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WROCŁAWSKA DRUKARNIA NAUKOWA

Uniformly convex norms on Banach lattices

by

T. FIGIEL (Gdańsk)

Abstract. Let X be a superreflexive Banach space of cotype f. We prove that if X has local unconditional structure, then X admits an equivalent norm with modulus of convexity $\geqslant cf$. Moreover, if f is, up to a constant, the best modulus of convexity for X, then there is $c > 0$ such that $f(ts) \geqslant cf(t)f(s)$ for $t, s \in [0, 1]$.

If X is only a complemented subspace of a Banach lattice and $\varepsilon > 0$, then X is complemented in a Banach lattice of cotype f and also in a superreflexive lattice of cotype $f(t)t^{\varepsilon}$.

0. Introduction. In the present paper we extend the results of [7]. We mostly work in the context of Banach lattices. We discuss some procedures to define a new norm that satisfies certain inequalities related to uniform convexity which are meaningful also in the case where there is no equivalent uniformly convex norm (cf. Theorems 4.2, 5.1 and corollaries). Our general reference is [21].

A Banach space $(X, \|\cdot\|)$ is said to be *of cotype* f, where f is a non-negative function on $[0, a)$ provided that there exist $b \in (0, a]$, $C < \infty$ such that if $x_1, \ldots, x_n$ is a finite sequence of elements of X and

$$\int_0^1 \Big\| \sum_{i=1}^n x_i r_i(t) \Big\| \, dt < b,$$

where r_i denotes the ith Rademacher function, then

$$\sum_{i=1}^n f(\|x_i\|) < C.$$

This property is independent of the choice of an equivalent norm on X. It was observed in [11] that X is of cotype δ where $\delta = \delta_{(X,\|\cdot\|)}$: $[0, 2] \to \boldsymbol{R}_+$ is the modulus of convexity of $(X, \|\cdot\|)$. (This is a generalization of Kadec's theorem on unconditionally convergent series [17], which can be deduced from the latter theorem and known results.)

In [7] we proved a partial converse to that result. Namely, the Banach space X can be given an equivalent norm, $|||\cdot|||$, such that

$$(*) \qquad \delta_{(X,|||\cdot|||)}(\varepsilon) \geqslant cf(\varepsilon),$$

for some $c > 0$ and sufficiently small $\varepsilon > 0$, provided that X is of cotype f, admits an equivalent uniformly convex norm and has an unconditional basis. The latter condition can be replaced by "X has l.u.st., i.e. local unconditional structure in the sense of [3]". This we do in Corollary 5.2 below. This condition cannot just be dropped; the corresponding example has been constructed in [30]. In fact we prove more than in [7], viz.

$$c_1 f(\varepsilon) \leqslant g(\varepsilon) \leqslant c_2 \delta(\varepsilon),$$

for small $\varepsilon > 0$, where $c_1, c_2 > 0$ and g is supermultiplicative, i.e. $g(ts) \geqslant g(t)g(s)$ for $0 \leqslant t, s \leqslant 1$ (Corollary 5.3). This answers a question asked in [5].

Let us remark that there exist uniformly convex Banach spaces with unconditional basis which do not admit best modulus of convexity in the sense defined in [5] (the construction given in [8] is motivated by the notion of cotype defined in Section 2).

We do not know whether it would suffice to assume only that X is a complemented subspace of a Banach lattice (this was asserted in [7], Remark 1). We can only prove a slightly weaker result (Proposition 5.4). For this we need, however, a stronger version of Proposition 2.6 (ii) from [10], which we prove in Section 4 using the Lions–Peetre interpolation technique. We obtain that an operator $T: X \to Y$, where the Banach spaces Y and X^* do not contain l_∞^n's uniformly (cf. [10]) factors through a superreflexive Banach lattice provided that it factors through a lattice (Theorem 4.6).

In Section 1 we prove that a Banach space of cotype f is also of cotype F, where the function F satisfies $F(t) \geqslant f(t)$ for small $t \geqslant 0$ and has some additional properties that are repeatedly used in the sequel (Theorem 1.8).

Sections 2, 3, 4 contain auxiliary results which can be applied also in other situations. For instance, we obtain a simple direct proof of B. Maurey's theorem [24] about (p, q)-absolutely summing operators defined on $C(K)$ spaces (cf. Corollary 4.3).

We also give a proof that a Banach space X with unconditional basis is complemented in a Banach space Y with symmetric basis, which admits moduli of convexity not worse than those of X. Namely, the space Y constructed in [2] has this property (cf. Remark 3 in [7]).

Let us mention that necessary and sufficient conditions for X to admit a norm satisfying (*) can be found in [29] and [12]. The general case requires martingale inequalities rather than cotype properties.

Our notation is standard except perhaps that occasionally A^J stands for the set of all functions $f: J \to A$ and $A^{(J)}$ denotes the set of those $f \in A^J$ that are 0 off a finite subset of J (here J can be an arbitrary set and A is either a linear space or $A = \mathbf{R}_+$). $\mathbf{N}$ denotes the set of positive integers.

1. Lower estimates for Rademacher averages. In this section $(X, \|\cdot\|)$ is a fixed Banach space with $\dim X \geqslant 1$. Let $\mathfrak{B}(X)$ be the set of those $t \in \mathbf{R}^{(\mathbf{N})}$ for which there exists $x \in X^{(\mathbf{N})}$ such that $\|x(i)\| = |t(i)|$ for each i and

$$\int_0^1 \Big\| \sum_i x(i) r_i(s) \Big\| \, ds \leqslant 1.$$

It follows from the contraction principle (cf. [25]) that $\mathfrak{B}(X)$ is solid, i.e. if $t, s \in \mathbf{R}^{(\mathbf{N})}$, $|s(i)| \leqslant |t(i)|$ for each i and $t \in \mathfrak{B}(X)$, then $s \in \mathfrak{B}(X)$.

Let $q_0: \mathbf{R}^{(\mathbf{N})} \to \mathbf{R}_+$ be the gauge of the set $\mathfrak{B}(X)$ and let q_p, $1 \leqslant p \leqslant \infty$, denote the gauge of $\mathfrak{B}\big(L_p(X)\big)$. One can show (cf. [6], Lemma 13) that, for $1 \leqslant p < \infty$, the set $\mathfrak{B}\big(L_p(X)\big)$ coincides with the p-convex hull of $\mathfrak{B}(X)$. Equivalently, the gauge q_p of $\mathfrak{B}\big(L_p(X)\big)$ is the greatest p-convex norm on $\mathbf{R}^{(\mathbf{N})}$ majorized by q_0. We shall be using only the norm q_1 and we shall write q instead of q_1.

For $t \in \mathbf{R}^{(\mathbf{N})}$ and $B \subseteq \mathbf{N}$ let $Bt \in \mathbf{R}^{(\mathbf{N})}$ be defined by $(Bt)(i) = t(i)$ for $i \in B$ and $(Bt)(i) = 0$ otherwise.

PROPOSITION 1.1. *Let $A_1, A_2, \ldots$ be a sequence of mutually disjoint finite non-empty subsets of $\mathbf{N}$. Let $t \in \mathbf{R}^{(\mathbf{N})}$ and set $t_j = A_j t$. Then*

(i) $q\big((q(t_j))\big) \leqslant q(t)$,

(ii) $q_0(t) \leqslant q_0\big((\|t_j\|_{l_1})\big)$,

(iii) $q(t) \leqslant b q\big((\|t_j\|_{l_2})\big)$,

where $b < \infty$ is a constant depending only on X.

Proof. (i) Given $\eta > 1$, pick $x \in X^{(\mathbf{N})}$ so that $\|x(i)\| = |t(i)|$ for each i and

$$\int_0^1 \Big\| \sum_i x(i) r_i(s) \Big\| \, ds \leqslant \eta q_0(t).$$

Using the convexity of q, we get

$$\begin{aligned} q\Big(\int_0^1 \Big(\Big\| \sum_{i\in A_j} x(i) r_i(s) \Big\|\Big) ds\Big) &\leqslant \int_0^1 q\Big(\Big(\Big\| \sum_{i\in A_j} x(i) r_i(s) \Big\|\Big)\Big) ds \\ &\leqslant \int_0^1 ds \int_0^1 \Big\| \sum_{j=1} r_j(u) \sum_{i\in A_j} x(i) r_i(s) \Big\| du \\ &= \int_0^1\int_0^1 \Big\| \sum_j \sum_{i\in A_j} x(i) r_j(u) r_i(s) \Big\| du\, ds \\ &= \int_0^1 \Big\| \sum_i x(i) r_i(v) \Big\| dv. \end{aligned}$$

The last equality follows from the fact that the two vector random variables have identical distribution. Since

$$q(t_j) \leqslant q_0(t_j) \leqslant \int_0^1 \Big\| \sum_{i\in A_j} x(i) r_i(s) \Big\| \, ds,$$

we obtain $q\big((q(t_j))\big) \leqslant \eta q_0(t)$ and, letting η approach 1, we get

$$q\big((q(t_j))\big) \leqslant q_0(t).$$

Since the left-hand side defines a seminorm on $\boldsymbol{R}^{(N)}$ that is dominated by q_0, it must also be dominated by q. This proves (i).

(ii) It is enough to prove that if $u \in \boldsymbol{R}^{(N)}$ and $\tilde{u}$ is obtained from u by changing just two coordinates, say $u(j)$ and $u(k)$, so that $\tilde{u}(j) = 0$ and $\tilde{u}(k) = |u(j)| + |u(k)|$, then $q_0(\tilde{u}) \geqslant q_0(u)$.

To prove this fix $\eta > 1$ and let $x \in X^{(N)}$ satisfy $\|x(i)\| = |\tilde{u}(i)|$ and

$$\int_0^1 \Big\| \sum_i x(i) r_i(t) \Big\| \, dt \leqslant \eta q_0(\tilde{u}).$$

We may assume that $\tilde{u}(k) \neq 0$. Let $\alpha = |u(j)|/\tilde{u}(k)$, $\beta = |u(k)|/\tilde{u}(k)$. Put $y(j) = x(k)$, $y(k) = 0$, $y(i) = x(i)$ for $i \in \boldsymbol{N} \setminus \{j, k\}$. Then $|u(i)| = \|(\alpha x + \beta y)(i)\|$ for each i and hence

$$\begin{aligned} q_0(u) &\leqslant \int_0^1 \Big\| \sum_i (\alpha x + \beta y)(i) r_i(t) \Big\| \, dt \\ &\leqslant \alpha \int_0^1 \Big\| \sum_i x(i) r_i(t) \Big\| \, dt + \beta \int_0^1 \Big\| \sum_i y(i) r_i(t) \Big\| \, dt \leqslant \eta q_0(\tilde{u}). \end{aligned}$$

Since $\eta > 1$ was arbitrary, this proves (ii).

(iii) If X contains l_∞^n's uniformly, then it follows from the contraction principle that $q_0(t) = \max_i |t(i)|$ for $t \in \boldsymbol{R}^{(N)}$ and hence $q = q_0$ and the estimate (iii) holds with $b = 1$.

Assume therefore that X does not contain l_∞^n's uniformly. Let $\gamma_1, \gamma_2, \ldots$ be a sequence of independent normalized Gaussian random variables on the probability space $([0, 1], dt)$ and let $z \in X^{(N)}$. Then

$$\tfrac{1}{2}\sqrt{\pi} \int_0^1 \Big\| \sum_i z(i) r_i(t) \Big\| \, dt \leqslant \int_0^1 \Big\| \sum_i z(i) \gamma_i(t) \Big\| \, dt \leqslant B \int_0^1 \Big\| \sum_i z(i) r_i(t) \Big\| \, dt,$$

where $B \in [1, \infty)$ depends only on X. The first of these estimates is due to G. Pisier [28], while the second appears in [26]. We shall prove that (iii) holds with $b = 2B/\sqrt{\pi}$.

Let g be a real function on $\boldsymbol{N}$ such that $\sum_{i\in A_j} g(i)^2 = 1$ for each j.

Put for $u \in \boldsymbol{R}^{(N)}$

$$\tilde{q}(u) = b^{-1} q(\tilde{u}),$$

where $u \in \boldsymbol{R}^{(N)}$, $\tilde{u}(i) = g(i) u(j)$ if $i \in A_j$, and $\tilde{u}(i) = 0$ otherwise.

It is clear that $\tilde{q}$ is a norm on $\boldsymbol{R}^{(N)}$. Hence, if we prove that $\tilde{q}(u) \leqslant q_0(u)$ for $u \in \boldsymbol{R}^{(N)}$, then it will follow that $\tilde{q} \leqslant q_1$. The estimate (iii) is then obtained by setting $g(i) = t(i)/\|t_j\|_{l_2}$ if $i \in A_j$, $t(i) \neq 0$ and $g(i) = 0$ otherwise.

