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Construction of invariant sets for Anosov diffeomorphisms and hyperbolic attractors

by

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Abstract. For a class of hyperbolic attractors, in particular for any hyperbolic toral automorphism, we construct an invariant subset of an arbitrary (reasonable) topological dimension.

§ 1. Introduction. Statement of results. Main idea. In the paper we prove the following

THEOREM A. *For any Anosov diffeomorphism f of an n-dimensional torus T^n and for any integer k such that $0 \leqslant k \leqslant n$ and $k \neq n-1$ there exists a compact f-invariant subset N^k of topological dimension k.*

THEOREM A′. *For any Anosov diffeomorphism f of an n-dimenisonal manifold M^n and for any integer k such that $0 \leqslant k \leqslant n-2$ there exists a compact f-invariant subset N^k such that $k \leqslant \dim N^k \leqslant \min\big(k+s-1, k+u-1, [(k+n-1)/2]\big)$ where s and u denote dimensions of stable and unstable manifolds, respectively.*

Theorem A answers positively the question of S. Smale (see [6]). It is known [6] that dimension $n-1$ is not allowed for any compact invariant subset. The subsets N^k should be quite complicated, because it is known, for example, that for any hyperbolic toral automorphism f without a proper invariant toral subgroup no compact proper invariant subset (except fixed points) can be a connected C^1-submanifold [8], be connected and locally connected provided $s = 1$ [6], or contain a C^2-arc [2] or even a nonconstant differentiable arc provided there is no proper toral subgroup invariant under a power of f [9].

In order to construct our invariant subsets we improve here the idea of S. G. Hancock [4], [5]. Hancock has constructed invariant subsets of dimension between 1 and $n-2$ but has not computed dimension exactly.

Theorem A and Theorem A′ will be proved by using invariant sets constructed in the more general situation:

THEOREM B. *Let $\Lambda \subset M^n$ be a compact hyperbolic attractor for a diffeomorphism f such that property* (*) *(which will be defined below) is satisfied. Then for every integer k such that $0 \leqslant k < u$ there exists a compact f-invariant subset N^k of Λ of topological dimension k.*

Without assuming property (*), we have

THEOREM B′. *Let $\Lambda \subset M^n$ be a compact hyperbolic attractor for a diffeomorphism f. Then for every integer k such that $0 \leqslant k \leqslant u$ there exists a compact f-invariant subset N^k of Λ such that $k \leqslant \dim N^k \leqslant k + \sup_{p\in\Lambda} (\dim (W^s_{p,\mathrm{loc}} \cap \Lambda))$.*

COROLLARY. (a) *If $\Lambda \subset M^n$ is a compact expanding hyperbolic attractor, then for every integer k such that $0 \leqslant k \leqslant u$ there exists a compact f-invariant subset N^k of Λ of topological dimension k.*

(b) *If $\Lambda \subset T^n$ is a standard attractor for a* DA-*diffeomorphism f* (see Definition 6 below or [11] for the description), *then for every k such that $0 \leqslant k \leqslant n-1$ there exists a compact f-invariant subset N^k of Λ of topological dimension k.*

(If $s = 1$, Λ is an expanding hyperbolic attractor, and then we have the situation from (a). Observe that if $s > 1$, then Λ is not hyperbolic. I owe the last remark to Anthony Manning.)

Remark. (a) In each theorem stated above the sets N^k can be constructed in such a way that $N^k \subset N^l$ if $k \leqslant l$.

(b) In each theorem stated above for every reasonable k infinitely many different N^k's can be constructed.

The technique used in this paper allows us to answer the question of Hancock [4], namely the following theorem holds:

THEOREM C. *Let $f\colon T^n \to T^n$ be an Anosov diffeomorphism. Then for $k < \min(s, u)$,*

$$\Big\{g\colon D^k \to T^n\colon \dim\Big(\mathrm{cl} \bigcup_{m=-\infty}^{+\infty} f^m\big(g(D^k)\big)\Big) = k\Big\}$$

is dense in $C(D^k, T^n)$.

(Here D^k denotes a compact k-dimensional disc, and $C(D^k, T^n)$ denotes the space of all continuous functions from D^k into T^n with topology of uniform convergence.)

In order to define property (*) we introduce some notation. We shall also explain some terms used above.

NOTATION AND DEFINITIONS (see [7], [11], [12]).

1. A compact set $\Lambda \subset M^n$ invariant under a diffeomorphism f defined on a neighbourhood of Λ is called a *hyperbolic set* if there exists a splitting $TM = E^s \oplus E^u$ into subbundles of dimensions s and u respectively, invariant under Df and such that for some constants a, μ, $a > 0$, $0 < \mu < 1$, for every integer $n \geqslant 0$:

$$\|Df^n(v)\| \leqslant a\mu^n\|v\| \quad \text{for} \quad v \in E^s,$$
$$\|Df^{-n}(v)\| \leqslant a\mu^n\|v\| \quad \text{for} \quad v \in E^u.$$

2. If $\Lambda = M^n$, f is called an *Anosov diffeomorphism*.

3. By W^s_x, W^u_x ($x \in \Lambda$) we denote global stable and unstable manifolds, respectively.

4. A hyperbolic set Λ is called an *attractor* if there exists a neighbourhood U of Λ such that $f(\mathrm{cl}\, U) \subset U$ and $\Lambda = \bigcap_{m\geqslant 0} f^m(U)$. Notice that in this situation, for every $x \in \Lambda$, $W^u_x \subset \Lambda$.

5. A hyperbolic attractor is called an *expanding hyperbolic attractor* if $\dim W^u_x = \dim \Lambda$ for every $x \in \Lambda$.

6. Let $f\colon T^m \to T^m$ be a hyperbolic toral (algebraic) automorphism. A DA-*diffeomorphism* f' is a diffeomorphism obtained from f by a perturbation along W^s in a small neighbourhood U of a finite number of periodic orbits such that these orbits become sources. A *standard attractor* Λ from Corollary (b) is defined as $\bigcap_{m\geqslant 0} f'^m(T^n \setminus U)$.

7. For any $x \in \Lambda$ we denote by $k_x\colon \mathbf{R}^u \to M$ an immersion such that $k_x(\mathbf{R}^u) = W^u_x$, $k_x(0) = x$. We shall use also the notion $k^s_x\colon \mathbf{R}^s \to M$ for an embedding such that $k^s_x(\mathbf{R}^s) = W^s_{x,\mathrm{loc}}$, $k^s_x(0) = x$. (Such k_x, k^s_x exist, see [7], [11].)

8. A Riemannian metric on M induces Riemannian metrics on W^s_x and W^u_x which induce metrics ϱ^s and ϱ^u along W^s_x and W^u_x, respectively.

