j} r_{m_k}(A_k;\,x_1,y)$$
$$\leqslant ca^{(*)}(x_2)\,\frac{|x-x_1|}{|x-y|^2} + c\sum_{j=1}^{n} a_j^*(x_2)\,\frac{|x-x_1|}{|x-y|^2}\prod_{k\neq j} a_k^*(x_2)$$

$$\leqslant ca^{(*)}(x_2)\,\frac{|x-x_1|}{|x-y|^2}.$$

A standard argument tells us that for $|x-x_2| \leqslant |x-x_1|$,

$$\int\limits_{|x-y|>2|x-x_1|}\frac{|x-x_1|}{|x-y|^2}\,|f(y)|\,dy \leqslant cf^*(x_2)$$

which implies part (b) of Lemma (4.2). Finally, part (c) follows by the same argument as (a).

Before stating the next lemma it is necessary to introduce a Hardy–Littlewood maximal function and an appropriate multi-index notation.

For $(a,f) = (a_1, a_2, \ldots, a_n, f)$ define

$$A_{p_j}(a_j)(x) = \sup_{I\ni x}\Big\{\frac{1}{|I|}\int\limits_I |a_j(t)|^{p_j}\,dt\Big\}^{1/p_j}$$

and the corresponding multi-index notation

$$A_{(p)}(a)(x) = \prod_{j=1}^{n} A_{p_j}(a_j)(x),$$
$$A_{(p,p_0)}(a,f)(x) = A_{(p)}(a)A_{p_0}(f)(x).$$

(4.3) Lemma. *Assume $(a,f) \in L^{(p,p_0)}$, $p_j \geqslant 1$, $j = 1,2,\ldots,n$ and $1 \leqslant p_0 < \infty$. Further, assume $0 < \delta < q$ where $1/q = \sum_{j=0}^{n}(1/p_j)$. If $T^m(a,f)$ satisfies the weak type inequality:*

(4.4) $$|x\colon\ T^m(a,f)(x) > \lambda| \leqslant c\Big[\frac{\|(a,f)\|_{(p,p_0)}}{\lambda}\Big]^q,$$

then the maximal function $T_*^m(a,f)$ satisfies

$$T_*^m(a,f)(x) \leqslant c\big[A_\delta\big(T^m(a,f)\big)(x) + A_{(p,p_0)}(a,f)(x)\big]$$

and the same weak type inequality as in (4.4) is valid for $T_*^m(a,f)$.

Proof. Before proving Lemma (4.3) we note that Theorems I and II imply (4.4) if $\sum_{j=0}^{n}\Big(\frac{1}{p_j}\Big) < 1$, $1 < p_j < \infty$ and the a_j's and f are C^∞ with compact support.

The lemma is proved for $a_j \in \mathcal{D}$, $j = 1,\ldots,n$, $f \in \mathcal{D}$ and then it follows from standard arguments that the lemma can be extended to any $(a,f) \in L^{(p,p_0)}$.

For $\varepsilon > 0$, let χ_ε be the characteristic function of the interval $[x-\varepsilon,\ x+\varepsilon]$. Let $\eta_\varepsilon \in \mathcal{D}$ be a function which is one on the interval $[x-e,\ x+e]$, less than or equal to one, non-negative and everywhere vanishing outside the interval $[x-2\varepsilon, x+2\varepsilon]$. Then, for $x_1 \in [x-\varepsilon/4, x+\varepsilon/4]$, we get the identity

(4.5) $$T_\varepsilon^m(a,f)(x) = \int [t_m(x,y) - t_m(x_1,y)][1-\eta_\varepsilon(y)]f(y)\,dy + \int t_m(x_1,y)f(y)\,dy - \int t_m(x_1,y)\eta_\varepsilon(y)f(y)\,dy + \int t_m(x,y)[\eta_\varepsilon(y) - \chi_\varepsilon(y)]f(y)\,dy.$$

Taking absolute values, raising to the δ power, averaging in x_1 over the interval $[x-\varepsilon/4, x+\varepsilon/4]$ and taking the $1/\delta$ power gives the estimate

(4.6) $$|T^m(a,f)(x)| \leqslant c\left\{(a,f)^{(*)}(x) + \Lambda_\delta(T^m(a,f))(x) + \sup_{\varepsilon>0}\left(\frac{2}{\varepsilon}\int_{x-\varepsilon/4}^{x+\varepsilon/4} |T^m(a,\eta_\varepsilon f)(x_1)|^\delta\,dx_1\right)^{1/\delta}\right\}.$$