Fix $u \in \boldsymbol{R}^{(N)}$ and let $\eta > 1$. Pick $x \in X^{(N)}$ so that $\|x(i)\| = |u(i)|$ for each i and

$$\int_0^1 \Big\| \sum_i x(i) r_i(t) \Big\| \, dt \leqslant \eta q_0(u).$$

Put $\tilde{x}(i) = g(i) x(j)$ for $i \in A_j$, $\tilde{x}(i) = 0$ otherwise. Then $\|\tilde{x}(i)\| = |\tilde{u}(i)|$ and, by the properties of Gaussian random variables,

$$\int_0^1 \Big\| \sum_i \tilde{x}(i) \gamma_i(t) \Big\| \, dt = \int_0^1 \Big\| \sum_j x(j) \gamma_j(t) \Big\| \, dt.$$

Using the estimates of Maurey and Pisier we get

$$(\sqrt{\pi}/2B) \int_0^1 \Big\| \sum_i \tilde{x}(i) r_i(t) \Big\| \, dt \leqslant \eta q_0(u)$$

and hence $b^{-1} q_0(\tilde{u}) \leqslant \eta q_0(u)$. Letting η tend to 1 we obtain the estimate $\tilde{q}(u) \leqslant q_0(u)$, which completes the proof.

Let f be a non-negative function on $[0, a)$ and let $0 < d \leqslant a$. We put

$$\omega_d(f) = \sup \Big\{ \sum_i f(t(i)) : \ t \in \boldsymbol{R}_+^N, \ q(t) < d \Big\}.$$

Clearly, if $\omega_d(f) < \infty$ for some $d > 0$, then X is of cotype f in the sense defined in the introduction. The converse is also true (cf. Remark 1.4).

LEMMA 1.2. *Let f_1 be defined for $0 \leqslant t < d$ by the formula*

$$f_1(t) = \sup \{ f(u) : \ 0 \leqslant u \leqslant t \}.$$

Then $\omega_d(f_1) = \omega_d(f)$.

Proof. Clearly, $f_1 \geqslant f$ and hence $\omega_d(f_1) \geqslant \omega_d(f)$. On the other hand, if $s, t \in \boldsymbol{R}^{(N)}$ and $0 \leqslant s(i) \leqslant t(i)$ for each i, then $q(s) \leqslant q(t)$ and hence, if $q(t) < d$, then we have

$$\sum_i f(s(i)) \leqslant \omega_d(f).$$

Taking the supremum of the left-hand side over all the choices of s, we obtain

$$\sum_i f_1(t(i)) \leqslant \omega_d(f),$$

which proves that $\omega_d(f_1) \leqslant \omega_d(f)$.

LEMMA 1.3. *Let* $f_2(t) = \sup_{n \geqslant 1} n f_1(t/\sqrt{n})$ *and* $f_3(t) = \sup_{u \geqslant 1} u f_1(t/\sqrt{u})$ *for* $0 \leqslant t < d$. *Then* $f_3(t) \leqslant 2 f_2(t)$ *and* $\omega_{d/b}(f_2) \leqslant \omega_d(f)$, b *being that of Proposition* 1.1 (iii).

Proof. Given $0 \leqslant t < d$ and $u \geqslant 1$, let n be an integer such that $n \leqslant u < 2n$. Then

$$u f_1(t/\sqrt{u}) \leqslant 2n f_1(t/\sqrt{n}) \leqslant 2 f_2(t).$$

The first assertion follows from this estimate.

Now let $t \in \mathbf{R}_+^{(N)}$ satisfy $q(t) < d/b$. Given any sequence $(n(j))$ of positive integers, pick disjoint subsets A_j of N such that A_j has exactly $n(j)$ elements and put $\tilde{t}(i) = n(j)^{-1/2} t(j)$ if $i \in A_j$ and $\tilde{t}(i) = 0$ otherwise. By Proposition 1.1 (iii) we have $q(\tilde{t}) \leqslant b\, q(t) < d$ and hence

$$\sum_j n(j) f_1(t(j) n(j)^{-1/2}) = \sum_i f_1(\tilde{t}(i)) \leqslant \omega_d(f).$$

Again by taking the supremum of the left-hand side over all the choices of the $n(j)$'s we get the desired estimate

$$\sum_j f_2(t(j)) \leqslant \omega_d(f).$$

Remark 1.4. Let $\omega_d^{\sim}(f)$ denote the number obtained when q is replaced by q_0 in the definition of $\omega_d(f)$. We shall prove that $\omega_{d/2}(f) \leqslant 2\, \omega_d^{\sim}(f)$.

Indeed, the proof of Lemma 1.2 shows also that $\omega_d^{\sim}(f_1) = \omega_d^{\sim}(f)$. Let $f_2^{\sim}(t) = \sup_{n \geqslant 1} n f_1(t/n)$ and $f_3^{\sim}(t) = \sup_{u \geqslant 1} u f_1(t/u)$ for $0 \leqslant t < d$. Using Proposition 1.1 (ii) instead of 1.1 (iii) in the proof of Lemma 1.3, one gets

$$\omega_d^{\sim}(f_2^{\sim}) = \omega_d^{\sim}(f_1^{\sim}), \quad f_3^{\sim}(t) \leqslant 2 f_2^{\sim}(t),$$

for $0 \leqslant t < d$. Observe that, if $t_j \geqslant 0$, $0 \leqslant s_j < d/2$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} t_j = 1$, then

$$f_3^{\sim}\Big(\sum_{j=1}^{n} t_j s_j\Big) \leqslant \sum_{j=1}^{n} t_j f_3^{\sim}(2 s_j).$$

(This estimate follows e.g. from Lemma 2 in [6].)

Now let $s \in \mathbf{R}_+^{(N)}$ satisfy $q(s) < d/2$. There exist $s_1, \ldots, s_n \in \mathbf{R}_+^{(N)}$ with $q_0(s) < d/2$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ such that $\sum_{j=1}^{n} t_j = 1$ and $\sum_{j=1}^{n} t_j s_j = s$. Therefore

$$\sum_i f_3^{\sim}(s(i)) = \sum_i f_3^{\sim}\Big(\sum_j t_j s_j(i)\Big) \leqslant \sum_i \sum_j t_j f_3^{\sim}(2 s_j(i)) \leqslant \sum_j t_j \omega_d^{\sim}(f_3) \leqslant 2 \omega_d^{\sim}(f).$$

Since $f \leqslant f_3$, this proves our claim that

$$\omega_{d/2}(f) \leqslant 2 \omega_d^{\sim}(f).$$

In the sequel we shall write $\omega(f)$ instead of $\omega_1(f)$.

LEMMA 1.5. *Assume that* $\omega(F), \omega(G) < \infty$ *and let*

$$h(t) = \sup\{F(2^{-j}) G(2^j t) \colon t < 2^{-j} \leqslant 1\},$$

for $0 \leqslant t < 1$, *where* $F(1) = \sup_{0 \leqslant t < 1} F(t)$. *Then*

$$\omega(h) \leqslant 4 \omega(F) \omega(G).$$

Proof. Fix $t \in \mathbf{R}_+^{(N)}$ such that $q(t) < 1$. Define the sets $K(j)$, $j = 0, 1, \ldots$, so that $i \in K(j)$ if j is the least index for which $h(t) = F(2^{-j}) G(2^j t)$. Obviously, for $i \in K(j)$ we have $t(i) < 2^{-j}$.

For each j we fix a partition

$$K_j = B_{j,0} \cup B_{j,1} \cup \ldots \cup B_{j,m(j)}$$

of K_j into disjoint sets such that

$$q(B_{j,s} t) \geqslant 2^{-j},$$

for $1 \leqslant s \leqslant m(j)$, but $q(Bt) < 2^{-j}$, if either $B = B_{j,0}$ or B is a proper subset of a $B_{j,s}$. Clearly, we have

$$\sum_{i \in B_{j,0}} G(2^j t(i)) \leqslant \omega(G)$$

and, splitting $B_{j,s}$ into two proper subsets, we get

$$\sum_{i \in B_{j,s}} G(2^j t(i)) \leqslant 2 \omega(G)$$

for $1 \leqslant s \leqslant m(j)$. These estimates yield

$$\sum_i h(t(i)) = \sum_j \sum_{i \in K(j)} F(2^{-j}) G(2^j t(i)) \leqslant \sum_j F(2^{-j}) (2m(j) + 1) \omega(G)$$
$$\leqslant \Big(F(1) + \sum_{j=1}^{\infty} F(2^{-j}) + 2 \sum_j m(j) F(2^{-j})\Big) \omega(G).$$

Clearly, $F(1) \leqslant \omega(F)$. Also $\sum_{j=1}^{\infty} F(2^{-j}) \leqslant \omega(F)$ because, if $s(j) = 2^{-j}$ for $1 \leqslant j \leqslant m$ and $s(j) = 0$ otherwise, then $s \in \mathfrak{B}(X)$ and hence $q(s) < 1$.

Thus it remains to prove that

$$\sum_{j=0}^{\infty} m(j)F(2^{-j}) \leqslant \omega(F).$$

This, however, follows from Proposition 1.1 (i) and Lemma 1.2. Indeed, arrange the vectors $B_{j,s}t$, where $j \geqslant 0$ and $1 \leqslant s \leqslant m(j)$, into a single sequence (z_k) and note that

$$q\big((q(z_k))\big) \leqslant q\Big(\sum_k z_k\Big) \leqslant q(t) < 1.$$

Since

$$\sum_j m(j)F(2^{-j}) \leqslant \sum_k F_1\big(q(z_k)\big) \leqslant \omega(F_1) = \omega(F),$$

the proof is complete.

Let us recall an operation considered in [6]. If f, g are non-negative functions on $[0,1)$, we put for $0 \leqslant t < 1$

$$(f \bar{*} g)(t) = \sup\{f(u)g(v)\colon 0 \leqslant u, v < 1,\ uv = t\}.$$

LEMMA 1.6. *Let F, G be as in Lemma 1.5 and let $H = F \bar{*} G$. Assume that $F(x) \leqslant \beta F(\tfrac{1}{2}x)$ for $0 \leqslant x \leqslant 1$. Then*

$$\omega(H) \leqslant 4\beta\omega(F)\omega(G).$$

Proof. Observe that the function F_1 defined by the formula of Lemma 1.2 also satisfies the assumptions of our lemma. Since $F \bar{*} G \leqslant F_1 \bar{*} G$, it suffices to prove that

$$\omega(F_1 \bar{*} G) \leqslant 4\beta\omega(F)\omega(G).$$

Put for $0 \leqslant t < 1$

$$h(t) = \sup\{F_1(2^{-j})G(2^j t)\colon t < 2^{-j} \leqslant 1\},$$

$$h_1(t) = \sup\{h(u)\colon 0 \leqslant u \leqslant t\}.$$

It follows from Lemmas 1.2 and 1.5 that

$$\omega(h_1) = \omega(h) \leqslant 4\omega(F_1)\omega(G) = 4\omega(F)\omega(G),$$

hence it suffices to prove that $F_1 \bar{*} G \leqslant \beta h_1$.

To this end let $0 < t \leqslant u < 1$ and let k be the integer such that $2^k \leqslant t/u < 2^{k+1}$. We have

$$F_1(t/u)G(u) \leqslant \beta F_1(2^k)G(u) \leqslant \beta h(2^k u) \leqslant \beta h_1(t).$$

Since $u \in (t, 1)$ was arbitrary, this completes the proof.

LEMMA 1.7. *Let F be a function on $[0,1)$ such that $0 < \omega(F) = A < \infty$ and $F(x) \leqslant \beta F(\tfrac{1}{2}x)$ for $0 \leqslant x \leqslant 1$. Put $F_1 = F$ and let $F_{n+1} = F \bar{*} F_n$ for*

$n \geqslant 1$. Let G be defined for $0 \leqslant t < 1$ by the formula

$$G(t) = \sum_{n=1}^{\infty} n^{-2} C^{1-n} F_n(t),$$

where $C = 4A\beta$. Then, for $t, s \in [0,1)$,

$$G(t) \geqslant F(t), \quad \omega(G) \leqslant \tfrac{1}{6}\pi^2\omega(F), \quad G(t)G(s) \leqslant \tfrac{4}{3}\pi^2 CG(ts).$$

Proof. The inequality $G(t) \geqslant F(t)$ is obvious. It follows from the previous lemma that $\omega(F_n) \leqslant \omega(F)C^{n-1}$, for $n = 1, 2, \ldots$, which yields $\omega(G) \leqslant \tfrac{1}{6}\pi^2\omega(F)$.