9. $W^{s(u)}_{x,a} = \{y \in W^{s(u)}_x\colon \varrho^{s(u)}(x, y) < a\}$. Sometimes we use the notion of local stable (unstable) manifolds $W^{s(u)}_{x,\mathrm{loc}}$.

10. We shall use the following definition of *topological dimension* of a separable metric space (X, ϱ) (see [1]):

(i) $\dim X = -1$ if and only if $X = \emptyset$;

(ii) $\dim X \leqslant n$ if for every $x \in X$ and every $\varepsilon > 0$ there exists a neighbourhood $U \subset X$ of x such that $\dim \mathrm{Fr}(U) \leqslant n-1$ and $\mathrm{diam}\, U \leqslant \varepsilon$;

(iii) $\dim X = n$ if $\dim X \leqslant n$ and the inequality $\dim X \leqslant n-1$ does not hold;

(iv) $\dim X = \infty$ if the inequality $\dim X \leqslant n$ does not hold for any n.

11. By the order of a family $\mathscr{A}$ ($\mathrm{ord}\, \mathscr{A}$) of subsets of X we mean the largest integer n such that the family $\mathscr{A}$ contains n sets with a non-empty intersection, or ∞ if no such number exists. We shall use also the notion of *diameter* of $\mathscr{A}$, $\mathrm{diam}\, \mathscr{A} = \sup\{\mathrm{diameter}\, A\colon A \in \mathscr{A}\}$.

The following fact will be useful, see [1], p. 492 (it is connected with the so called *covering dimension*):

$\dim X \leqslant n$ *iff there exists a sequence* $\mathscr{W}_1, \mathscr{W}_2, \ldots$ *of open covers of the space* X *such that* $\operatorname{ord}\mathscr{W}_i \leqslant n+1$, $\operatorname{diam}\mathscr{W}_i \leqslant 1/i$ *and* $\mathscr{W}_{i+1}$ *is a refinement of* $\mathscr{W}_i$.

Definition of property (*). We say that a hyperbolic attractor $\Lambda \subset M$ *satisfies property* (*) if for a point $p \in \Lambda$ the following conditions are satisfied:

(a) For every $x \in \Lambda$, W^u_x is dense in Λ;

(b) There exists a local homeomorphism $h\colon \Lambda_p \times \boldsymbol{R}^u \to \Lambda$ (we write $\Lambda_p = (k^s_p)^{-1}(\Lambda) \subset \boldsymbol{R}^s$) which satisfies the following conditions: $h(0,0) = p$, $h|\Lambda_p \times \{0\} = k^s_p|\Lambda_p$, $h|\{0\}\times \boldsymbol{R}^u = k_p$; for every $q \in \Lambda_p$ $h(\{q\}\times \boldsymbol{R}^u) = W^u_{h(q,0)}$; for every $q \in \boldsymbol{R}^u$ $h(\Lambda_p \times \{q\}) \subset W^s_{h(0,q)}$; for every $l_1 > 0$ there exists an $l_2 > 0$ such that if $q \in B(0, l_1) \subset \boldsymbol{R}^u$, then $\operatorname{diam}_{\varrho^s} h(\Lambda_p \times \{q\}) < l_2$.

(c) There are compact sets $Q_1, Q_2, \ldots, Q_J \subset W^s_{p,\mathrm{loc}}$ such that $\bigcup_{j=1}^{J} Q_j$ disconnects $W^s_{p,\mathrm{loc}}$ and, for every $j = 1, \ldots, J$ and $q \in \Lambda$, the intersection $W^u_q \cap Q_j$ consists of at most one point.

Conditions (b) and (c) are considered only if $s > 1$ and $\dim \Lambda > u$.

It would be a good thing to check property (*) for Anosov diffeomorphisms of infranilmanifolds (in this paper we check it only for hyperbolic toral automorphisms). This would give the proof that $\dim N^k = k$ in this situation. It is also interesting to know whether N^k can be locally maximal (¹), whether the equality $\Omega(f|N^k) = N^k$ can hold or what periodic points in N^k can occur.

In order to explain the main idea of the proofs in this paper without getting into technical difficulties we give first the proof of Theorem A assuming f to be a hyperbolic toral automorphism with $s = \dim W^s_x = 1$.

Proof. Denote by $\pi\colon \boldsymbol{R}^n \to \boldsymbol{R}^n/Z^n$ the standard covering projection. We may lift f to $\tilde f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$, $f(0) = 0$. Denote by E^u the expanding eigenspace covering W^u_0. One may assume that the orthogonal projection $P\colon E^u \to \boldsymbol{R}^{n-1} = \{x \in \boldsymbol{R}^n\colon x_n = 0\}$ is an isomorphism. Fix k $(1 \leqslant k \leqslant n-2)$. One may consider $\boldsymbol{R}^{n-1}$ as a union of $(n-1)$-dimensional cubes

$$\{x = (x_1, \ldots, x_{n-1})\colon m_1 \leqslant x_1 \leqslant m_1+1, \ldots, m_{n-1} \leqslant x_{n-1} \leqslant m_{n-1}+1;\ m_i \in Z\}$$

with edges of length 1. Denote by $\mathscr{K}$ the union of $(n-k-2)$-dimensional skeletons of our cubes. Let D be a k-dimensional disc embedded by g into E^u. There exists a continuous mapping $g_0\colon D \to E^u$ such that $P\circ g_0$ is $C\cdot\varepsilon$ close to $P\circ g$ and $P\circ g_0(D)$ is disjoint with $B(\mathscr{K}, \varepsilon)$, where

$$B(\mathscr{K}, \varepsilon) = \{x \in \boldsymbol{R}^{n-1}\colon \varrho(\mathscr{K}, x) < \varepsilon\}$$

(¹) In the case of a hyperbolic toral automorphism if there is no proper invariant toral subgroup, the answer is negative. It follows from the paper by R. Mañé [8].

and C is a constant coefficient. Let $\lambda > 1$, $a > 0$ satisfy the condition $a\|\tilde f^n(v)\| \geqslant \lambda^n\|v\|$, $v \in E^u$. There exists a positive integer q which satisfies the inequality

$$(1) \qquad 1 - \|P^{-1}\|\cdot a\cdot C\cdot \sum_{i=1}^{\infty} (1/\lambda^q)^i = \delta > 0.$$

Assume that a continuous mapping $g_i\colon D \to E^u$ such that $P\circ\tilde f^{q\cdot i}\circ g_i(D) \cap B(\mathscr{K}, \varepsilon) = \emptyset$ is defined. We define $g_{i+1} = \tilde f^{-q(i+1)}\circ P^{-1}\circ h$, where h is an arbitrary continuous mapping $h\colon D \to \boldsymbol{R}^{n-1}$ such that $h(D) \cap B(\mathscr{K}, \varepsilon) = \emptyset$ and h is $C\cdot\varepsilon$ close to $P\circ\tilde f^{q(i+1)}\circ g_i$.