The $(a,f)^{(*)}(x)$ comes from the first and last term of (4.5) after applying Lemma (4.2). That is,

$$\left|\int [t_m(x,y) - t_m(x_1,y)][1-\eta_\varepsilon(y)]f(y)\,dy\right| \leqslant \int |t_m(x,y) - t_m(x_1,y)|[1-\chi_\varepsilon(y)]\,|f(y)|\,dy \leqslant c(a,f)^{(*)}(x)$$

and

$$\left|\int t_m(x_1,y)[\eta_\varepsilon(y) - \chi_\varepsilon(y)]f(y)\,dy\right| \leqslant ca^{(*)}(x)\int_{2\varepsilon>|x-y|>\varepsilon} \frac{|x-x_1|}{|x-y|^2}\,|f(y)|\,dy \leqslant c(a,f)^{(*)}(x).$$

Both estimates are independent of x_1 and are left unaffected by averaging in x_1.

To evaluate the second term on the right hand side of (4.6) we use the fact that the usual Hardy–Littlewood maximal function is weak type 1-1. If we set $E_\lambda = \{x\colon |T^m(a,f)(x)|^{\delta^*} > \lambda^\delta\}$, then

$$|E_\lambda| \leqslant \frac{2}{\lambda^\delta}\int_{E_\lambda} |T^m(a,f)(x)|^\delta\,dx$$

using Kolmogorov's inequality

$$\leqslant \frac{2}{\lambda^\delta}\,|E|^{1-\delta/q}\,\|(a,f)\|^\delta_{(p,p_0)}.$$

Taking the $1/\delta$ power of both sides and simplifying we get

(4.7) $$|E_\lambda| \leqslant \left\{\frac{2\,\|(a,f)\|_{(p,p_0)}}{\lambda}\right\}^q.$$

To evaluate the last term in (4.6) we need to be a little careful. We let $\eta_{2\varepsilon} = \eta_\varepsilon(t/2)$. Then $\eta_{2\varepsilon}(t) = 1$ for $|t-x| < 2\varepsilon$. If we adopt the notation $\eta_{2\varepsilon}a = (\eta_{2\varepsilon}a_1, \ldots, \eta_{2\varepsilon}a_n)$, then for $|x_1 - x| < \varepsilon/4$,

$$T^m(a,\eta_\varepsilon f)(x_1) = T^m(\eta_{2\varepsilon}a, \eta_\varepsilon f)(x_1).$$

Applying Kolmogorov's inequality again (the weak type property is guaranteed since $\eta_{2\varepsilon}a_j \in \mathcal{D}$ and $\eta_\varepsilon f \in \mathcal{D}$) we have,

$$\left\{\frac{2}{\varepsilon}\int_{x-\varepsilon/4}^{x+\varepsilon/4} |T^m(\eta_{2\varepsilon}a, \eta_\varepsilon f)(x_1)|^\delta\,dx_1\right\}^{1/\delta} \leqslant \left\{\frac{2}{\varepsilon}c_\delta\left(\frac{\varepsilon}{2}\right)^{\delta/q}\|(\eta_{2\varepsilon}a, \eta_\varepsilon f)\|^\delta_{(p,p_0)}\right\}^{1/\delta}$$

$$\leqslant c\prod_{j=1}^n\left\{\frac{1}{8\varepsilon}\int_{x-4\varepsilon}^{x+4\varepsilon} |a_j(t)|^{p_j}\,dt\right\}^{1/p_j}\left\{\frac{1}{4\varepsilon}\int_{x-2\varepsilon}^{x+2\varepsilon}|f(t)|^{p_0}\,dt\right\}^{1/p_0} \leqslant c\Lambda_{(p,p_0)}(a,f)(x).$$

Since $\delta < \delta_1 \Rightarrow \Lambda_\delta(f)(x) \leqslant \Lambda_{\delta_1}(f)(x)$ we get the estimate

$$|T^m_*(a,f)(x)| \leqslant c[\Lambda_\delta(T^m(a,f)(x)) + \Lambda_{(p,p_0)}(a,f)(x)].$$

The weak type estimate for T^m_* then follows from (4.7) and

(4.8) OBSERVATION. *If $T_1, \ldots, T_n$ are weak type (p_j, p_j) operators with $1/q = \sum_{j=1}^n (1/p_j)$ and $\infty > p_j \geqslant 1$ for $j = 1, 2, \ldots, n$, then $\prod_{j=1}^n |T_j a_j|$ satisfies the weak estimate of (4.4).*