Now observe that, if k, j are positive integers and $j \leqslant \tfrac{1}{2}k$, then

$$\sum_{n=1}^{j} n^{-2}(k-n)^{-2} \leqslant 4k^{-2}\sum_{n=1}^{\infty} n^{-2} \leqslant \tfrac{2}{3}\pi^2 k^{-2},$$

and therefore, if $0 \leqslant t, s < 1$,

$$G(t)G(s) \leqslant (G \bar{*} G)(ts) \leqslant \sum_{n,m=1}^{\infty} \big((n^{-2}C^{1-n}F_n) \bar{*} (m^{-2}C^{1-m}F_m)\big)(ts)$$

$$= \sum_{n,m=1}^{\infty} (nm)^{-2} C^{2-n-m} F_{n+m}(ts)$$

$$= C\sum_{k=2}^{\infty}\sum_{n+m=k} (nm)^{-2} C^{1-k} F_k(ts) \leqslant \tfrac{4}{3}\pi^2 CG(ts).$$

This completes the proof.

THEOREM 1.8. *Let X be a Banach space and let $f\colon [0,a) \to \mathbf{R}_+$. Assume that, whenever $x_1, \ldots, x_n \in X$ satisfy*

$$\int_0^1 \Big\|\sum_{i=1}^n x_i r_i(t)\Big\|\,dt < a,$$

one has $\sum_{i=1}^n f(\|x_i\|) \leqslant A$.

Then there exist $r, C < \infty$ and a function $F\colon \mathbf{R}_+ \to \mathbf{R}_+$ such that

(i) *$F(t) \geqslant f(t)$, for $0 \leqslant t < a$;*

(ii) *The function $t \mapsto F(t^{1/2})$ is convex;*

(iii) *The function $t \mapsto F(t)t^{-r}$ is non-increasing;*

(iv) *$F(t)F(s) \leqslant CF(ts)$, for $0 \leqslant t, s < 1$;*

(v) *If $x_1, \ldots, x_n \in X$ and*

$$\int_0^1 \Big\|\sum_{i=1}^n x_i r_i(t)\Big\|\,dt = \delta,$$

then $\sum_{i=1}^n F(\|x_i\|) \leqslant C(\delta^2 + \delta^r)$.

Moreover, r can be any number such that X is of cotype t^r and C depends on X, a, A, r but not on f.

Proof. Fix $r \in [2, \infty)$ such that X is of cotype g, where $g(t) = t^r$. It was proved by B. Maurey and G. Pisier [25] that if there is no such an r, then X contains l_∞^n's uniformly and hence our assumption on f yields $f(t) = 0$ for $0 \leqslant t < a$, which makes the theorem trivial.

We may, replacing perhaps f by $f+g$, assume that $f(t) > 0$ for $0 < t < a$. By Remark 1.4, we have $\omega_{a/2}(f) \leqslant 2A$. Hence, if f_1, f_2, f_3 are the functions defined in Lemmas 1.2 and 1.3, then

$$\omega_{a/2b}(f_3) \leqslant 2\omega_{a/2}(f) \leqslant 4A.$$

It is clear that f_3 satisfies (i). Observe that f_3 satisfies the following condition

(ii′) the function $t \mapsto f_3(t)t^{-2}$ is non-decreasing.

Let $d = a/2b$, and let

$$f_4(t) = \begin{cases} \sup\{f_3(t/u)u^r \colon t/d < u \leqslant 1\}, & \text{if } 0 \leqslant t < d, \\ (t/d)^r \lim\limits_{s\to d_-} f_3(s), & \text{if } t \geqslant d. \end{cases}$$

Using Lemma 1.6, with $F(t) = t^r$ and $G(t) = f_3(t/d)$, we get

$$\omega_d(f_4) \leqslant 4\cdot 2^r \omega_d(f_3)\,\omega(g).$$

Observe that $f_4(t) \geqslant f_3(t)$ for $0 \leqslant t < d$ and f_4 satisfies (ii′) and (iv). It is easy to check that for $c \geqslant d$

$$\omega_c(f_4) \leqslant (c/d)^2 \omega_d(f_4).$$

Letting $F = f_4$ in Lemma 1.7, we get the function $G = f_5$, which satisfies (iv). It also satisfies (ii′) and (iii) on $[0, 1)$, (v) for $\delta < 1$ and (i) on $[0, d_1)$ where $d_1 = \min(d, 1)$. The latter properties will be fulfilled on $[0, \infty)$ if we put for $t \geqslant 1$

$$f_5(t) = t^r \lim_{s\to 1_-} f_5(s).$$

Let f_6 be the supremum of all functions φ that satisfy (ii) and are majorized by f_5. Since f_5 satisfies (ii′) and (iii), Lemma 2 from [6] yields

$$2^{-r/2} f_5(t) \leqslant f_6(t) \leqslant f_5(t),$$

for $t \geqslant 0$. Therefore the function f_7 defined by the formula

$$f_7(t) = 2^{r/2} \sup\{(t/u)^r f_5(u) \colon u \geqslant t\}$$

satisfies (ii), (iii), (iv), (v) and also (i) for $t \in [0, d_1)$.

To make sure that (i) holds for all $t < d$ it suffices to multiply f_7 by a suitable constant.

The verification of the last statement of the theorem may be omitted.

2. Auxiliary facts about lattices. Let L be a vector lattice. A norm p on L is said to be a *lattice norm* if $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$. Lattice seminorms on L are defined analogously. A Banach lattice is a Banach space $(L, \|\cdot\|)$ with a fixed vector lattice structure such that $\|\cdot\|$ is a lattice norm on L. A linear subspace I of L is said to be an *ideal* (or *lattice ideal*) *of* L if $x \in I$, $y \in L$, $|y| \leqslant |x|$ implies $y \in I$. An ideal is called a *band* if $A \subset I$, $\sup A = x \in L$ implies $x \in I$.

A Banach lattice L is said to be *complete* (resp. *σ-complete*) if every non-empty (resp. every countably infinite) subset of L which is bounded from above has the least upper bound.

By a homomorphism we shall always mean a vector lattice homomorphism, i.e. a linear map $T \colon L \to M$ between vector lattices such that $T(x \wedge y) = Tx \wedge Ty$ for $x, y \in L$.

Elements $x, y \in L$ are said to be *disjoint* if $|x| \wedge |y| = 0$.

Let L be a σ-complete Banach lattice and let $x, z \in L$ be non-negative. Then there exists $P_x z = \sup\limits_{n\geqslant 1}(nx \wedge z)$. We put

$$P_x y = P_{|x|} y^+ - P_{|x|} y^-$$

if $x, y \in L$ are arbitrary. The operator $P_x \colon L \to L$, which we shall call the *projection onto the support of* x, is a homomorphism such that $P_x y = 0$ if x, y are disjoint and $P_x z = z$ if z is disjoint with each y that is disjoint with x (cf. [31], Proposition II.2.11).

Let us describe a general scheme for constructing functions of elements of a Banach latice (cf. [18], [19]). (In this scheme the operator P_x can be defined by the formula $P_x y = g(x, y)$, where $g \in \mathscr{B}^2$, $g(t, s) = 0$ if $t = 0$ and $g(t, s) = s$ otherwise.) We shall consider only the case of continuous functions.

Let $\varLambda^n$ denote the vector lattice of all functions $f \colon \boldsymbol{R}^n \to \boldsymbol{R}$ such that $f(tx) = tf(x)$ for $x \in \boldsymbol{R}^n$, $t \geqslant 0$, with the operations defined pointwise. Let $\varLambda_0^n$ denote the vector sublattice of $\varLambda^n$ generated by the elements $g_1, \ldots, g_n$ where $g_i(x) = x_i$ for $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$.

It was proved by A. I. Yudin [35] (cf. [34], Theorem V.7.2) that if L is a normed lattice then, given any $l_1, \ldots, l_n \in L$, there is a (unique) homomorphism $\varPhi \colon \varLambda_0^n \to L$ such that $\varPhi(g_i) = l_i$ for $1 \leqslant i \leqslant n$.

We shall usually write $f(l_1, \ldots, l_n)$ instead of $\varPhi(f)$, also when $\varPhi$ is extended to other sublattices of $\varLambda^n$. For instance, if $f = \max\limits_{1\leqslant i\leqslant n} |g_i| = \|\cdot\|_{l_\infty^n}$, then

$$\varPhi(f) = \max_{1\leqslant i\leqslant n} |g_i(l_1, \ldots, l_n)| = |l_1| \vee \ldots \vee |l_n| = \|(l_1, \ldots, l_n)\|_{l_\infty^n}.$$

Put for $f \in \varLambda^n$

$$\|f\| = \sup\{|f(x)| \colon x \in \boldsymbol{R}^n, \|x\|_{l_\infty^n} \leqslant 1\}.$$

Clearly, $\|\cdot\|$ is a lattice norm on the sublattice Λ_b^n of those $f \in \Lambda^n$ that are continuous at the origin, which makes Λ_b^n a Banach lattice. Since $\Phi\colon \Lambda_0^n \to L$ preserves the order, it is easy to see that $\|\Phi\| = \|(l_1, \ldots, l_n)\|_{l_\infty^n} < \infty$. Consequently, if L is a Banach lattice, then Φ extends to the closure $\mathscr{C}^n$ of Λ_0^n in Λ_b^n which, by Stone's theorem, consists of those $f \in \Lambda^n$ that are continuous. This extension will still be denoted by Φ.

It is easy to check that if M is another Banach lattice and $\psi\colon L \to M$ is a lattice homomorphism, then

$$\psi\big(f(l_1, \ldots, l_n)\big) = f\big(\psi(l_1), \ldots, \psi(l_n)\big)$$

for $f \in \mathscr{C}^n$, $l_1, \ldots, l_n \in L$. In particular, if L is represented as a vector lattice of real functions with the operations defined pointwise and $\psi\colon L \to \boldsymbol{R}$ is the evaluation at a point s, then

$$f(l_1, \ldots, l_n)(s) = f\big(l_1(s), \ldots, l_n(s)\big).$$

All properties of the maps Φ we need are made intuitive by this simple observation and representation theorems for lattices (cf. [31], [34]).

To illustrate this procedure let us discuss an example which is a variation of the above scheme since the functions f are defined only on a subset of $\boldsymbol{R}^n$ and may not be restrictions of elements of $\mathscr{C}^n$. (One may note that if L is σ-complete then, by extending Φ to the sublattice $\mathscr{B}^n$ of those $f \in \Lambda_b^n$ that are Baire functions, that example can be reduced to the general scheme. We prefer, however, not to introduce the assumption of σ-completeness where it is not necessary.) The specific computations we make below are to be used in the proof of Proposition 3.2.

Let $(L, \|\cdot\|)$ be a Banach lattice and let $x \in L$, $x > 0$. Put

$$I = \{z \in L\colon\ |z| \leqslant ax \text{ for some } a \in \boldsymbol{R}\}.$$

By Theorem II.7.4 of [31] there exists a compact space K and a lattice isomorphism Ψ of $C(K)$ onto I such that $\Psi(1) = x$, where $1 \in C(K)$ is the constant one function. Fix $y \in L$ with $0 \leqslant y \leqslant x$ and let $\eta = \Psi^{-1}(y)$. If $\varphi \in C([0,1])$, define $M_\varphi\colon I \to I$ by

$$M_\varphi z = \Psi\big((\varphi \circ \eta)\cdot \Psi^{-1}(z)\big).$$

Fix $q \geqslant 1$ and let $\psi(t) = (1-t^q)^{1/q}$ for $0 \leqslant t \leqslant 1$. In $C(K)$ there is a unique $\xi \geqslant 0$ such that $(\eta^q + \xi^q)^{1/q} = 1$, viz. $\xi = \psi \circ \eta$. Hence there is a unique nonnegative $z \in I$ such that $(|y|^q + |z|^q)^{1/q} = x$, viz. $z = \Psi(\xi) = M_\psi x$. Now, if $\varphi \in C([0,1])$ is nonnegative and $\varphi(t) = 0$ for $t < a$, then

$$\varphi(t)\psi(t) \leqslant \big(\psi(a)/a\big)\varphi(t)t,$$

for $0 \leqslant t \leqslant 1$, and thus

$$M_\varphi z = \Psi\big((\varphi\circ\eta)\cdot(\psi\circ\eta)\big) \leqslant \big(\psi(a)/a\big)\Psi\big((\varphi\circ\eta)\cdot\eta\big) = \big(\psi(a)/a\big)M_\varphi y.$$

LEMMA 2.1. *Let s be a positive integer and let $a_j = 1-e^{-j}$, for $0 \leqslant j \leqslant s+1$, $a_{s+2} = 1$. Put $Q_j = M_{\varphi_j}$, where $\varphi_0, \varphi_1, \ldots, \varphi_s$ is the partition of unity on $[0,1]$ which consists of continuous piecewise linear functions with nodes at the a_j's and such that* $\operatorname{supp}\varphi_j = [a_j, a_{j+2}]$, *for $0 \leqslant j \leqslant s$. Then*

(i) $\sum_{j=0}^{s} Q_j w = w$, $|Q_j w| \wedge |Q_k w| = 0$, *for* $w \in I$, $k > j+1$,

(ii) $Q_j(x-y) \geqslant e^{-j-2}Q_j x$, *for* $0 \leqslant j \leqslant s-1$,

(iii) $0 \leqslant Q_j z \leqslant C_1 e^{-j/q} Q_j y$, *for* $1 \leqslant j \leqslant s$,

where $C_1 = \big(e(1-e^{-1})^{-q}-1\big)^{1/q}$. *One has* $1 < C_1 < e^2/(e-1)$.