By (1) there exists a continuous mapping $G = \lim\limits_{i\to\infty} g_i$ and for every $i = 0, 1, \ldots$ $\tilde f^{q\cdot i}G(D) \cap P^{-1}B(\mathscr{K}, \varepsilon\cdot\delta) = \emptyset$. If ε is sufficiently small, then $G(D)$ (and so $\operatorname{cl}\bigcup_{i\geqslant 0} \pi(\tilde f^{q\cdot i}G(D))$) is at least k-dimensional. (This follows easily from the definition of covering dimension.) Denote $L = \bigcup_{i\geqslant 0} \tilde f^{-q\cdot i}P^{-1}B(\mathscr{K}, \varepsilon\cdot\delta)$. Observe that

$$(2) \qquad \dim(E^u \setminus L) \leqslant k.$$

Indeed, closed $(n-1)$-dimensional cubes in $\boldsymbol{R}^{n-1}$, with edges of length 2^{n-2}, of the form

$$2^{n-2}\cdot\bigl(\{x = (x_1, \ldots, x_{n-1})\colon p_i \leqslant x_i \leqslant p_i+1,\ i = 1, \ldots, n-1\} + y_{(p_i)_i}\bigr)$$

where p_i are integers and $y_{(p_i)_{i=1}^{n-1}}$ is the vector $(y_1, \ldots, y_{n-1})$, $y_i = \sum_{j=i+1}^{n-1} (1/2)^{j-i}p_j$, intersected with $\boldsymbol{R}^{n-1}\setminus P(L)$ and slightly thickened give us an open cover $\mathscr{A}$ of $\boldsymbol{R}^{n-1}\setminus P(L)$ of order $\leqslant k+1$ with a nonzero Lebesgue number. Covers $f^{-q\cdot i}(P^{-1}\mathscr{A})$ (for $i \geqslant 0$) also have orders $\leqslant k+1$ and their diameters converge to zero. Now (2) follows from the fact 11.

Denote

$$D' = \bigcup_{i\geqslant 0} f^{q\cdot i}\pi G(D).$$

$D' \cap \pi\bigl(B(\mathscr{K}, \varepsilon\cdot\delta)\times \boldsymbol{R}\bigr) = \emptyset$; hence, for sufficiently small $\xi > 0$, $\zeta > 0$, $D' \cap \bigcup \{W^s_{x,\zeta}\colon\ x \in \pi P^{-1}B(\mathscr{K}, \xi)\} = \emptyset$. This implies $D' \cap \bigcup \{W^s_{x,\zeta}\colon x \in \bigcup_{i\geqslant 0} f^{-q\cdot i}\bigl(\pi P^{-1}B(\mathscr{K}, \xi)\bigr)\} = \emptyset$; $\operatorname{cl} D' = D' \cup \omega$, where

$$\omega = \bigl\{x \in T^n\colon \text{there exists a sequence of points } (x_t),\ x_t \in f^{q\cdot m_t}\bigl(\pi G(D)\bigr), \text{ such that } m_t \xrightarrow{t\to\infty} \infty \text{ and } x_t \xrightarrow{t\to\infty} x\bigr\}.$$

By the construction the set ω is disjoint from a neighbourhood of $\pi(0) \in T^n$; hence ω is disjoint from $W^u_{\pi(0)}$. Since $W^u_{\pi(0)}$ is dense in T^n, we know that ω

is contained locally in the product of 0-dimensional (along W^s) and k-dimensional (along W^u) sets. Thus $\dim\omega \leqslant k$. The dimension of a union of a countable family of compact sets in $\boldsymbol{R}^n$ is equal to the maximum of their dimensions, and so the dimension of $\mathrm{cl}\,D'$ (and hence the dimension of the compact f-invariant set $\bigcup_{i=0}^{q-1} f^i(\mathrm{cl}\,D')\cup\bigcup_{i\geqslant 0} f^{-i}\pi(G(D))$) is equal to k.

§ 2. Proofs. If one deals with a mapping of an arbitrary manifold, one should replace the skeleton $\mathscr{K}$ (see Proof in § 1) connected with the global structure of the torus by one constructed in a local manner. The Topological Lemma which follows will be used in future for the estimation of the topological dimension of $W^u\setminus\bigcup_{i\geqslant 0} f^{-i}(\mathscr{K})$. But first let us introduce some

ADDITIONAL NOTATION.

12. If $x\in\Lambda$ where Λ is a hyperbolic attractor, then for $A\subset W^s_{x,\mathrm{loc}}\cap\Lambda$, $B\subset W^u_{x,\mathrm{loc}}\cap\Lambda$ we denote

$$A\times_{\mathrm{rect}}B = \{y\in\Lambda:\ W^u_{y,\mathrm{loc}}\cap A\neq\emptyset\ \&\ W^s_{y,\mathrm{loc}}\cap B\neq\emptyset\}$$

and call it a *rectangle product*.

For a small number $\mathscr{R}>0$ there exists a continuous strictly increasing function L defined on the interval $(0, 2\mathscr{R})$ such that $\lim_{t\to 0} L(t) = 0$ and

$$L(t) > \sup\{\varrho^u(W^s_{y,\mathrm{loc}}\cap W^u_{q,\mathrm{loc}},\ W^s_{z,\mathrm{loc}}\cap W^u_{q,\mathrm{loc}}):\ y, z\in W^u_{x,\mathscr{R}},\ \varrho^u(y, z)\leqslant t,\ q\in W^s_{x,\mathscr{R}},\ x, y, z, q\in\Lambda\}.$$

13. Let $\mathscr{A} = (A_t)_{t\in T}$ be a cover of a metric space (X, ϱ). We denote

$$C_k\mathscr{A} = \inf\{a:\ a = \max_{1\leqslant i,j\leqslant k}\varrho(x_i, x_j),\quad x_i\in A_{t_i} \text{ for } i = 1,\ldots,k\ \&\ t_i\neq t_j \text{ if } i\neq j\},\qquad k>1.$$

Observe that $C_k\mathscr{A}>0$ implies $\mathrm{ord}\,\mathscr{A}\leqslant k-1$.