(4.9) LEMMA. *If $(a,f) \in L^{(p,p_0)}$, where $1/q = \sum_{j=0}^n (1/p_j)$, and (4.4) is satisfied for this set of p_j's, then there exist constants $\gamma_0 > 0$ and $c > 0$ so that for $0 < \gamma < \gamma_0$,*

(4.10) $$|x\colon T^m_*(a,f)(x) > 2\lambda,\ \Lambda_{(p,p_0)}(a,f)(x) \leqslant \gamma\lambda| \leqslant c\gamma^q|x\colon T^m_*(a,f)(x) > \lambda|.$$

Proof. $\{x\colon T^m_*(a,f)(x) > \lambda\} = \bigcup_j I_j$ where the I_j's are open disjoint intervals, $I_j = (a_j, a_j + \delta_j)$ and $T^m_*(a,f)(a_j) \leqslant \lambda$ since $a_j \notin \bigcup_j I_j$.

It will suffice to establish (4.10) for each I_j since the I_j's are disjoint. Choose an I_j with a point z satisfying $\Lambda_{(p,p_0)}(a,f)(z) \leqslant \gamma\lambda$. If I_j contains no such point, (4.10) is automatically satisfied. Let $\bar{I}_j = (a_j - 2\delta_j, a_j + 2\delta_j)$,

$f_1 = \chi_{\bar{I}_j} f$ ($\chi_{\bar{I}_j}$ is the characteristic function of the interval $\bar{I}_j$),

$$f_2 = f - f_1.$$

For $x \in I_j$, $T^m_*(a,f_1)(x) = T^m_*(\chi_{\bar{I}_j}a, \chi_{\bar{I}_j}f)(x)$. By virtue of Lemma (4.3) we have for $1/q = \sum_{j=0}^{n}(1/p_j)$,

(4.11) $$|x \in I_j: T_*^m(\chi_{\bar{I}_j}a, \chi_{\bar{I}_j}f)(x) > \beta\lambda| \leqslant c\left[\frac{\|(\chi_{\bar{I}_j}a, \chi_{\bar{I}_j}f)\|_{(p,p_0)}}{\beta\lambda}\right]^q$$
$$\leqslant c\left[\prod_{k=1}^{n}|\bar{I}_j|^{1/p_k}\left\{\frac{1}{|\bar{I}_j|}\int_{\bar{I}_j}|a_k(t)|^{p_k}dt\right\}^{1/p_k}\right]\left[|\bar{I}_j|^{1/p_0}\left\{\frac{1}{|\bar{I}_j|}\int_{\bar{I}_j}|f(t)|^{p_0}dt\right\}^{1/p_0}\right]$$
$$\leqslant c\left\{\frac{A_{(p,p_0)}(a,f)(z)|\bar{I}_j|^{\sum_{j=0}^{n}(1/p_j)}}{\beta\lambda}\right\}^q \leqslant c(\gamma/\beta)^q|I_j|.$$

For the f_2 part

(4.12) $$|T_\varepsilon^m(a,f_2)(x)| \leqslant |T_\varepsilon^m(a,f_2)(x) - T_\varepsilon^m(a,f_2)(a_j)| + |T_\varepsilon^m(a,f_2)(a_j)|$$
$$\leqslant \lambda + \left|\int[t_m(x-y) - t_m(a_j,y)]f_2(y)\,dy\right| +$$
$$+ \int_{x-\varepsilon}^{a_j-\varepsilon}|t_m(x,y)|\,|f_2(y)|\,dy + \int_{x+\varepsilon}^{a_j+\varepsilon}|t_m(x,y)|\,|f_2(y)|\,dy$$
$$\leqslant \lambda + c_1(a,f)^{(*)}(z) \leqslant \lambda + c_1\gamma\lambda.$$

The estimate for the first integral in (4.12) follows from Lemma (4.2). The estimate for the second and third integral follow from the observation that
$$|t_m(x,y)| \leqslant c\frac{a^{(*)}(z)}{|I_j|+\varepsilon}.$$

Combining the f_1 and f_2 estimates,
$$|x \in I_j: T^m(a,f_1)(x) > 2\lambda,\ A_{(p,p_0)}(a,f)(x) \leqslant \gamma\lambda|$$
$$\leqslant |x \in I_j: T_*^m(a,f_1)(x) > \lambda(1-c_1\gamma)| +$$
$$+ |x \in I_j: T_*^m(a,f_2)(x) > \lambda(1+c_1\gamma)|$$
$$\leqslant c\big(\gamma/(1-c_1\gamma)\big)^q|I_j|.$$

Since the constants c_1, c are independent of the choice of I_j we can add the estimate for each interval to establish (4.10).