Proof. (i) is obvious. (ii) follows from

$$Q_j(x-y) = \Psi\big((\varphi_j\circ\eta)\cdot(1-\eta)\big) \geqslant \Psi(e^{-j-2}\varphi_j\circ\eta) = e^{-j-2}Q_j x.$$

To obtain (iii) observe that if $a_1 \leqslant a < 1$, then

$$\big(\psi(a)/a\big)^q = (1-a)\,[(a^{-q}-1)/(1-a)] \leqslant (1-a)\,[(a_1^{-q}-1)/(1-a_1)];$$

hence, letting $a = a_j = 1-e^{-j}$, we get

$$\psi(a_j)/a_j \leqslant e^{-j/q}C_1$$

which yields (iii), because $Q_j z \leqslant \big(\psi(a_j)/a_j\big)Q_j y$. The estimates of C_1 are left to the reader (in fact $C_1 < 2.03$).

In Section 3 we shall need the following generalization of the decomposition lemma for lattices.

LEMMA 2.2. *Assume that L is a Banach lattice and $p, q \geqslant 1$. If $x_1, \ldots, x_n, y_1, \ldots, y_m \in L$ satisfy*

$$\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} = x = \Big(\sum_{j=1}^{m} |y_j|^q\Big)^{1/q},$$

then there exist $z_{i,j} \in L$ such that

$$|x_i| = \Big(\sum_{j=1}^{m} |z_{i,j}|^q\Big)^{1/q}, \qquad |y_j| = \Big(\sum_{i=1}^{n} |z_{i,j}|^p\Big)^{1/p},$$

for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$.

Proof. The lemma is obvious if L is a $C(K)$ space and $x(w) = 1$ for $w \in K$. (One can take $z_{i,j}(w) = x_i(w)y_j(w)$.) The general case reduces to this one immediately by considering the isomorphism $\Psi\colon C(K) \to I$ we have discussed above.

There is a natural generalization of the notion of (q,p)-summing operators to the case of Banach lattices. We recall some definitions and results from [19] and [24].

Let $1 \leqslant p \leqslant q \leqslant \infty$. A linear operator u mapping a Banach lattice L into a Banach space X is said to be *of type* $\leqslant (p,q)$ provided there is a constant $C \geqslant 0$ such that

$$\Big(\sum_{i=1}^{n} \|ux_i\|^q\Big)^{1/q} \leqslant C \Big\|\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}\Big\|,$$

for every finite sequence $x_1, \ldots, x_n \in L$. The smallest C with this property is denoted by $K_{p,q}(u)$. Obviously, if $L = C(K)$, then $u\colon L \to X$ is of type $\leqslant (p, q)$ if and only if it is (q, p)-absolutely summing and $K_{p,q}(u) = \pi_{q,p}(u)$.

If $I\colon L \to L$ is the identity operator and $K_{q,q}(I) < \infty$, then L is said to be *of type* $\leqslant q$ (cf. [19]); we say that L has *q-concave norm* (cf. [9]) if $K_{q,q}(I) \leqslant 1$. Finally, L is said to be q-Besselian (with constant C) provided that

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \leqslant C \Big\|\sum_{i=1}^{n} x_i\Big\|,$$

for every disjoint finite sequence $x_1, \ldots, x_n \in L$. (This means that every sequence of disjoint non-zero vectors in L is a q-Besselian basic sequence with constant C.)

An operator v mapping a Banach space X into a Banach lattice L is said to be *of type* $\geqslant (q, p)$ provided that for some $C \geqslant 0$

$$\Big\|\Big(\sum_{i=1}^{n} |vx_i|^q\Big)^{1/q}\Big\| \leqslant C \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p},$$

for every finite sequence $x_1, \ldots, x_n \in X$. $K^{p,q}(v)$ is the least C with this property. The lattice L is *of type* $\geqslant p$, if $K^{p,p}(I) < \infty$, I being the identity on L. We say that L has *p-convex norm* if $K^{p,p}(I) \leqslant 1$ and L is *p-Hilbertian* (with constant C) if

$$\Big\|\sum_{i=1}^{n} x_i\Big\| \leqslant C \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}$$

for disjoint sequences $x_1, \ldots, x_n \in L$.

One can prove that $K^{q,p}(v) = K_{q',p'}(v^*)$, where $p' = p/(p-1)$, $q' = q/(q-1)$ and $v^*\colon L^* \to X^*$ is the adjoint of v, and similarly for $u\colon L \to X$ one has $K_{p,q}(u) = K^{p',q'}(u^*)$ (cf. [19], [24]).

The following two lemmas will be used in Section 4.

LEMMA 2.3. *Let L be a q-Besselian Banach lattice, $1 \leqslant q < \infty$. Then L is a complete lattice and the canonical image $\varkappa(L)$ of L in L^{**} is complemented.*

Proof. Clearly, no Banach sublattice of L can be isomorphic (as a vector lattice) to c_0. Therefore, by Proposition II.5.15 of [31], $\varkappa(L)$ is a band in L^{**}. Since L^{**} is complete, so is $\varkappa(L)$ and hence L. By Proposition II.5.2 of [31], the band projection of L^{**} onto $\varkappa(L)$ is continuous. This completes the proof of the lemma.

LEMMA 2.4. *Let $T\colon L \to \Lambda$ be a homomorphism of Banach lattices. If $T(L)$ is (norm) dense in Λ and order intervals in L are weakly compact, then $T(L)$ is a lattice ideal in Λ, and hence $T^*\colon \Lambda^* \to L^*$ is a lattice homomorphism.*

Proof. Let $y \in \Lambda$, $x \in L$, $|y| \leqslant Tx$. We are to prove that $y = Tz$ for some $z \in L$. There exists a sequence (x_n) in L such that $\|Tx_n - y\| \to 0$. Let $z_n = (x_n \wedge x) \vee (-x)$. Clearly, $Tz_n = (Tx_n \wedge Tx) \vee T(-x)$ also converges to y. The order interval $[-x, x]$ being weakly compact, there is a subsequence (z_{n_i}) weakly convergent to a $z \in [-x, x]$. Since T is weakly continuous, $Tz = \lim Tz_{n_i} = y$. This proves that $T(L)$ is a lattice ideal in Λ. The remaining assertion follows from this fact (cf. Exercise III.24 (c) in [31]).

Let us now introduce a more general notion of type and cotype of a Banach space X. If $\boldsymbol{R}^{(N)} \subseteq E \subseteq \boldsymbol{R}^N$ and $(E, \|\cdot\|_E)$ is a Banach lattice, then X is said to be *of cotype E* (with constant C) provided that

$$\|(\|x(i)\|)\|_E \leqslant C \int_0^1 \Big\|\sum_i x(i) r_i(t)\Big\| dt,$$

for $x \in X^{(N)}$. Analogously, X is said to be *of type E* if

$$\int_0^1 \Big\|\sum_i x(i) r_i(t)\Big\| dt \leqslant C \|(\|x(i)\|)\|_E.$$

Observe that, if X is of cotype f in the sense defined in the introduction, then it is of cotype F, F being the Orlicz function constructed in Theorem 1.8, and hence it is of cotype E, where E is the Orlicz sequence space l_F. The converse implication is obvious.

This definition can be given another useful form if $X = L$ is a Banach lattice (cf. [24]). It was observed by Maurey that, if $x_1, \ldots, x_n \in L$, then

$$2^{-1/2}\Big\|\Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}\Big\| \leqslant \Big\|\int_0^1 \Big|\sum_{i=1}^{n} x_i r_i(t)\Big| dt\Big\| \leqslant \int_0^1 \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\| dt \leqslant B \Big\|\Big(\sum_{i=1}^{n} |x_i|^2\Big)^{1/2}\Big\|.$$

The first estimate is a consequence of Khintchine's inequality (cf. [33], [13]) and the second follows from the convexity of the lattice norm $\|\cdot\|$. The constant B in the third inequality can be chosen independently of the x_i's if and only if the lattice L does not contain l_∞^n's uniformly. The latter condition is equivalent to L being of type $\leqslant q$ for some $q < \infty$.

Let us recall a construction of Banach lattices. If $(L, \|\cdot\|_L)$ is a Banach lattice, J is a set and $\|\cdot\|_E$ is a lattice norm on $\boldsymbol{R}^{(J)}$, we put for $x \in L^{(J)}$

$$\|x\|_{L(E)} = \|\,\|x(j)\|_E\|_L,$$

$$\|x\|_{E(L)} = \|\,\|x(j)\|_L\|_E.$$

Clearly, $\|\cdot\|_{L(E)}$ and $\|\cdot\|_{E(L)}$ are lattice norms on $L^{(J)}$, which are p-convex if so are $\|\cdot\|_E$ and $\|\cdot\|_L$. We let $L(E)$ (resp. $E(L)$) denote the completion of $L^{(J)}$ with respect to $\|\cdot\|_{L(E)}$ (resp. $\|\cdot\|_{E(L)}$). The spaces $L(E)$ and $E(L)$ can be naturally regarded as lattice ideals of the vector lattice L^J.

This construction can be iterated. Obviously, it is associative, i.e.

$$E(F(G)) = (E(F))(G)$$

whenever E, F, G are Banach lattices and two of them are atomic.

In Section 5 we shall need the following simple fact.

LEMMA 2.5. *Let X, G be Banach lattices and let E be a sequence lattice. If $G(l_2) \subseteq l_2(G)$, $X(l_2) \subseteq E(X)$ and either X or G is an atomic lattice, then*

$$(X(G))(l_2) \subseteq E(X(G)).$$

Proof. Using the associativity one obtains easily

$$(X(G))(l_2) = X(G(l_2)) \subseteq X(l_2(G)) \subseteq E(X(G)).$$

The next lemma contains B. Maurey's characterization of cotype of Banach lattices.

LEMMA 2.6. *A Banach lattice L is of cotype E, where E is a sequence lattice, if and only if*

$$L(l_2) \subseteq E(L).$$

Proof. Assume first that $L(l_2) \subseteq E(L)$. By the closed graph theorem, there is $K < \infty$ such that

$$\|x\|_{E(L)} \leqslant K\|x\|_{L(l_2)}$$

for $x \in L(l_2)$. Using the estimate

$$\|x\|_{L(l_2)} = \Big\|\Big(\sum_i x(i)^2\Big)^{1/2}\Big\|_L \leqslant 2^{1/2}\int_0^1 \Big\|\sum_j x(j)r_j(t)\Big\|\,dt$$

we get that L is of cotype E with constant $\sqrt{2}\,K$.

Now assume that L is of cotype E with constant C. If L does not contain l_∞^n's uniformly, then the other estimate of Maurey yields for $x \in L^{(N)}$

$$\|x\|_{E(L)} \leqslant C\int_0^1 \Big\|\sum_j x(j)r_j(t)\Big\|\,dt \leqslant BC\Big\|\Big(\sum_j x(j)^2\Big)^{1/2}\Big\| = BC\|x\|_{L(l_2)},$$

and therefore $L(l_2) \subseteq E(L)$.

The case where L does contain l_∞^n's uniformly is trivial, because then one easily obtains that $\|t\|_E \leqslant C$ for each sequence $t \in \mathbf{R}^{(N)}$ whose terms are either 0 or 1, and hence

$$\|x\|_{E(L)} \leqslant C\max_j \|x(j)\|_L \leqslant C\|x\|_{L(l_2)},$$

for $x \in L(l_2)$. This completes the proof.

Remark. If the lattice L is of type E, then $E(L) \subseteq L(l_2)$. The converse implication holds provided that L does not contain l_∞^n's uniformly.