14. Let $Y\subset X$. For $\varepsilon>0$ we denote $B_\varrho(Y,\varepsilon) = \{x\in X:\ \varrho(x, Y)<\varepsilon\}$, $B_\varrho(Y, -\varepsilon) = X\setminus B_\varrho(X\setminus Y, \varepsilon)$, if $Y=\emptyset$ we assume that $B_\varrho(Y,\varepsilon) = \emptyset$. Sometimes we omit the index ϱ.

15. We define the property $P(k)$ (or $P(k, (d_i)_{i=1}^I, (c_i)_{i=1}^I)$) for a metric space X as follows:

There exist a finite cover of X by nonempty sets U_i, $i = 1,\ldots,I$ and a family $\mathscr{K}_i = \{K_{i,t}\}_{t\in T_i}$ of subsets of U_i covering U_i, for $i = 1,\ldots,I$, such that $c_i = C_{k+2}\mathscr{K}_i>0$, $d_i\geqslant\mathrm{diam}\,\mathscr{K}_i$ and the following conditions hold:

(i) $\min(c_i, d_i) > 3^k\sum_{j=i+1}^{I} d_j$;

(ii) $X = \bigcup_{i=1}^{I} B(U_i, -2(k+1)d_i)$.

16. If $f\colon X\to Y$ is a mapping between metric spaces X and Y, then we denote

$$\lambda(f) = \sup\{\lambda:\ \varrho(f(x), f(y))\geqslant\lambda\varrho(x, y) \text{ for every } x, y\in X\}.$$

TOPOLOGICAL LEMMA. *Let $(X_n)_{n\geqslant 0}$ be a sequence of metric spaces which satisfy the properties $P(k, (d_{n,i})_{i=1}^{I_n}, (c_{n,i})_{i=1}^{I_n})$, respectively, where the set $\{d_{n,i}:\ n\geqslant 0,\ 1\leqslant i\leqslant I_n\}$ is upper bounded by γ. Let $\min(c_{n,i}, d_{n,i})\geqslant\beta\sum_{j=1}^{I_s} d_{s,j}$ for every $0\leqslant n<s$, where $\beta>0$ is a constant number.*

If there exists a sequence of continuous mappings $f_n\colon X_n\to X_{n+1}$, $n = 0, 1, \ldots$ such that $\lambda(f_n)\geqslant\lambda>1$, λ is a constant, then $\dim X_0\leqslant k$.

Proof. One may assume that λ is arbitrarily large using compositions of mappings f_i. We proceed inductively. If $k=0$, the lemma is trivial.

Assume that the lemma holds for an arbitrary integer $k\geqslant 0$. Take spaces (X_n) which satisfy the assumptions of the lemma for $k+1$. Let x be an arbitrary point in X_0. If we show for any $n\geqslant 0$ a neighbourhood V_n of $f_{n-1}\circ\ldots\circ f_0(x)$ in X_n such that $\dim\mathrm{Fr}(V_n)\leqslant k$ and diameters of V_n are bounded, then $\mathrm{diam}((f_{n-1}\circ\ldots\circ f_0)^{-1}(V_n))\xrightarrow{n\to\infty} 0$ and $\dim\mathrm{Fr}((f_{n-1}\circ\ldots\circ f_0)^{-1}(V_n))\leqslant k$, which implies $\dim X_0\leqslant k+1$. But for every $n_0\geqslant 0$ the sequence $(X_n)_{n\geqslant n_0}$ satisfies the assumptions of the lemma. Thus it suffices to show only how to find $V = V_0$ with its diameter upper bounded by a constant depending only on the γ.

We define inductively sets $A_{n,i}\subset X_n$ for $n = 0, 1, \ldots$ and $i = 1, \ldots, I_n$. Every object assumed to exist for X_n by the property $P(k+1)$ is additionally indexed by n as a first index. Fix numbers $\eta_{n,i}>0$ such that $\eta_{n,i}\leqslant\alpha\cdot d_{n,i}$ for a small constant α (further it will be clear how small the α should be)

$$A_{0,1} = \bigcup\{K_{0,1,t}\in\mathscr{K}_{0,1}:\ t\in T_{0,1}\ \&\ x\in K_{0,1,t}\}.$$

Let $A_{n,i}$, for $i<I_n$ be already defined. Define

$$A_{n,i+1} = \begin{cases}\bigcup\{K_{n,i+1,t}\in\mathscr{K}_{n,i+1}:\ t\in T_{n,i+1}\ \&\ K_{n,i+1,t}\cap B(A_{n,i}, \eta_{n,i+1})\neq\emptyset\}\cup\\ \qquad\cup B(A_{n,i}, \eta_{n,i+1}) \quad\text{if}\quad A_{n,i}\neq\emptyset,\\ \bigcup\{K_{n,i+1,t}\in\mathscr{K}_{n,i+1}:\ t\in T_{n,i+1}\ \&\ x\in K_{n,i+1,t}\}\\ \qquad\text{if}\quad n=0 \text{ and } A_{n,i}=\emptyset.\end{cases}$$

Let $A_{n,I_n}\subset X_n$ be already defined. Define $A_{n+1,1}\subset X_{n+1}$ as follows:

$$A_{n+1,1} = \bigcup\{K_{n+1,1,t}\in\mathscr{K}_{n+1,1}:\ t\in T_{n+1,1}\ \&\ \&\ K_{n+1,1,t}\cap B(f_n(A_{n,I_n}), \eta_{n+1,1})\neq\emptyset\}\cup B(f_n(A_{n,I_n}), \eta_{n+1,1}).$$

Define $Y_n \subset X_n$ for $n = 0, 1, \ldots$ as follows:

$$Y_n = \operatorname{Fr} \bigcup_{m > n} (f_{m-1} \circ \ldots \circ f_n)^{-1}(A_{m,I_m}).$$

$\bigcup_{m>0} (f_{m-1} \circ \ldots \circ f_0)^{-1}(A_{m,I_m})$ is our set V and we want to show that $\dim Y_0 = \dim \operatorname{Fr} V \leqslant k$. It is obvious that $f_n(Y_n) \subset Y_{n+1}$. Thus we have the sequence $Y_0 \xrightarrow{f_0|Y_0} Y_1 \xrightarrow{f_1|Y_1} \ldots$, for which we hope to be able to use the induction hypothesis. Define

$$\mathscr{K}'_{n,i} = \{K_{n,i,t} \cap Y_n \colon t \in T_{n,i} \ \& \ K_{n,i,t} \subset B(U_{n,i}, -d_{m,i})\}.$$

Define $U'_{n,i} = \bigcup \mathscr{K}'_{n,i}$. (We consider further only nonempty sets $U'_{n,i}$, and so formally we ought to reindex them. However, we will not do it is order not to complicate our notation.)