(4.13) Corollary. *$\int|T_*^m(a,f)(x)|^q dx \leqslant c\int|A_{(p,p_0)}(a,f)(x)|^q dx$ for any $0 < q$.*

(4.14) Corollary. *For $a \in L^{(\infty)}$ (i.e. $a_j \in L^\infty$, $j = 1, \ldots, n$) and $f \in L^{p_0}$, $p_0 > 1$,*
$$\|T_*^m(a,f)\|_{p_0} \leqslant c\|a\|_{(\infty)}\|f\|_{p_0}$$
where $\|a\|_{(\infty)} = \prod_{j=1}^{n}\|a_j\|_\infty$.

Corollary (4.13) is the standard result of a good λ inequality (see remarks at the end of this section). Corollary (4.14) follows from the fact that any function in L^∞ is locally in L^p and the fact that $A_p(f)(x) \leqslant \|f\|_\infty$.

To finish the proof of Proposition (4.1) we need one more lemma which extends the range of p's.

(4.15) Lemma. *If $a_j \in L^1$ for $j = 1, 2, \ldots, n$ and $f \in L^1$, then*
$$|x: T^m(a,f)(x) > \lambda| \leqslant c\left\{\frac{\|(a,f)\|_{(1,1)}}{\lambda}\right\}^{1/(n+1)}$$
where $\|(a,f)\|_{(1,1)} = \{\prod_{j=1}^{n}\|a_j\|_1\}\|f\|_1$.

Proof. The proof consists of showing the following. For any $j = 0, 1, \ldots, n$, $\|a_1\|_1 = \|a_2\|_1 = \ldots = \|a_j\|_1 = \|f\|_1 = 1$ and $\|a_{j+1}\|_\infty = \ldots = \|a_n\|_\infty = 1$, then

(4.16) $$|x: T^m(a,f)(x) > \lambda| \leqslant \frac{c}{\lambda^{1/(j+1)}}.$$

Choosing $j = n$ and using the multilinearity of $T^m(a,f)$, (4.16) will imply the lemma. We proceed by induction.

First, the estimate (4.16) is valid for $j = 0$ (i.e. for all the a_j's in L^∞) since by Lemma (4.2)
$$|t_m(x,y) - t_m(x,y_1)| \leqslant ca^{(*)}(y_2)\frac{|y-y_1|}{|x-y|^2} \quad \text{for} \quad |x-y| > 2|y-y_2|,$$
and T^m is bounded in L^2 as an operator acting on f whenever all the a_j's are in L^∞. By a standard Calderón–Zygmund argument as given in [9] we get that T^m is weak type 1-1 as an operator on f.

Next assume (4.16) is valid for $1, 2, \ldots, j-1$ and assume $\|a_1\|_1 = \ldots = \|a_j\|_1 = \|f\|_1 = 1$ and $\|a_{j+1}\|_\infty = \ldots = \|a_n\|_\infty = 1$. Let $\Omega = \{x: a_j^*(x) > \lambda^{1/(j+1)}\}$. It follows that $\Omega = \bigcup_k I_k$ and $a_j = g + b$ where

(i) The I_k's are open and disjoint.

(ii) $\sum_k |I_k| < c/\lambda^{1/(j+1)}$.

(iii) $\frac{1}{|I_k|}\int_{I_k}|a_j(t)|\,dt \leqslant \lambda^{1/(j+1)}$.

(iv) $|g(x)| \leqslant \lambda^{1/(j+1)}$.

(v) $b = \sum_k b_k$ where b_k is supported in I_k.

This follows by defining g by
$$g(x) = \begin{cases} a_j(x), & x \notin \Omega, \\ \frac{1}{|I_k|}\int_{I_k} a_j(t)\,dt, & x \in I_k \end{cases}$$

and defining $b = a_j - g$. Then if χ_{I_k} denotes the characteristic function of the interval I_k, we define $b_k = b\chi_{I_k}$.

Let y_k be the center of the inetrval I_k and let $2I_k$ denote the interval which is twice the length of I_k and also centered at y_k.