3. Estimates of lattice norms. Our estimates of the moduli of convexity will be based on the following simple proposition.

PROPOSITION 3.1. *Let $(L, \|\cdot\|)$ be a p-convex Banach lattice, $1 \leqslant p \leqslant 2$, and let $h\colon [0,1]\to \mathbf{R}$. Assume that, if $u, v \in L$ satisfy $\|(u^2+v^2)^{1/2}\| \leqslant 1$, then*

$$h(\|v\|) \leqslant 1-\|u\|.$$

Then the modulus of convexity of $(L, \|\cdot\|)$ satisfies the estimate

$$\delta_L(\varepsilon) \geqslant h(\tfrac{1}{2}\sqrt{p-1}\,\varepsilon) \quad \textit{for} \quad 0 \leqslant \varepsilon \leqslant 2.$$

Proof. Let $x, y \in L$, $\|x\|, \|y\| \leqslant 1$. Consider functions $G, H\colon \mathbf{R}^2 \to \mathbf{R}$ given by

$$G(t,s) = \big((\tfrac{1}{2}(|t|^p+|s|^p))^{2/p} - \tfrac{1}{4}(p-1)(t-s)^2\big)^{1/2},$$
$$H(t,s) = |\tfrac{1}{2}(t+s)|.$$

Clearly, $G, H \in \mathscr{C}^2$ (cf. Section 2). Since $H \leqslant G$ (cf. [6] Lemma 25), it follows that

$$0 \leqslant |\tfrac{1}{2}(x+y)| = H(x,y) \leqslant G(x,y).$$

Put $u = G(x,y)$, $v = \frac{1}{2}\sqrt{p-1}\,|x-y|$. Then

$$(u^2+v^2)^{1/2} = \big(\tfrac{1}{2}(|x|^p+|y|^p)\big)^{1/p}$$

and hence

$$\|(u^2+v^2)^{1/2}\| \leqslant \big(\tfrac{1}{2}(\|x\|^p+\|y\|^p)\big)^{1/p} \leqslant 1.$$

Therefore our assumption on h gives

$$\|\tfrac{1}{2}(x+y)\| \leqslant \|u\| \leqslant 1-h(\|v\|) = 1-h(\tfrac{1}{2}\sqrt{p-1}\,\|x-y\|),$$

which completes the proof.

Remark. In [7] we use a slightly different argument which yields the estimate $\delta_L(\varepsilon) \geqslant \frac{1}{2}(p-1)h(\frac{1}{2}\varepsilon)$.

The next proposition is a partial generalization of Proposition 27 in [6] (the variant with a general function M is not needed in the present context).

PROPOSITION 3.2. *Let $(L, |||\cdot|||)$ be a Banach lattice and let $p, q \geqslant 1$. Let $\|\cdot\|$ be a lattice seminorm on L such that*

$$|||u+v|||^p \geqslant \|u\|^p + |||v|||^p,$$

whenever $u, v \in L$ are disjoint.

Then there exists a constant $C = C(p,q)$ such that, if $y, z \in L$ satisfy

$|||(|y|^q+|z|^q)^{1/q}|||\leqslant 1$, *then*

$$\|z\|\leqslant C\varphi(1-|||y|||),$$

where

$$\varphi(t)=\begin{cases}t^{1/p} & \text{if } q<p,\\ t^{1/p}(\log(e/t))^{1-1/p} & \text{if } q=p,\\ t^{1/q} & \text{if } q>p.\end{cases}$$

Proof. Write $x=(|y|^q+|z|^q)^{1/q}$, $t=1-|||y|||$ and assume, without loss of generality, that $|||x|||=1$, $0<|||y|||<1$, $y\geqslant 0$, $z\geqslant 0$. Let s be the positive integer such that $e^{-s}\leqslant t<e^{1-s}$ and let $Q_0,\ldots,Q_s$ be the homomorphisms constructed in Lemma 2.1. Put

$$\eta_0=\|Q_0x\|,\qquad \eta_j=\|Q_jy\| \text{ for } 1\leqslant j\leqslant s,$$

and observe that, by Lemma 2.1 (iii),

$$\|Q_jz\|\leqslant C_1e^{-j/q}\eta_j \quad \text{for} \quad 0\leqslant j\leqslant s.$$

Hence, using Lemma 2.1 (i) and the triangle inequality for $\|\cdot\|$, we obtain

$$\|z\|\leqslant\sum_{j=0}^{s}\|Q_jz\|\leqslant C_1\Big(e^{-s/q}+\sum_{j=0}^{s-1}e^{-j/q}\eta_j\Big).$$

Clearly, $e^{-s/q}\leqslant t^{1/q}\leqslant\varphi(t)$. Suppose we know that

$$(*)\qquad \sum_{j<s}e^{-j}\eta_j^p\leqslant pC_2t,$$

where C_2 is an absolute constant. Then the estimate of $\|z\|$ can be completed as follows. Assume first that $p>1$ and let r satisfy $p^{-1}+r(1-p^{-1})=q^{-1}$. Using the Hölder inequality we get

$$\sum_{j=0}^{s-1}e^{-j/q}\eta_j\leqslant\Big(\sum_{j=0}^{s-1}e^{-j}\eta_j^p\Big)^{1/p}\Big(\sum_{j=0}^{s-1}e^{-jr}\Big)^{1-1/p}\leqslant p^{1/p}C_2^{1/p}t^{1/p}\Big(\sum(r,s)\Big)^{1-1/p},$$

where $\sum(r,s)$ denotes the sum of the geometric progression, which is

$$\begin{aligned}&\leqslant(1-e^{-r})^{-1}<1+r^{-1}=(pq-2q+p)/(p-q) && \text{if } q<p,\\ &\leqslant s<\log(e/t) && \text{if } q=p,\\ &<e^{-r(s-1)}/(1-e^{-r})<t^r/(1-e^r)\leqslant(1-1/r)t^r\\ &=t^rp(q-1)/(q-p) && \text{if } q>p.\end{aligned}$$

In each case we get the right estimate for $\|z\|$. The case $p=1$ follows now easily by continuity (a direct argument is also available), hence it remains to prove $(*)$.

To this end pick a positive functional $x^*\in L^*$ so that $|||x^*|||=1$, $x^*(y)=|||y|||=1-t$. Write $a_j=x^*(Q_jy)$ for $0\leqslant j\leqslant s$. Using Lemma 2.1 (i) and (ii), we get

$$t\geqslant x^*(x)-x^*(y)=x^*(x-y)=\sum_{j\leqslant s}x^*(Q_j(x-y))$$
$$\geqslant\sum_{j<s}e^{-j-2}x^*(Q_jx)\geqslant e^{-2}\sum_{j<s}e^{-j}a_j.$$

Now, since Q_jx and Q_kx are disjoint if $k>j+1$,

$$\begin{aligned}\eta_0^p&=\|Q_0x\|^p\leqslant|||x|||^p-\Big\|\sum_{k=2}^{s}Q_kx\Big\|^p\\ &\leqslant p(1-|||x-Q_0x-Q_1x|||)\leqslant p(1-|||y-Q_0y-Q_1y|||)\\ &\leqslant p\big(1-x^*(y-Q_0y-Q_1y)\big)\leqslant p(t+a_0+a_1),\end{aligned}$$

and similarly, for $1\leqslant j\leqslant s-1$,

$$\begin{aligned}\eta_j^p&=\|Q_jy\|^p\leqslant|||y|||^p-|||y-Q_{j-1}y-Q_jy-Q_{j+1}y|||^p\\ &\leqslant p(|||y|||-|||y-Q_{j-1}y-Q_jy-Q_{j+1}y|||)\\ &\leqslant p\big(x^*(y)-x^*(y-Q_{j-1}y-Q_jy-Q_{j+1}y)\big)\\ &=p(a_{j-1}+a_j+a_{j+1}).\end{aligned}$$

These estimates yield

$$\sum_{j<s}e^{-j}\eta_j^p\leqslant pt+p(e+1+e^{-1})\sum_{j<s}e^{-j}a_j\leqslant pt(1+e+e^2+e^3),$$

which completes the proof.

In the next three lemmas let $(L,\|\cdot\|)$ be a Banach lattice and let q be a continuous seminorm on L. Let F be a function on $[0,\infty)$ such that $F(1)>0$, the function $t\mapsto F(t^{1/p})$ is convex for some $p\geqslant 1$ and for some $r<\infty$ one has $F(ts)\geqslant t^rF(s)$, if $s\geqslant 0$, $0\leqslant t\leqslant 1$. We assume that there is C_1, $F(1)\leqslant C_1<\infty$, such that

$$\sum_{i=1}^{n}F(q(x_i))\leqslant C_1$$

for all finite sequences $x_1,\ldots,x_n\in L$ with $\|(\sum_{i=1}^{n}x_i^2)^{1/2}\|\leqslant 1$, and set for $x\in L$

$$|||x|||=\inf\Big\{t>0:\sum_{i=1}^{n}F(q(x_i)/t)\leqslant F(1) \text{ whenever } |x|=\Big(\sum_{i=1}^{n}|x_i|^2\Big)^{1/2}\Big\}.$$

Lemma 3.3. *$|||\cdot|||$ is a lattice seminorm on L such that*

$$q(x)\leqslant|||x|||\leqslant(C_1/F(1))^{1/p}\|x\| \quad \textit{for} \quad x\in L.$$

Proof. Suppose $x \in L \setminus \{0\}$, $|x| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$. Since $C_2 = (C_1/F(1))^{1/p} \geqslant 1$, we have

$$\sum_{i=1}^{n} F\left(q(x_i)/C_2\|x\|\right) \leqslant C_2^{-p} \sum_{i=1}^{n} F\left(q(x_i)/\|x_i\|\right) \leqslant C_2^{-p} C_1 = F(1).$$

This proves that $|||x||| \leqslant C_1\|x\|$ for $x \in L$. It is clear that $|||\cdot|||$ is homogeneous and that $|||x||| \geqslant q(x)$ and $|||x||| = |||\,|x|\,|||$ for $x \in L$. To prove that $|||\cdot|||$ satisfies the triangle inequality one may apply Lemma 3.5 below with $p = 1$ and q replaced by the functional q' defined for $x \in L$ by $q'(x) = \sup\{q(y)\colon |y| \leqslant |x|\}$, which obviously is 1-convex. Indeed, the seminorm $|||\cdot|||'$ defined by the formula, analogous to that for $|||\cdot|||$, in which q is replaced by q' is identical with $|||\cdot|||$.

LEMMA 3.4. *If $x, y \in L$ satisfy $|||(x^2+y^2)^{1/2}||| \leqslant 1$, then*

$$F\left(q(y)\right) \leqslant rF(1)\left(1-|||x|||\right).$$

Furthermore, if for $t, s \in [0, 1)$

$$F(t)F(s) \leqslant C_3F(ts),$$

then $F(|||y|||) \leqslant C_3 r(1-|||x|||)$ and also one has

$$\sum_{i=1}^{k} F(|||z_i|||) \leqslant C_3$$

whenever $z_1, \ldots, z_k \in L$ and $|||\left(\sum_{i=1}^{k} z_i^2\right)^{1/2}||| \leqslant 1$.

Proof. Fix elements $x, y \in L$ such that $|||(x^2+y^2)^{1/2}||| \leqslant 1$. Consider a pair of decompositions

$$|x| = \left(\sum_{j=1}^{m} |x_j|^2\right)^{1/2}, \qquad |y| = \left(\sum_{k=1}^{n} |y_k|^2\right)^{1/2},$$

Write $A = \sum_{j=1}^{m} F\left(q(x_j)\right)$, $B = \sum_{k=1}^{n} F\left(q(y_k)\right)$. Clearly, $A+B \leqslant F(1)$. Now, if $\left(1-B/F(1)\right)^{1/r} < a \leqslant 1$, then

$$\sum_{j=1}^{m} F\left(q(x_j)/a\right) \leqslant a^{-r} \sum_{j=1}^{m} F\left(q(x_j)\right) \leqslant F(1),$$

since obviously $r \geqslant p \geqslant 1$. It follows that

$$|||x||| \leqslant \left(1-B/F(1)\right)^{1/r} \leqslant 1-B/rF(1),$$

or $B \leqslant rF(1)(1-|||x|||)$, which in the special case $n = 1$, $y_1 = y$ implies the first assertion of the lemma.

If F is supermultiplicative and $|||y||| \neq 0$, then

$$\sum F\left(q(y_k)/|||y|||\right)F(|||y|||) \leqslant C_3 \sum F\left(q(y_k)\right) \leqslant C_3 rF(1)(1-|||x|||).$$

Taking the supremum over all the decompositions of y and then dividing by $F(1)$ we get

$$F(|||y|||) \leqslant C_3 r(1-|||x|||).$$

This inequality is obvious, if $|||y||| = 0$.