Now we claim that the numbers $c'_{n,i} = C_{k+2}\mathscr{K}'_{n,i}$ satisfy the inequalities

(1) $$c'_{n,i} \geqslant c_{n,i}/3.$$

Fix, for the time being, the indexes n and i. Suppose on the contrary that there are points $z_1, \ldots, z_{k+2}$ such that $z_j \in K'_{n,i,t_j} \in \mathscr{K}'_{n,i}$ where $t_{j_1} \neq t_{j_2}$ if $j_1 \neq j_2$ and $\max_{j_1,j_2} \varrho(z_{j_1}, z_{j_2}) < c_{n,i}/3$. There exists a point $w_i \in A_{n,i}$ such that

$$\varrho(w_i, z_1) < \min(d_{n,i}, c_{n,i}/2).$$

This follows from the fact that

$$\sum_{j=i+1}^{I_n} \eta_{n,j} + \sum_{j=i+1}^{I_n} d_{n,j} + \sum_{m=n+1}^{\infty} \lambda^{n-m}\Big(\sum_{j=1}^{I_m} (d_{m,j} + \eta_{m,j})\Big)$$

$$\leqslant (\alpha+1)\Big(\sum_{j=i+1}^{I_n} d_{n,j} + \sum_{m=n+1}^{\infty} \lambda^{n-m}\Big(\sum_{j=1}^{I_m} d_{m,j}\Big)\Big) < \min(d_{n,i}, c_{n,i}/2).$$

The last inequality holds provided the appropriate α and λ have been set. Thus $w_i \in K_{n,i,t}$ for an index $t \in T_{n,i}$. We know also that this $K_{n,i,t}$ is disjoint from Y_n (due to our thickening the set $A_{n,i}$ by $\eta_{n,i+1}$ or the set $f_n(A_{n,i})$ by $\eta_{n+1,1}$ if $i = I_n$). So we have the points $w_i, z_1, \ldots, z_{k+2}$ belonging to different sets of the family $\mathscr{K}_{n,i}$ and

$$\max_{j_1,j_2} \big(\varrho(z_{j_1}, z_{j_2}), \varrho(w_i, z_{j_1})\big) \leqslant c_{n,i}/3 + c_{n,i}/2 < c_{n,i}.$$

This gives a contradiction with the definition of $c_{n,i}$.

Set $d'_{n,i} = d_{n,i}$. Now it is clear that the inequality from the statement of the lemma for the numbers $d'_{n,i}$ and $c'_{n,i}$ holds (with, possibly, another coefficient β). Also the properties $P(k, (d'_{n,i}), (c'_{n,i}))$ are obviously satisfied (in view of (1) and the definion of $\mathscr{K}'_{n,i}$). So, by the induction hypothesis, $\dim Y_0 \leqslant k$. This ends our proof.

Proof of Theorem B. Recall that since Λ is an attractor the manifolds W^u_x are contained in Λ for $x \in \Lambda$. Since there exists an ω-limit point in Λ, there exists also a periodic point $p \in \Lambda$ (this follows from the theorem on ε-trajectories [7]). We may assume p to be fixed because it suffices to find an f^q-invariant compact set Y ($\dim \bigcup_{i=0}^{q-1} f^i(Y) = \dim Y$).

Let $R > r > 0$ be some numbers and $R < \mathscr{R}$ (see Notation 12). Assume additionaly that for every $y, z \in \Lambda$ diameter in the metric ϱ^u of every component of $W^u_z \cap \big((W^s_{y,R} \cap \Lambda) \times_{\mathrm{rect}} W^u_{y,R}\big)$ is less than $\mathscr{R}$. We can find a finite cover of Λ by open (in Λ) sets $(W^s_{y_i,r} \cap \Lambda) \times_{\mathrm{rect}} W^u_{y_i,r}$, $i = 1, \ldots, I$. If $\mathscr{R}$ is sufficiently small, then there are some standard smooth mappings $h_i \colon W^u_{y_i,\mathscr{R}} \to E^u_{y_i}$ such that the Lipschitz constants of h_i and h_i^{-1} are less than 2. (We consider here the metric ϱ^u on $W^u_{y_i,\mathscr{R}}$ and a Euclidean metric on $E^u_{y_i}$.) Denote $\hat{h_i} = h_i | W^u_{y_i,r}$.

Similarly to the manner in the proof in § 1, we construct the cover $\mathscr{A}_i$ of $E^u_{y_i}$ for $i = 1, \ldots, I$. Each $\mathscr{A}_i$ consists of cubes $A_{i,\tau}$ with edges of length $2^{u-1} \cdot \alpha_i$. (The numbers α_i and β_i, which will appear in a moment, will be defined later.) The cubes $A_{i,\tau}$ are clusters of u-dimensional cubes with edges of length α_i of the form

(1) $$\{x = (x_1, \ldots, x_u) \in E^u_{y_i} \colon \alpha_i \cdot m_j \leqslant x_j \leqslant \alpha_i(m_j + 1)\},$$

where m_j's are integers.

After removing from $E^u_{y_i}$ the set $B(S_{\alpha_i}, \beta_i)$ which is the $(u-k-1)$-dimensional skeleton of the partition into cubes of form (1) thickened by β_i, we obtain from the cover $\mathscr{A}_i$ the cover $\mathscr{A}'_i$ (for every $i = 1, \ldots, I$). We have $\operatorname{diam} \mathscr{A}'_i < \sqrt{u} \cdot \alpha_i \cdot 2^{u-1}$ and $C_{k+2}\mathscr{A}'_i > H \cdot \beta_i$ for a constant H.

There exists a smooth immersion $k_p \colon \boldsymbol{R}^u \to M$ and a diffeomorphism $g \colon \boldsymbol{R}^u \to \boldsymbol{R}^u$ such that $k_p(\boldsymbol{R}^u) = W^u_p$ and $k_p \circ g = (f | W^u_p) \circ k_p$ (see Definition 7). Consider $\boldsymbol{R}^u$ as a metric space with the metric induced by k_p from the metric ϱ^u on W^u_p. Let $(\hat{K_{i,t}})_{t \in T_i}$ be defined as a family of connected components of the sets $k_p^{-1}\big((W^s_{y_i,r} \cap \Lambda) \times_{\mathrm{rect}} \hat{h_i}^{-1}(A_{i,\tau})\big)$ for $i = 1, \ldots, I$. Define $X \subset \boldsymbol{R}^u$ as follows:

$$X = \bigcap_{j=0}^{\infty} g^{-j}\Big(\boldsymbol{R}^u \setminus \bigcup_{i=1}^{I} k_p^{-1}\Big((W^s_{y_i,r} \cap \Lambda) \times_{\mathrm{rect}} \hat{h_i}^{-1}\big(B(S_{\alpha_i}, \beta_i)\big)\Big)\Big).$$