For $x \notin 2I_k$ we observe:

(1) $$\Big|\int_y^x b_k(t)(x-t)^m\,dt\Big| \leqslant 2|x-y|^m |I_k| \lambda^{1/(j+1)}$$

since

$$\frac{1}{|I_k|}\int_{I_k} |b_k(t)|\,dt \leqslant 2\Big\{\frac{1}{|I_k|}\int_{I_k} |a_j(t)|\,dt\Big\} \leqslant 2\lambda^{1/(j+1)}.$$

(2) $$|T^m(a_1, \ldots, a_{j-1}, b_k, a_{j+1}, \ldots, a_n, f)(x)| \leqslant c \prod_{i=1}^{j-1} a_i^*(x) \prod_{i=j+1}^{n} \|a_i\|_\infty \frac{|I_k|\lambda^{1/(j+1)}}{|I_k|^2 + |x-y_k|^2} \int_{I_k} |f(t)|\,dt.$$

(3) The map
$$L\colon f \to \sum_k \frac{|I_k|}{|I_k|^2 + |x-y_k|^2} \int_{I_k} |f(t)|\,dt$$
satisfies the following weak type estimate:

When $f \in L^1$,

(4.17) $$|x \notin \bigcup_k 2I_k\colon Lf(x) > \lambda| \leqslant c\|f\|_1/\lambda.$$

This follows from using Fubini's theorem to get

$$\int_{x \notin \bigcup_k 2I_k} |Lf(x)|\,dx \leqslant \sum_k \int_{I_k} |f(y)| \Big\{|I_k| \int_{x \notin \bigcup_k 2I_k} \frac{dx}{|x-y_k|^2 + |I_k|^2}\Big\}\,dy \leqslant c\int_\Omega |f(y)|\,dy \leqslant c\|f\|_1.$$

This strong type result implies the weak estimate (4.17). Putting (1), (2) and (3) together we see that for $x \notin \bigcup_k 2I_k$,

$$|T(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n, f)(x)| \leqslant ca_1^*(x), \ldots, a_{j-1}^*(x)\lambda^{1/(j+1)}Lf(x).$$

So we have the product of j operators which are all weak type $(1, 1)$ and the functions $a_1, \ldots, a_{j-1}, f$ all have an L^1 norm of one. This implies that

(4.18) $$\begin{aligned}|x \notin \bigcup_k 2I_k\colon T^m(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n, f)(x) > \lambda| &\leqslant |x \notin \bigcup 2I_k\colon ca_1^*(x), \ldots, a_{j-1}^*(x)Lf(x) > \lambda^{j/(j+1)}| \\ &\leqslant c/\lambda^{1/(j+1)}.\end{aligned}$$

This follows from observation (4.8) about weak type estimates for the product of weak type operators.

We next use the inductive hypothesis to make a weak type estimate for g.

(4.19) $$\begin{aligned}|x\colon T^m(a_1, \ldots, a_{j-1}, g, a_{j+1}, \ldots, a_n, f)(x) > \lambda| &= \Big|x\colon T^m\Big(a_1, \ldots, a_{j-1}, \frac{g}{\|g\|_\infty}, a_{j+1}, a_n, f\Big)(x) > \frac{\lambda}{\|g\|_\infty}\Big| \\ &\leqslant c[\|g\|_\infty/\lambda]^{1/j} \leqslant c/\lambda^{1/(j+1)}.\end{aligned}$$

Since $\|g\|_\infty \leqslant \lambda^{1/(j+1)}$, we have

(5) There is

(4.20) $$\Big|\bigcup_k 2I_k\Big| \leqslant 2|\Omega| \leqslant c/\lambda^{1/(j+1)}.$$

Putting together estimates (4.18), (4.19) and (4.20) we have shown

$$|x\colon T^m(a, f)(x) > \lambda| \leqslant c/\lambda^{1/(j+1)}.$$

If the L^1 and L^∞ norms of the functions are arbitrary, by dividing by the appropriate norms and using the multilinearity of T^m we get the estimate:

$$|x\colon T^m(a, f)(x) > \lambda| \leqslant c\left\{\frac{\|f\|_1 \prod_{i=1}^{j} \|a_i\|_1 \prod_{i=j+1}^{n} \|a_i\|_\infty}{\lambda}\right\}^{1/(j+1)}.$$

In particular, for $(a, f) \in L^{(1,1)}$,

$$|x\colon T^m(a, f)(x) > \lambda| \leqslant c\left\{\frac{\|(a, f)\|_{(1,1)}}{\lambda}\right\}^{1/(n+1)}.$$