Finally, similar estimates give

$$\sum_{j=1}^{m} F\left(q(x_j)/|||x|||\right)F(|||x|||) + \sum_{k=1}^{n} F\left(q(y_k)/|||y|||\right)F(|||y|||)$$
$$\leqslant C_3 \sum_{j=1}^{m} F\left(q(x_j)\right) + C_3 \sum_{k=1}^{n} F\left(q(y_k)\right) = C_3(A+B) \leqslant C_3F(1),$$

which yields again

$$F(|||x|||) + F(|||y|||) \leqslant C_3.$$

The case where the pair x, y is replaced by an arbitrary finite sequence (z_j) with $|||\left(\sum z_j^2\right)^{1/2}||| \leqslant 1$ can be handled in a similar manner. This completes the proof of the lemma.

LEMMA 3.5. *If q is p-convex lattice norm, then so is $|||\cdot|||$. If q is p-Hilbertian with constant 1, then so is $|||\cdot|||$.*

Proof. We shall consider both cases simultaneously. Let $x, y, z \in L$, $|z| = (|x|^p+|y|^p)^{1/p}$. We are to prove that $|||z||| \leqslant (|||x|||^p+|||y|||^p)^{1/p}$. Assume, without loss of generality, that $|||x|||^p+|||y|||^p = 1$. Consider a decomposition $|z| = \left(\sum_{j=1}^{n} |z_j|^2\right)^{1/2}$. By Lemma 2.2, there exist $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in L$ such that $|x| = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$, $|y| = \left(\sum_{j=1}^{n} |y_j|^2\right)^{1/2}$ and $(|x_j|^p + |y_j|^p)^{1/p} = |z_j|$, for $1 \leqslant j \leqslant n$. Thus, if q is p-convex, then

$$q(z_j) \leqslant \left(q(x_j)^p + q(y_j)^p\right)^{1/p}.$$

The same estimate holds if x, y are disjoint and q is p-Hilbertian with constant 1, because $0 \leqslant |x_j| \wedge |y_j| \leqslant |x| \wedge |y| = 0$ and $|z_j| = |x_j|+|y_j|$ (in fact $z_j = x_j+y_j$). Hence, by the convexity of $F(t^{1/p})$, we have

$$F\left(q(z_j)\right) \leqslant F\left(\left(q(x_j)^p + q(y_j)^p\right)^{1/p}\right)$$
$$\leqslant F\left(\left(|||x|||^p\left(q(x_j)/|||x|||\right)^p + |||y|||^p\left(q(y_j)/|||y|||\right)^p\right)^{1/p}\right)$$
$$\leqslant |||x|||^p F\left(q(x_j)/|||x|||\right) + |||y|||^p F\left(q(y_j)/|||y|||\right).$$

Summing over j, we obtain the estimate

$$\sum F(q(z_j)) \leqslant |||x|||^p F(1) + |||y|||^p F(1) = F(1),$$

which implies that $|||z||| \leqslant 1$ and thus completes the proof.

Remark. It is easy to see that if q' is another seminorm on L such that $Aq(x) \leqslant q'(x) \leqslant Bq(x)$ for $x \in L$, then the corresponding norm $|||\cdot|||'$ satisfies $A|||x||| \leqslant |||x|||' \leqslant B|||x|||$ for $x \in L$. Therefore, if q' is p-Hilbertian with constant C', then so is $|||\cdot|||'$.

4. Factorization and interpolation. Let L be a Banach lattice and let X be a Banach space. Let $T\colon L \to X$ be a linear operator. For $x \in L$ let $\mathfrak{A}(x)$ denote the set of all sequences (x_i) of elements of L with finitely many non-zero terms such that $\sum_i |x_i| \leqslant |x|$ and the x_i's are mutually disjoint. We put for $x \in L$ and $1 \leqslant q \leqslant \infty$

$$\|x\|_{(q)} = \sup\{\|(\|Tx_i\|)\|_{l_q}\colon (x_i) \in \mathfrak{A}(x)\},$$
$$L_{(q)} = \{x \in L\colon \|x\|_{(q)} < \infty\}.$$

LEMMA 4.1. *$\|\cdot\|_{(q)}$ is a lattice seminorm on $L_{(q)}$ and, if $q < \infty$, then*

$$\|x\|_{(q)}^q + \|y\|_{(q)}^q \leqslant \|x+y\|_{(q)}^q,$$

whenever $x, y \in L_{(q)}$ are disjoint.

Proof. The only thing that needs checking is the triangle inequality for $\|\cdot\|_{(q)}$. Let $x, y \in L_{(q)}$ and let $(z_i) \in \mathfrak{A}(x+y)$. It follows from the decomposition lemma for lattices (cf. [31], Proposition II.1.6) that there exist $(x_i) \in \mathfrak{A}(x)$, $(y_i) \in \mathfrak{A}(y)$ such that $x_i + y_i = z_i$ for each i. Thus

$$\|(\|Tz_i\|)\|_{l_q} \leqslant \|(\|Tx_i\| + \|Ty_i\|)\|_{l_q} \leqslant \|(\|Tx_i\|)\|_{l_q} + \|(\|Ty_i\|)\|_{l_q} \leqslant \|x\|_{(q)} + \|y\|_{(q)}.$$

This shows that $\|x+y\|_{(q)} \leqslant \|x\|_{(q)} + \|y\|_{(q)}$ and completes the proof.

THEOREM 4.2. *Let $(L, \|\cdot\|)$ be a Banach lattice, let $(X, \|\cdot\|_X)$ be a Banach space and let $T\colon L \to X$ be an operator such that*

$$\Big(\sum_{i=1}^n \|Tx_i\|_X^q\Big)^{1/q} \leqslant B\Big\|\sum_{i=1}^n x_i\Big\|,$$

whenever $x_1, \ldots, x_n \in L$ are disjoint.

Then L admits an equivalent norm $|||\cdot|||$ such that, if either $1 \leqslant p < q \leqslant r$ or $p = q = 1$, one has, for $x, y \in L$,

$$\|Tx\|_X^q \leqslant C(q,p)^q\big(|||(|x|^p + |y|^p)^{1/p}|||^q - |||y|||^q\big),$$
$$\|Tx\|_X^r \leqslant C(q,r)^r\big(|||(|x|^r + |y|^r)^{1/r}|||^r - |||y|||^r\big),$$

the constants being those of Proposition 3.2.

One can take $|||x||| = (\|x\|^q + \|x\|_{(q)}^q)^{1/q}$ for $x \in L$.

Proof. The assumptions imply that $\|x\|_{(q)} \leqslant B\|x\|$, for $x \in L$, and hence $L_{(q)} = L$. It follows from Lemma 4.1 that Proposition 3.2 may be applied with $|||\cdot|||$ defined as above, $\|\cdot\|$ replaced by $\|\cdot\|_{(q)}$ and p by q. Since $\|Tx\|_X \leqslant \|x\|_{(q)}$, the theorem follows immediately.

COROLLARY 4.3 [24]. *Let T satisfy the assumptions of Theorem 4.2 and let u, v satisfy either $1 \leqslant u < v = q$, or $u = v > q$ or else $u = v = q = 1$. Then*

$$K_{u,v}(T) \leqslant 2^{1/q} C(q,u) B.$$

Proof. Assume, without loss of generality, that $B = 1$. Then the norm $|||\cdot|||$ defined in Theorem 4.2 satisfies $|||x||| \leqslant 2^{1/q}\|x\|$, for $x \in L$.

If $x_1, \ldots, x_n \in L$ and $1 \leqslant i \leqslant n$, then

$$\|Tx_i\|^v \leqslant C(q,u)^v\Big(\Big|\Big|\Big|\Big(\sum_{j \leqslant i} |x_j|^u\Big)^{1/u}\Big|\Big|\Big|^v - \Big|\Big|\Big|\Big(\sum_{j<i} |x_j|^u\Big)^{1/u}\Big|\Big|\Big|^v\Big).$$

Adding up these estimates, we get

$$\sum_{i=1}^n \|Tx_i\|^v \leqslant C(q,u)^v \Big|\Big|\Big|\Big(\sum_{j \leqslant n} |x_j|^u\Big)^{1/u}\Big|\Big|\Big|^v,$$

which combined with the previous estimate for $|||\cdot|||$, proves the corollary.

COROLLARY 4.4 (cf. [9]). *A q-Besselian Banach lattice L admits for each $r > q$ an equivalent r-concave norm.*

Proof. The required r-concave norm $|||\cdot|||$ can be defined by the usual formula

$$|||x||| = \sup\Big\{\Big(\sum \|x_i\|^r\Big)^{1/r}\colon x_i \in L,\ |x| = \Big(\sum |x_i|^r\Big)^{1/r}\Big\}.$$

It follows from Corollary 4.3 that $|||x||| \leqslant 2^{1/q} C(q,r) C\|x\|$ for $x \in L$, if L is q-Besselian with constant C. The proof that $|||\cdot|||$ is a norm mimics that of Lemma 3.5.

The norms $|||\cdot|||_{(q)}$ we have defined above lead to some useful factorizations of the operator $T\colon L \to X$, generalizing those in [10].

The subspace $I = \{x \in L\colon \|x\|_{(q)} = 0\}$ does not depend on q and is a lattice ideal of L. Fix $1 \leqslant q \leqslant \infty$ and let $\Lambda_{(q)}$ denote the completion of the quotient lattice $L_{(q)}/I$ with respect to the norm induced by $\|\cdot\|_{(q)}$. We make $\Lambda_{(q)}$ a Banach lattice by extending the lattice operations and the norm from $L_{(q)}/I$. The norm of $\Lambda_{(q)}$ will still be written as $\|\cdot\|_{(q)}$. There is a natural lattice homomorphism $A\colon L_{(q)} \to \Lambda_{(q)}$ which is the composition of the quotient map of $L_{(q)}$ onto $L_{(q)}/I$ and the embedding into the completion. Since $\|Ax\|_{(q)} \geqslant \|Tx\|_X$ for $x \in L_{(q)}$, there is a unique continuous operator $B\colon \Lambda_{(q)} \to X$ such that $BAx = Tx$ for $x \in L_{(q)}$. Clearly, $\|B\| \leqslant 1$.

If $L_{(q)} = L$, then the pair A, B will be called the *q-factorization* of T.

The obvious estimate $\|x\|_{(\infty)} \leqslant \|T\| \|x\|$ for $x \in L$, implies that $L_{(\infty)} = L$, i.e. the operator T admits ∞-factorization. If $C < \infty$ is the $(q, 1)$-absolutely summing norm of the identity on L, then $\|x\|_{(q)} \leqslant C\|x\|$, hence $L_{(q)} = L$ and L is q-Besselian with constant C. More generally, $L_{(q)} = L$ if and only if T is of type $\leqslant (1, q)$. In fact $K_{1,q}(T)$ is the least constant C such that $\|x\|_{(q)} \leqslant C\|x\|$, for $x \in L$ (cf. B. Maurey [24]).

Remark 4.5. The ∞-factorization $A\colon L \to \Lambda_{(\infty)}$, $B\colon \Lambda_{(\infty)} \to X$ of the operator $T\colon L \to X$ has the following interesting property.

If φ, ψ are lattice seminorms on $\boldsymbol{R}^n$ such that

$$\psi(\|Tx_1\|, \ldots, \|Tx_n\|) \leqslant \|\varphi(x_1, \ldots, x_n)\|,$$

for all $x_1, \ldots, x_n \in L$, then one also has

$$\psi(\|Ax_1\|, \ldots, \|Ax_n\|) \leqslant \|\varphi(x_1, \ldots, x_n)\|.$$

An analogous assertion holds if $x_1, \ldots, x_n$ are supposed to be disjoint.

This fact is an immediate consequence of the definitions, because if $z_i \in L$ and $|z_i| \leqslant |x_i|$ for $1 \leqslant i \leqslant n$, then

$$0 \leqslant \varphi(z_1, \ldots, z_n) \leqslant \varphi(x_1, \ldots, x_n).$$

In particular, letting $\psi(t_1, \ldots, t_n) = \|(t_i)\|_{l_q^n}$ and $\varphi(t_1, \ldots, t_n) = K_{p,q}(T)\|(t_i)\|_{l_p^n}$, we obtain that $K_{p,q}(A) = K_{p,q}(T)$ (the estimate "$\geqslant$" is obvious). This shows that the study of operators of type $\leqslant (p, q)$ can be reduced to the case of lattice homomorphisms.