We check that $\dim X \leqslant k$. We use the Topological Lemma. We set $X_n = X$ and $f_n = g|X$ for every $n \geqslant 0$. To check property $P(k)$ one can put

$$U_i = \bigcup_{t \in T_i} \hat{K_{i,t}} \cap X, \quad K_{i,t} = \hat{K_{i,t}} \cap X \quad \text{for} \quad t \in T_i,$$

$$\mathscr{K}_i = \{K_{i,t}\}_{t \in T_i} \quad \text{and} \quad d_i = L\big(\sqrt{u} \cdot 2^u \cdot \alpha_i\big)$$

(see Notation 12). We have $c_i = C_{k+2}(\mathscr{K}_i) \geqslant L^{-1}(H\beta_i/2)$. Now one can see that if we have taken the α_i, β_i such that

$$(2) \qquad I\cdot 3^k \cdot L(\sqrt{u}\cdot 2^u\cdot \alpha_i) \leqslant \min\big(L(\sqrt{u}\cdot 2^u \cdot \alpha_{i-1}),\ L^{-1}(H\beta_{i-1}/2)\big)$$

for $i = 2, \ldots, I$ and $4(k+1)\cdot L(\sqrt{u}\cdot 2^u\cdot \alpha_i)$ is less than the Lebesgue number of the cover $\big((W^s_{y_j,r}\cap\Lambda)\times_{\mathrm{rect}} W^u_{y_j,r}\big)_{j=1}^I$ for $i = 1, \ldots, I$, then property $P(k)$ is satisfied.

Now we want to define a disc similar to the disc $G(D)$ in the proof in § 1. But first we ought to show a set from which the disc D should be removed.

For any $\beta > 0$ define the set $Y_i^\beta \subset \boldsymbol{R}^u$, $i = 1, \ldots, I$, as follows:

$$Y_i^\beta = k_p^{-1}\big(W^s_{y_i,R}\cap\Lambda\times_{\mathrm{rect}} h_i^{-1}\big(B(S_{\alpha_i}\cap h_i W^u_{y_i,r}),\ \beta)\big)\big).$$

Omitting the sets $Y_i^{\beta_i}$ with images of a disc under forward iterations of g will allow us to estimate the dimension of a final N^k along W^u.

Now we use property $(*)$ (see the definition of property $(*)$ in § 1). We want to find a set the omitting of which with a forward f-orbit of a disc will allow us to estimate the dimension of N^k along W^s. Let a point $p' \in \Lambda$ play the role of p from property $(*)$. Fix compact subsets Q_j, $j = 1, \ldots$ $\ldots, J$, of $W^s_{p',\mathrm{loc}}$ and a mapping $h\colon \Lambda_{p'}\times\boldsymbol{R}^u\to\Lambda$ which satisfy the properties described in property $(*)$. Denote by V the bounded (interior) component of $W^s_{p',\mathrm{loc}}\setminus\bigcup_{j=1}^J Q_j$. Using $(*)$ (a) one can check that there exists $l_1 > 0$ such that

$$h\big(((k^s_{p'})^{-1}(V)\cap\Lambda_{p'})\times B(0, l_1/2)\big) = \Lambda.$$

By $(*)$ (b) there exists a number $l_2 > 0$ such that

$$\operatorname{diam}_{\varrho^s} h(\Lambda_{p'}\times\{q\}) < l_2$$

for every $q\in B(0, l_1)$. One can assume that l_2 is arbitrarily small because one can iterate forward the whole structure by f. Take a number l_3 such that

$$l_3 > \sup\big\{\operatorname{diam}_{\varrho^u} h\big(B(q, l_1)\big)\colon k^s_{p'}(q)\in\big(V\cup\bigcup_{j=1}^J Q_j\big)\cap\Lambda\big\}.$$

Let exist such numbers $\varepsilon_j > 0$, $j = 1, \ldots, J$ that for every $q\in\boldsymbol{R}^u$ the intersection $k_p\big(B(q, 2^{J+2}\cdot l_3)\big)\cap B_{\varrho^s}(Q_j, \varepsilon_j)$ consists of at most one point (by $(*)$ (c)).

Define the sets $Z_j^\beta\subset\boldsymbol{R}^u$, $j = 1, \ldots, J$ as follows:

$$Z_j^\beta = k_p^{-1}\big(\bigcup\{W^u_{q,\beta}\colon q\in B_{\varrho^s}(Q_j, \varepsilon_j)\}\big).$$

Let $G_{-1}\colon D^k\to\boldsymbol{R}^u$ be a compact k-dimensional disc embedded into $\boldsymbol{R}^u$. Assume that a mapping $G_m\colon D^k\to\boldsymbol{R}^u$ for an index $m\geqslant -1$ is defined. We define G_{m+1} as follows: $G_{m+1} = g^{-(m+1)}\circ\mathscr{H}$ where $\mathscr{H}$ is a perturbed $g^{m+1}\circ G_m$. Now we shall describe it more carefully. We start with the mapping $g^{m+1}\circ G_m$. For an arbitrary small $\eta > 0$, by successive perturbations of sizes not bigger than $2^J\cdot l_3+\eta$, $2^{J-1}\cdot l_3+\eta, \ldots, 2\cdot l_3+\eta$, $L(2^{\binom{u}{k+1}+1}\times\beta_1)+\eta, \ldots, L(2^{\binom{u}{k+1}+1}\cdot\beta_I)+\eta$ we obtain images of the disc disjoint from the sets $Z_1^{2^J\cdot l_3}, \ldots, Z_J^{2\cdot l_3}$, $Y_1^{2\cdot\beta_1}, \ldots, Y_I^{2\cdot\beta_I}$, respectively. The above is possible provided

$$(3) \qquad \alpha_i > 2^{\binom{u}{k+1}+2}\cdot\beta_i \quad\text{and}\quad \mathscr{R}-r > 3\cdot\alpha_i.$$

$\mathscr{H}$ is defined as a mapping after the last $((J+I)\mathrm{th})$ perturbation. This requires some explanation. The removing from every Y_i^β can be made successively as well, by using the formula

$$S_{\alpha_i} = \bigcup\{S_{\alpha_i}^{(m_1,\ldots,m_{k+1})}\colon 1\leqslant m_t\leqslant u \text{ for } t = 1, \ldots, k+1 \ \&\ m_{t_1}\neq m_{t_2} \text{ if } t_1\neq t_2\}$$

where

$$S_{\alpha_i}^{(m_1,\ldots,m_{k+1})} = \bigcup_{(n_1,\ldots,n_u)\in Z^u}\big(\{x\in\boldsymbol{R}^u\colon\ x_m = 0 \ \text{ if }\ m\in\{m_1, \ldots, m_{k+1}\}\} + \\ + \alpha_i\cdot(n_1, \ldots, n_u)\big).$$