P r o o f o f T h e o r e m III. Proposition (4.1) then follows by applying Lemmas (4.3) and (4.9) to Lemma (4.15). In other words we have the estimate:

$$|x\colon T_*^m(a, f)(x) > 2\lambda, (a, f)^{(*)}(x) \leqslant \gamma\lambda| \leqslant c\gamma^{n+1}|x\colon T_*^m(a, f)(x) > \lambda|.$$

Theorem III then follows by the following argument. If $a_j \in L^1$ for $j = 1, \ldots, n$ and $f \in L^1$, then $(a, f)^{(*)}(x) \simeq c/|x|^{n+1}$ as $x \to \infty$. Hence $\int_R [(a, f)^{(*)}(x)]^q\,dx = +\infty$ for $q \leqslant 1/(n+1)$. For $q > 1/(n+1)$, Lemma (4.3) guarantees the existence of I_N^{ε} where

$$\begin{aligned}I_N^{\varepsilon} &= q\int_\varepsilon^N \lambda^{q-1}|x\colon T_*^m(a, f)(x) > \lambda|\,d\lambda \\ &\leqslant c\gamma^{n+1}\int_\varepsilon^N \lambda^{q-1}|x\colon T_*^m(a, f)(x) > \lambda|\,d\lambda + c\int_\varepsilon^N \lambda^{q-1}|x\colon (a, f)^{(*)}(x) > \lambda|\,d\lambda \\ &\leqslant c\gamma^{n+1}I_N^{\varepsilon} + c_\gamma \int_R [(a, f)^{(*)}(x)]^q\,dx.\end{aligned}$$

Choosing γ so that $c\gamma^{n+1} \leqslant \frac{1}{2}$, letting $N \to \infty$ and $\varepsilon \to 0$ completes the proof of Theorem III.

By standard arguments Theorem III implies the following important corollary.

(4.21) Corollary. *For* $(a, f) \in L^{(p,p_0)}$, $\sum_{j=0}^{n} (1/p_j) = 1/q < n+1$, $1 < p_0 < \infty$, $1 < p_j \leqslant \infty$, *and* $T_\varepsilon^m(a, f)$ *and* $T_*^m(a, f)$ *the operators defined in Theorem* III, *the following properties are satisfied*:

(1) $\|T_*^m(a, f)\|_q \leqslant c\|(a, f)\|_{(p,p_0)}$.

(2) $\lim_{\varepsilon \to 0} T_\varepsilon^m(a, f)(x)$ *exists almost everywhere.*

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An analog of the Marcinkiewicz integral in ergodic theory

by

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Abstract. Let T be an invertible measure preserving point transformation from a space X onto itself. Define $\tau_B(x) = \inf\{n \geqslant 0 \mid T^n x \in B\}$. The analog of the classical Marcinkiewicz integral $I(f)(x)$, is defined by

$$I(f)(x) = \sum_{k=1}^{\infty} \frac{\tau_B(T^k x) f(T^k x)}{k^2}.$$

If f is the characteristic function of a set B, then this integral, like its classical analog, gives a measure of the distance from a point x to the set B. Intuitively it is the average amount of time the point spends outside the set B during its orbit. It is used to give a direct proof that the ergodic Hilbert transform is weak type (1, 1).

Theorems. Let (X, Σ, m) denote a complete nonatomic probability space, and T an ergodic measure preserving invertible point transformation from X onto itself. For $B \in \Sigma$, with $0 < m(B) < 1$ and a point x, consider the orbit, $x, Tx, T^2x, \ldots$ Following this orbit we will enter and leave the set B infinitely often. In the following we will be interested in various measures of the distance from the point x to the set B.

A natural measure is the recurrence time, defined by

$$\nu_B(x) = \begin{cases} \inf\{n > 0 \mid T^n x \in B\}, & x \in B, \\ 0, & x \notin B. \end{cases}$$

This function has been previously studied by Kac [6] and Blum and Rosenblatt [1]. Kac has shown that $\|\nu_B\|_1 = 1/m(B)$, and Blum and Rosenblatt have studied the higher moments.

A second measure, related to the recurrence time, is defined by

$$\tau(x) = \inf\{n \geqslant 0 \mid T^{-n} x \in B\}.$$

It is not hard to see that $\tau(x)$ may fail to be in $L^1(X)$. In fact $\tau(x) \in L^1(X)$ if and only if $\nu_B(x)$ has a finite second moment.

Both of the above measurements are local in the sense that after a return to B, they fail to observe the remainder of the orbit. However,