THEOREM 4.6. *Let X, Y be Banach spaces such that X^* and Y do not contain l_∞^n's uniformly. If a linear operator $T\colon X \to Y$ factors through a Banach lattice, then it factors through a superreflexive Banach lattice.*

Proof. We shall first deduce the result (using the methods of [10]) from the following special case.

(4.6′) *If $R\colon \Lambda \to \Lambda_1$ is a homomorphism of Banach lattices, Λ is p-Hilbertian and Λ_1 is q-Besselian, where $1 < p, q < \infty$, then R factors through a superreflexive Banach lattice Z.*

Our assumptions imply that there exists $q < \infty$ such that each operator mapping a Banach lattice into either X^* or Y admits q-factorization. Therefore, since T factors through a Banach lattice, it also factors through a q-Besselian Banach lattice L, say $T = B \circ A$, where $A\colon X \to L$, $B\colon L \to Y$. By Lemma 2.3, there exists $P\colon L^{**} \to L$ such that $P \circ \varkappa_L$ is the identity on L.

Let $C\colon L^* \to L^*_{(q)}$ and $D\colon L^*_{(q)} \to X^*$ be the q-factorization of A^*. By Corollary 4.3, $L^*_{(q)}$ is of type $\leqslant (1, q)$, hence its dual Λ is p-Hilbertian for $p = q/(q-1)$.

Let $R\colon \Lambda \to \Lambda_1$ and $S\colon \Lambda_1 \to Y$ be the q-factorization of the operator $B \circ P \circ C^*$. It follows from (4.6′) that R factors through a superreflexive Banach lattice, hence so does $S \circ R \circ D^* \circ \varkappa_X = T$.

The proof of (4.6′) requires more work and we postpone it till the end of this section.

Theorem 4.6 yields, by letting T be the identity on X, the following result found in [10] (it has been noticed in [16] that the proof in [10] is wrong).

COROLLARY 4.7. *If X and X^* do not contain l_∞^n's uniformly and X is isomorphic to a complemented subspace of a Banach lattice, then X is isomorphic to a complemented subspace of a superreflexive Banach lattice.*

Remark. The example of a non-reflexive Banach space X which does not contain l_1^n's uniformly, due to R. C. James [14], shows that, if T fails the factorization assumption, then it may even not be weakly compact.

To prove (4.6′) we reduce it first to a problem about interpolation spaces.

Thus we are given Banach lattices $(\Lambda, \|\cdot\|)$ and $(\Lambda_1, \|\cdot\|_1)$ and a homomorphism $R\colon \Lambda \to \Lambda_1$. We may and do assume that $\|R\| = 1$. Moreover, Λ is p_0-Hilbertian for some $p_0 > 1$ (with constant C) and Λ_1 is q-Besselian for some $q < \infty$ (and hence, by Lemma 2.3, Λ_1 is complete). Put

$$\Lambda_0 = \{x \in \Lambda_1\colon |x| \leqslant Ry \text{ for some } y \in \Lambda\},$$

$$\|x\|_0 = \inf\{\|y\|\colon y \in \Lambda,\ |x| \leqslant Ry\},$$

for $x \in \Lambda_0$. It is easy to check that $\|\cdot\|_0$ is a lattice norm on Λ_0, $\|x\|_1 \leqslant \|x\|_0$ for $x \in \Lambda_0$ and $\|Rx\|_0 \leqslant \|x\|$ for $x \in \Lambda$. In particular, $R(\Lambda) \subseteq \Lambda_0$. Since Λ_0 is a lattice ideal in Λ_1 we see that $(\Lambda_0, \|\cdot\|_0)$ is a complete Banach lattice. Let us check that Λ_0 is also p_0-Hilbertian with constant C. Indeed, let $x_1, \ldots, x_n \in \Lambda_0$. Fix $b > 1$ and pick $y_1, \ldots, y_n \in \Lambda$ so that $|x_i| \leqslant Ry_i$ and $\|y_i\| \leqslant b\|x_i\|_0$ for $1 \leqslant i \leqslant n$. If $x = |x_1| \vee \ldots \vee |x_n| \in \Lambda_0$ and $y = |y_1| \vee \ldots \vee |y_n| \in \Lambda$, then

$$\|y\| \leqslant C\Big(\sum \|y_i\|^{p_0}\Big)^{1/p_0} \leqslant Cb\Big(\sum \|x_i\|_0^{p_0}\Big)^{1/p_0}.$$

Since $|x| \leqslant Ry$, the right-hand side gives an estimate for $\|x\|_0$. Letting b tend to 1, we get the desired estimate.

Now let $\vartheta \in (0, 1)$ and $p \geqslant 1$ be fixed. Let us recall a definition of the intermediate space $Z = (\Lambda_0, \Lambda_1)_{\vartheta,p}$ of Lions and Peetre [22]. Fix $\xi > 0$ and let $\eta = \xi - \xi/\vartheta$. Set for $x \in \Lambda_1$ and $n = 0, \pm 1, \pm 2, \ldots$

$$|\!|x|\!|_n = \inf\{\max(e^{n\xi}\|b\|_1, e^{n\eta}\|w\|_0)\colon x = b + w,\ w \in \Lambda_0\},$$

$$|||x||| = \Big(\sum_{n=-\infty}^{\infty} |\!|x|\!|_n^p\Big)^{1/p},$$

$$Z = (\Lambda_0, \Lambda_1)_{\vartheta,p} = \{x \in \Lambda_1\colon |||x||| < \infty\}.$$

The set Z depends only on ϑ and p. Since A_0 is a lattice ideal in A_1, it follows from the decomposition lemma for lattices that the $\|\cdot\|_n$'s are lattice norms on A_1 equivalent to $\|\cdot\|_1$, and hence Z is a lattice ideal in A_1. Since $(Z, |||\cdot|||)$ is a Banach space, we infer that it is a complete Banach lattice.

We should mention that $|||\cdot|||$ is equivalent to (but differs slightly from) the norms used by Lions and Peetre. The reader may consult Beauzamy [1] where this subject is discussed in more detail.

Now we can formulate a technical lemma.

LEMMA 4.8. *Let A_0, A_1, Z be as above. Then*

(i) *the lattice Z is q'-Besselian for some $q' < \infty$,*

(ii) *if $(A_1, \|\cdot\|_1)$ is p_1-Hilbertian for some $p_1 \geqslant 1$ and $p^{-1} = (1-\vartheta)p_0^{-1} + \vartheta p_1^{-1}$, then Z is p-Hilbertian.*

Proof. (ii) B. Beauzamy has proved an analogous result ([1], Proposition 4) that the space $A = (A_0, A_1)_{\vartheta,p}$ interpolating between Banach spaces A_0, A_1 (A_i being of (Rademacher) type p_i, for $i = 0, 1$) is of type p if $p^{-1} = (1-\vartheta)p_0^{-1} + \vartheta p_1^{-1}$. In our case both the hypotheses and the conclusion are weaker (A_0 may contain l_∞^n's uniformly). However, Beauzamy's proof can easily be adapted to the present situation. Changes to be made consist in taking a disjoint sequence $u_1, \ldots, u_n$ in Z, choosing the function $u(t)$ so that $\int_{-\infty}^{\infty} u(t)\,dt = u = \sum_{i=1}^{n} u_i$ and letting $u_i(t)$ be the image of $u(t)$ under the projection onto the support of u_i. (Those projections map A_0 into itself, because A_0 is an ideal.)

Some (mostly notational) simplifications are possible in our case, but the use of the results of [22] and [27] cannot be avoided. (Beauzamy uses another norm on Z whose equivalence with $|||\cdot|||$ is a result of [22].)

(i) It is enough to prove that there exist $\tau \in (0, 1)$ and a positive integer m such that, if $z^{(1)}, \ldots, z^{(m)}$ are disjoint vectors in Z, then

$$|||z^{(j)}||| \leqslant \tau \Big|\Big|\Big| \sum_{j=1}^{m} z^{(j)} \Big|\Big|\Big|$$

for some $j \leqslant m$. A direct proof that this property implies that Z is q'-Besselian for some $q' < \infty$ can be found in [15] or [24].

Since A_1 is q-Besselian, there exists a positive integer k such that, if $b^{(1)}, \ldots, b^{(k)}$ are disjoint vectors in A_1, then

$$\min_{j \leqslant k} \|b^{(j)}\|_1 \leqslant e^{\eta-\xi} \Big\| \sum_{j \leqslant k} b^{(j)} \Big\|_1 .$$

Let $z^{(1)}, \ldots, z^{(2k)}$ be disjoint vectors in Z and let

$$\varrho = 1 - \tfrac{1}{2}(1 - e^{\eta p/2})^2 .$$

We shall prove that

$$\sum_{j \leqslant 2k} |||z^{(j)}|||^p \leqslant 2k\varrho \Big|\Big|\Big| \sum_{j \leqslant 2k} z^{(j)} \Big|\Big|\Big|^p .$$

This will show that the numbers $m = 2k$ and $\tau = \varrho^{1/p}$ satisfy the property formulated above and thus it will complete the proof of the lemma.

Fix $\beta > 1$ and write $z = \sum_{j \leqslant 2k} z^{(j)}$. For each integer n fix a decomposition $z = b_n + w_n$ so that the numbers

$$a_n = \max(e^{n\xi}\|b_n\|_1, e^{n\eta}\|w_n\|_0)$$

satisfy

$$\sum_{n=-\infty}^{\infty} a_n^p \leqslant \beta |||z|||^p .$$

Put

$$S = \{n \in \mathbf{Z}: a_{n-1} > e^{-\eta/2} a_n\} .$$

Let P_j, $1 \leqslant j \leqslant 2k$, denote the projection (in A_1) onto the support of $z^{(j)}$ (here we use the σ-completeness of A_1). Write

$$w_n^{(j)} = P_j w_n, \quad b_n^{(j)} = P_j b_n .$$

Clearly, $w_n^{(j)} \in A_0$ (because A_0 is a lattice ideal) and

$$z^{(j)} = P_j z = P_j(b_n + w_n) = b_n^{(j)} + w_n^{(j)} .$$

Observe that, if

$$(**) \qquad \|b_{n-1}^{(j)}\|_1 \leqslant e^{\eta-\xi} \|b_{n-1}\|_1 ,$$

then one has

$$\begin{aligned} \|z^{(j)}\|_n &\leqslant \max(e^{n\xi}\|b_{n-1}^{(j)}\|_1, e^{n\eta}\|w_{n-1}^{(j)}\|_0) \\ &\leqslant e^{\eta} \max(e^{(n-1)\xi}\|b_{n-1}\|_1, e^{(n-1)\eta}\|w_{n-1}\|_0) \\ &= e^{\eta} a_{n-1} . \end{aligned}$$

Since $\sum_{j \leqslant 2k} b_{n-1}^{(j)} = b_n$, the definition of k yields that $(**)$ is satisfied for at least k values of j. Consequently, if $n \in \mathbf{Z} \setminus S$, then

$$\begin{aligned} \sum_{j \leqslant 2k} \|z^{(j)}\|_n^p &\leqslant k\|z\|_n^p + k(e^{\eta} a_{n-1})^p \\ &< k a_n^p + k(e^{\eta/2} a_n)^p = k(1 + e^{\eta p/2}) a_n^p . \end{aligned}$$

For $n \in S$ we have $\sum_{j \leqslant 2k} \|z^{(j)}\|_n^p \leqslant 2k a_n^p$, hence writing

$$A = \sum_{n \in S} a_n^p, \quad B = \sum_{n \in \mathbf{Z} \setminus S} a_n^p ,$$

we have

$$\sum_{j\leqslant 2k}|||z^{(j)}|||^p = \sum_{j\leqslant 2k}\sum_{n}\|z^{(j)}\|_n^p$$
$$= \sum_{n\in S}\sum_{j\leqslant 2k}\|z^{(j)}\|_n^p + \sum_{n\notin S}\sum_{j\leqslant 2k}\|z^{(j)}\|_n^p$$
$$\leqslant 2kA + k(1+e^{\eta p/2})B$$
$$= 2k(A+B)\big(1-\tfrac{1}{2}(1-e^{\eta p/2})B/(A+B)\big).$$

If we know that

$$(\overset{*}{**})\qquad B \geqslant (1-e^{\eta p/2})(A+B),$$

then the right-hand side can be estimated by $2k\beta|||z|||^p\varrho$ and $(*)$ follows by letting β tend to 1.