In order to remove a k-dimensional disc from $B(S_{\alpha_i}^{(m_1,\ldots,m_{k+1})}, \beta)$ we smooth it, remove from $S_{\alpha_i}^{(m_1,\ldots,m_{k+1})}$ by Thom's Lemma and eventually compose with the orthogonal projection of

$$B(S_{\alpha_i}^{(m_1,\ldots,m_{k+1})}, \beta)\setminus S_{\alpha_i}^{(m_1,\ldots,m_{k+1})} \quad\text{onto}\quad \operatorname{Fr} B(S_{\alpha_i}^{(m_1,\ldots,m_{k+1})}, \beta).$$

If we assume about β_i additionally that for a number $\delta > 0$

$$(4) \quad l_3 - 2(I+J)\eta - \sum_{i=1}^{I} L(2^{\binom{u}{k+1}+1}\cdot\beta_i) > \delta \quad\text{and}$$

$$L^{-1}(\beta_i/2) - I\eta - \sum_{s=i+1}^{I} L(2^{\binom{u}{k+1}+1}\cdot\beta_s) > L(2\delta),$$

then no step spoils the previous steps. This means at the end that $\mathscr{H}(D^k)$ is disjoint from the sets

$$Z_1^{l_3+\delta}, \ldots, Z_J^{l_3+\delta},\ Y_1^{\beta_1+\delta}, \ldots, Y_I^{\beta_I+\delta}.$$

The numbers α_i, β_i satisfying the conditions (2), (3), (4), can be found successively (i.e. α_1, β_1, α_2, β_2, etc.).

Now if $\lambda(g)$, which is the expanding coefficient of g, is sufficiently large, the sequence (G_m) converges to a mapping G and for every $m\geqslant 0$ $g^m G(D^k)$ is disjoint from the sets

$$Z_1^{l_3}, \ldots, Z_J^{l_3},\ Y_1^{\beta_1}, \ldots, Y_I^{\beta_I}.$$

Now our invariant set N^k may be defined as follows:

$$N^k = \mathrm{cl}\Big(\bigcup_{m=-\infty}^{+\infty} k_p g^m G(D^k)\Big).$$

We prove that $\dim N^k = k$. First of all observe that

$$N^k = \bigcup_{m=-\infty}^{+\infty} f^m\big(k_p G(D^k)\big) \cup \omega\big(k_p G(D^k), f\big) \cup \{p\}$$

where for any mapping F of a metric $\mathscr{X}$ into itself and for any subset $A \subset \mathscr{X}$ we denote

(5) $\omega(A, F) = \{x \in \mathscr{X}$: there exists a sequence of points

$$x_t \in F^{m_t}(A) \text{ such that } m_t \xrightarrow{t\to\infty} \infty \;\&\; x_t \xrightarrow{t\to\infty} x\}.$$

Therefore it suffices to study the dimension of our f-invariant set $\omega = \omega\big(k_p G(D^k), f\big)$. By the construction we obtain

(6) $$\omega \cap \bigcup\{W^s_{q,\mathscr{L}^{-1}(R-r)}:\ q \in W^u_p \setminus k_p(X)\} = \emptyset.$$

Recall that X has been defined on p. 207. Function $\mathscr{L}$ is assumed to have similar properties to those of function L but with interchanged roles of W^s and W^u. Disjointness in (6) follows from the fact that

$$\Big(\bigcup_{m\geqslant 0} k_p g^m G(D^k)\Big) \cap \Big(\bigcup\{W^s_{q,\mathscr{L}^{-1}(R-r)}:\ q \in W^u_p \setminus k_p(X)\}\Big) = \emptyset.$$

ω is also disjoint from $f^m\Big(h\big((k^s_{p'})^{-1}\big(\bigcup_{j=1}^{J} Q_j \cap A\big) \times B(0, l_1)\big)\Big)$ for every integer m. We are especially interested in large positive integers here because the thicknesses of our pipes $f^m\Big(h\big((k^s_{p'})^{-1}(V \cap A) \times B(0, l_1)\big)\Big)$ converge to 0 if m's converge to ∞. Recall that the thickness of the 0th pipe, $m = 0$, is less than l_2. These pipes, solid in A, are spread onto the whole A and their walls are disjoint from ω. Thus, locally, ω is contained in a rectangle product of 0-dimensional and k-dimensional sets.

$$\dim N^k = \max\Big(\big(\dim k_p g^m G(D^k)\big)_{m\in Z},\ \dim \omega\Big) = k.$$

Proof of Theorem B′. This is a subproof of the proof of Theorem B.

Part (a) of the Corollary follows immediately from Theorem B, as well as from Theorem B′.

Proof of Theorem A. Since f is topologically conjugate with a hyperbolic toral automorphism, we may assume f to be algebraic (see [3], [10]). We ought to check property (*) (c) ((*) (a) and (*) (b) are obviously satisfied). As in the proof in § 1 denote by $\pi\colon \mathbf{R}^n \to \mathbf{R}^n/Z^n = T^n$ the standard covering projection. Denote by E^s_0 and $E^u_0 \subset \mathbf{R}^n$, respectively, the stable, and unstable spaces at $0 \in \mathbf{R}^n$. We take $\pi(0)$ as p' in property (*).

Denote

$$V = \{v \in E^s_0:\ \pi(v) \in \pi(E^u_0) = W^u_{\pi(0)}\}.$$

The set V is countable and hence there exist $w_1, w_2, \ldots, w_s \in E^s_0$ such that no w_i is orthogonal to any element of V, $\mathrm{span}(w_1, \ldots, w_s) = E^s_0$. Denote by $L_1, \ldots, L_s$ the subspaces of E^s_0 orthogonal to $w_1, \ldots, w_s$, respectively. Now, the subspaces $L_1, L_1 + cw_1, L_2, L_2 + cw_2, \ldots, L_s, L_s + cw_s$ for a small number $c > 0$ bound a small s-dimensional parallelepiped. The sets $Q_1, \ldots, Q_{2s}$ may be defined as its walls projected by π into T^m.

If $k < \max(s, u)$, then the set N^k may be obtained directly from the proof of Theorem B by using a disc D^k embedded into W^s or W^u. If $2 \leqslant k \leqslant n-2$ we write $k = k_1 + k_2$ where $1 \leqslant k_1 < s$ and $1 \leqslant k_2 < u$. By the proof of Theorem B there exist sets $D^{k_1}, D^{k_2} \subset T^m$ which are continuous images of k_1 and k_2-dimensional discs, respectively,

$$D^{k_1} \subset W^s_{\pi(0),\mathrm{loc}}, \qquad D^{k_2} \subset W^u_{\pi(0),\mathrm{loc}},$$

$$\dim \omega(D^{k_1}, f^{-1}) \leqslant \dim D^{k_1} = k_1 \quad \text{and} \quad \dim \omega(D^{k_2}, f) \leqslant \dim D^{k_2} = k_2$$

(see (5) in the proof of Theorem B for the definition of ω). Define

(1) $$N^{(k_1,k_2)} = \mathrm{cl} \bigcup_{m=-\infty}^{+\infty} f^m(D^{k_1} \times_{\mathrm{rect}} D^{k_2}).$$

We have $\dim N^{(k_1,k_2)} = k_1 + k_2 = k$ because

$$N^{(k_1,k_2)} = \bigcup_{m=-\infty}^{+\infty} f^m(D^{k_1} \times_{\mathrm{rect}} D^{k_2}) \cup \omega(D^{k_1}, f^{-1}) \cup \omega(D^{k_2}, f).$$

Proof of part (b) of the Corollary. For any hyperbolic toral automorphism f and its periodic orbits $\gamma_1, \ldots, \gamma_m$ we can find the sets N^k in Theorem A in such a way that they are disjoint from $\bigcup_{j=1}^{m} \gamma_j$. The mapping $f|N^k$ does not change after a perturbation of f in a sufficiently small neighbourhood of $\bigcup_{j=1}^{m} \gamma_j$. So after changing f into a DA-diffeomorphism g the same N^k are g-invariant sets.

Proof of Theorem A′. Proceeding as in the proof of Theorem B (but not using property (*)), one obtains a set $D^k \subset W^u_p$, where $p \in \mathrm{Perf}$, which is a continuous image of a k-dimensional disc, $\dim D^k = k$ and $\dim \omega(D^k, f) \leqslant k + s$.

(a) If $\Omega(f) = M^n$, then there exists a number $a > 0$ such that for every $q \in M^n$ $W^s_{q,a} \cap B(x, r) \neq \emptyset$, where $B(x, 2r)$ is a ball in M^n disjoint from $\omega(D^k, f)$. So $\omega(D^k, f)$, by its f-invariance, omits a dense subset of $W^s_{q,\mathrm{loc}}$ thickened by r in the direction of W^u, for every $q \in M^n$. So $\omega(D^k, f)$ is contained locally in a rectangle product of $s-1$ and k-dimensional sets.

(We used here the fact that $\dim(W^s_{q,\mathrm{loc}} \setminus \{\text{a dense subset}\}) \leqslant s-1$. Observe that for attractors the analogous inequality $\dim((W^s_{p,\mathrm{loc}} \cap \Lambda) \setminus \{\text{a dense subset}\}) \leqslant \dim(W^s_{p,\mathrm{loc}} \cap \Lambda) - 1$ can be false.) Thus we have the inequality

$$\dim \omega(D^k, f) \leqslant k+s-1. \tag{1}$$

(b) If $\Omega(f) \neq M^n$ (it is an open problem whether that is possible), in order to obtain inequality (1) it is necessary to assume something additional about the construction of D^k. It suffice to know that, for each basic set $\Omega(f)_i$ which is a repellor, $\omega(D^k, f)$ is disjoint from an open non-empty subset of $\Omega(f)_i$. However, the necessity of omitting additionally a finite number of small open sets with forward f-images of a disc in the proof of Theorem B does not spoil this proof.

Let $u \geqslant s$. For $k < u$ we have constructed the sets N^k such that $k \leqslant \dim N^k \leqslant k+s-1$. If $k > u-s$, then one can set as N^k the sets $N^{(k_1,k_2)}$ which were constructed in the proof of Theorem A, where

$$k_1 = \frac{k-(u-s)}{2}, \quad k_2 = \frac{k+u-s}{2} \quad \text{if} \quad k+n \text{ is an even number}$$

or

$$k_1 = \left[\frac{k-(u-s)}{2}\right]+1, \quad k_2 = \left[\frac{k+u-s}{2}\right] \quad \text{if} \quad k+n \text{ is odd.}$$

Then

$$k \leqslant \dim N^k \leqslant \max(k_1+k_2, k_1+u-1, k_2+s-1) = \left[\frac{k+n-1}{2}\right].$$

Proof of statement (a) in the Remark. In the proof of Theorem B one can start with a $(u-1)$-dimensional cube as a disc D^{u-1} and choose a sequence of its k-dimensional walls ($k = 1, \ldots, u-1$) such that $D^1 \subset D^2 \subset \ldots \subset D^{n-1}$. Now, each perturbation should be done as a composition of removing successively the discs $D^1, \ldots, D^{u-1}$ from skeletons of dimensions $u-2, \ldots, 0$, respectively (one must remember to prolong mappings defined on D^k to the whole cube D^{u-1} at each step). In order to construct N^k for $k \geqslant u$ one can use the sets $N^{(k_1,k_2)}$.

The proof of statement (b) in the Remark is straightforward and will be omitted.

Proof of Theorem C. Assume that f is algebraic. We start with an arbitrary mapping $g\colon D^k \to T^n$ and lift it to $\tilde{g}\colon D^k \to \boldsymbol{R}^n$. Let P^s, P^u denote the projections of $\boldsymbol{R}^n$ onto E^s_0, E^u_0 along E^u_0, E^s_0, respectively. We perturb $P^s \circ \tilde{g}$ and $P^u \circ \tilde{g}$ to such mappings which are embeddings into E^s_0, E^u_0 on a smooth subdisc of D^k and after that to mappings $g_s\colon D^k \to E^s_0$ and $g_u\colon D^k \to E^u_0$ such that $\dim(\mathrm{cl} \bigcup_{m=-\infty}^{+\infty} f^m \pi g_s D^k) = k$ and $\dim(\mathrm{cl} \bigcup_{m=-\infty}^{+\infty} f^m \pi g_u D^k) = k$. Now π composed with the diagonal product of g_s and g_u, $g_s \Delta g_u\colon D^k \to \boldsymbol{R}^n$ (where $\boldsymbol{R}^n$ is considered as a Cartesian product of E^s_0 and E^u_0) gives us the required perturbation.

Added in proof. Now I am able to prove Theorem B for Anosov diffeomorphisms assuming property (*) without the item (c). This gives Theorem A for infranilmanifolds.

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