To prove $(\overset{*}{**})$ write for $n\in \boldsymbol{Z}$

$$\psi(n) = \sup\{m\in \boldsymbol{Z}\setminus S\colon\ m\leqslant n\}$$

(the latter set cannot be empty). Observe that, if $i = n-\psi(n)$, then

$$a_n \leqslant e^{i\eta/2}a_{\psi(n)}.$$

Using this estimate, we get easily

$$A+B = \sum_{n\in\boldsymbol{Z}}a_n^p = \sum_{m\in\psi(S)}\ \sum_{n\in\psi^{-1}(\{m\})}a_n^p$$
$$\leqslant \sum_{m\in\psi(S)}a_m^p\sum_{i=0}^{\infty}e^{i\eta p/2} = B(1-e^{\eta p/2})^{-1},$$

which proves $(\overset{*}{**})$ and completes the proof of the lemma.

Now we can complete the proof of Theorem 4.6 by establishing Proposition (4.6′). In fact, we shall obtain a slightly stronger result.

Corollary 4.9. *If $1<p'<p$, then the lattice Z in (4.6′) can be chosen to be p'-Hilbertian.*

Proof. The homomorphism R of (4.6′) factors through the embedding $\Lambda_0\to\Lambda_1$, hence it factors through the interpolating space $Z=(\Lambda_0,\Lambda_1)_{\vartheta,p'}$, where $\vartheta=(p-p')/p'(p-1)$. Since the lattice Λ in (4.6′), and hence also Λ_0 is p-Hilbertian, and obviously, Λ_1 is 1-Hilbertian, it follows from Lemma 4.8 (ii) that Z is p'-Hilbertian. By part (i) of that lemma, Z is also q'-Besselian for some $q'<\infty$. Therefore Z is superreflexive (cf. [15], or Proposition 24 and Corollary 28 in [6]). (This follows also from Corollaries 4.4 and 5.2 in the present paper.)

5. Some applications. There are many instances where the methods developed in previous sections yield equivalent renorming of a Banach lattice L, after which some geometric properties of L become obvious consequences of the corresponding special properties of the new norm. In the case of cotype this is done in the following result.

Theorem 5.1. *Let $(L,\|\cdot\|)$ be a Banach lattice. Assume that L is of cotype f. Then there exists an equivalent lattice norm $|||\cdot|||$ on L and a constant C such that, if $x,y\in L$ satisfy $|||(|x|^2+|y|^2)^{1/2}|||\leqslant 1$, then*

$$f(|||y|||)\leqslant C(1-|||x|||).$$

Moreover, if L admits an equivalent p-convex norm, $1\leqslant p\leqslant 2$, then $|||\cdot|||$ can be chosen to be p-convex.

Proof. We may assume that $\|\cdot\|$ is p-convex, $1\leqslant p\leqslant 2$. By Theorem 1.8 we may also assume that f satisfies the conditions (ii), (iii), (iv), (v) of that theorem. Therefore the norm $|||\cdot|||$ can be defined by the formula of Section 3, in which q is to be replaced by $\|\cdot\|$, (the existence of the constant C_1 follows from Lemma 2.6). It follows from Lemmas 3.3, 3.4 and 3.5 that $|||\cdot|||$ satisfies the required properties.

Corollary 5.2. *Let X be a superreflexive Banach space with* l.u.st. *Assume that X is of cotype f. Then X admits an equivalent norm $|||\cdot|||$ such that for some $c'>0$ the modulus of convexity of $(X,|||\cdot|||)$ satisfies the estimate*

$$\delta(\varepsilon)\geqslant c'f(\varepsilon),$$

for small ε.

Proof. By Corollary 2.2 in [10] there exists a Banach lattice L and a projection P onto a subspace Y such that L is finitely representable in Y and Y is isomorphic to X. This implies that L is superreflexive and of cotype f. Consequently, L admits an equivalent norm that is p-convex for some $p>1$ (cf. [9]). Indeed, L^* is superreflexive too, hence it is q-Besselian for some $q<\infty$. By Corollary 4.4, L^* admits an equivalent r-concave norm if $r>q$. The dual norm on L is p'-convex, where $p'=r/(r-1)$. Let $p=\min(2,p')$ and let $|||\cdot|||$ be the p-convex norm on L yielded by Theorem 5.1. It follows from Proposition 3.1 that the modulus of convexity of $(L,|||\cdot|||)$ satisfies the required estimate. X being isomorphic to a subspace of L, this completes the proof.

Corollary 5.3. *Let $(X,\|\cdot\|)$ be a uniformly convex Banach space with* l.u.st. *Then there exist an equivalent norm on X, a supermultiplicative function F and positive constants c_1,c_2 such that the corresponding moduli of convexity satisfy*

$$c_1\delta_{(X,\|\cdot\|)}(\varepsilon)\leqslant F(\varepsilon)\leqslant c_2\delta_{(X,|||\cdot|||)}(\varepsilon),$$

for $0\leqslant\varepsilon\leqslant 1$.

Proof. By the result of [11], X is of cotype $\delta_{(X,\|\cdot\|)}$. Therefore the corollary follows from Theorem 1.8 and Corollary 5.2.

It is not known whether or not a (superreflexive) complemented subspace of a Banach lattice must have l.u.st. We could not prove the exact analogue of Corollary 5.2 for those spaces. Still the following, slightly weaker result holds.

PROPOSITION 5.4. *Let X be a superreflexive Banach space isomorphic to a complemented subspace of a Banach lattice. Put*

$$q^X = \inf\{q < \infty: X \text{ is of cotype } t^q\}.$$

Then X admits an equivalent norm whose modulus of convexity satisfies

$$\lim_{\varepsilon\to 0+} \delta(\varepsilon)/\varepsilon^q = \infty,$$

for each $q > q^X$.

Proof. Fix numbers $q > q' > q^X$ and let $p = q/(q-1)$, $p' = q'/(q'-1)$. By Corollary 4.7, there exist a superreflexive lattice L and operators $U: X\to L$, $V: L\to X$ such that $V\circ U$ is the identity on X. Let $A: L\to L_{(q')}$, $B: L_{(q')}\to X$ be the q'-factorization of V. It follows from Lemma 2.4 that the adjoint $A^*: (L_{(q')})^*\to L^*$ of the homomorphism A is a homomorphism with a trivial kernel. Since $(L_{(q')})^*$ is p'-Hilbertian and $1 < p < p'$, we may apply Corollary 4.9 and obtain that A^* factors through a p-Hilbertian superreflexive Banach lattice Z. Hence X^* is isomorphic to a complemented subspace of Z, whence X is isomorphic to a complemented subspace of the q-Besselian superreflexive Banach lattice Z^*. Since $q > q^X \geqslant 2$, it follows from Proposition 3.1 that Z^* is of type $\leqslant (2, q)$, hence it is of cotype t^q, by Lemma 2.6. By Corollary 5.2, Z^* (and hence also X) admits an equivalent norm whose modulus of convexity satisfies $\delta(\varepsilon) \geqslant c_q\varepsilon^q$, where $c_q > 0$. Since q may be an arbitrary number greater than q^X, we can complete the proof by using Proposition 18 from [6].

Remark. Let X be a Banach space of cotype f such that $X = P(L)$ where L is a superreflexive Banach lattice and $P: L\to L$ is a projection. A natural approach to prove for X an analogue of Corollary 5.2 would be to use the norm constructed in Section 3 with $q(x) = \|Px\|$ for $x\in L$. This does yield a factorization of P through a lattice Λ of cotype f.

Indeed, if $x_1, \ldots, x_n \in L$, then

$$\int_0^1 \Big\| \sum_i Px_i r_i(t) \Big\| dt \leqslant \|P\| \int_0^1 \Big\| \sum_i x_i r_i(t) \Big\| dt \leqslant B\|P\| \Big\| \Big(\sum_i |x_i|^2\Big)^{1/2} \Big\|,$$

where B depends only on L, by the result of Maurey used in Lemma 2.6. Hence we are in a position to apply Lemmas 3.3 and 3.4. There is, however, no reason for the new norm on L to be p-Hilbertian for some $p > 1$. Let us describe an example.

Let $p > 1$ be fixed. Given a positive integer n, let $L_{(n)}$ be the subspace of $L_p([0,1])$ spanned by $\{e_1^{(n)}, \ldots, e_{2^n}^{(n)}\}$, where $e_k^{(n)}$ denotes the characteristic function of the interval $[(k-1)2^{-n}, k2^{-n})$ for $1 \leqslant k \leqslant 2^n$, and let $X_{(n)}$ be the subspace of $L_{(n)}$ spanned by the first n Rademacher functions. Let $P_{(n)}$ denote the orthogonal projection of $L_{(n)}$ onto $X_{(n)}$. Consider the norm $|||\cdot||| = |||\cdot|||_{(n)}$ on $L_{(n)}$ defined by the formula of Section 3 with $q(x) = \|P_{(n)}x\|$ for $x \in L_{(n)}$ and $F(t) = t^2$. It is easy to check that

$$\Big|\Big|\Big| \sum_{j=1}^{2^n} e_j^{(n)} \Big|\Big|\Big| = \Big|\Big|\Big| \sum_{j=1}^{2^{n-1}} e_j^{(n)} - \sum_{j=2^{n-1}+1}^{2^n} e_j^{(n)} \Big|\Big|\Big| = |||r_1||| \geqslant \|Pr_1\| = 1,$$

$$|||e_j^{(n)}||| = \|P_{(n)}e_j^{(n)}\| = \Big\| 2^{-n} \sum_{i=1}^n \pm r_i \Big\|_{L_p} \leqslant C_p n^{1/2} 2^{-n},$$

for $1 \leqslant j \leqslant 2^n$, where C_p depends only on p.

Now let L (resp. X) denote the l_2-sum of the sequence $(L_{(n)})$ (resp. $(X_{(n)})$) and let $P((x_n)) = (P_{(n)}(x_n))$ for $(x_n) \in L$. It follows from Khintchine's inequality that X is isomorphic to a Hilbert space, hence one can take $F(t) = t^2$ to define the norm $|||\cdot|||$ on L and it follows from our earlier estimates that the unconditional basis

$$\{e_j^{(n)}: 1 \leqslant j \leqslant 2^n, \ n = 1, 2, \ldots\}$$

is not p'-Hilbertian with respect to the norm $|||\cdot|||$ for any $p' > 1$.

Another application, mentioned already in [7] is concerned with the properties of the spaces constructed in [2]. W. J. Davis has proved, extending earlier results of J. Lindenstrauss [20] and A. Szankowski [32], that a Banach space X with an unconditional basis is isomorphic to a complemented subspace of a space Y which has a symmetric basis and is uniformly convex if X is. We shall prove that his construction leads to no loss of convexity properties (cf. [6], Remark on p. 148).

COROLLARY 5.5. *The space Y constructed in* [2] *has the same cotype as X. If δ_X is the modulus of convexity corresponding to an equivalent norm on X, then Y can be given an equivalent norm such that*

$$\delta_Y(\varepsilon) \geqslant c\delta_X(\varepsilon),$$

for $0 \leqslant \varepsilon \leqslant 2$ and some positive constant c.

Proof. The space Y is a sublattice of $X(G)$, where G is the l_2-sum of a sequence of spaces (W_{m_k}) which are isometric to quotients of the space $l_2 \oplus l_{3/2}$ and have monotonely unconditional bases. It follows (cf. e.g. [4], Added in proof) that G has modulus of convexity $\geqslant ct^2$ and hence, by the result of [11] and Lemma 2.6, we have $G(l_2) \subseteq l_2(G)$. Therefore Lemma 2.5 yields that $X(l_2) \subseteq E(X)$ implies $(X(G))(l_2) \subseteq E(X(G))$, i.e., if X is of cotype E (in the sense defined in Section 2), then so is $X(G)$. Since Y is a subspace of $X(G)$, this proves the first part of the corollary.

To prove the second part we may assume that $\delta_X(\varepsilon) > 0$ for some $\varepsilon \in (0, 2)$. Since, by [11], X is of cotype δ_X, Theorem 1.8 yields an Orlicz function F such that $F(\varepsilon) \geqslant \delta_X(\varepsilon)$ for $0 \leqslant \varepsilon \leqslant 2$ and X is of cotype F. By the first part of the proof, $X(G)$ is also of cotype F.

The assumption $\delta_X(\varepsilon) > 0$ implies that X is superreflexive, and hence it admits an equivalent p-convex norm for some $p \in (1, 3/2)$. The same being true about G, the space $X(G)$ also admits a p-convex norm. Therefore, by Proposition 3.1, the norm on $X(G)$ constructed in Theorem 5.1 has modulus of convexity $\geqslant cF(\varepsilon)$. This yields the required renorming of the subspace Y.

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INSTYTUT MATEMATYCZNY
POLSKA AKADEMIA NAUK
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

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