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Multiplier criteria of Hörmander type for Jacobi expansions

by

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Abstract. It is shown how the multiplier criteria of Hörmander type derived in Connett and Schwartz [7] for Jacobi expansions by the use of finite differences can be substantially improved by using fractional differences. The main result, stated n Theorem 1, is in a certain sense best possible.

1. Introduction. In this paper we show how the multiplier criteria of Hörmander type [14] derived in Connett and Schwartz [7] for Jacobi expansions by the use of finite differences can be substantially improved by using fractional differences.

To state our results we shall employ the following notation which, for the convenience of the reader, is essentially that in [7]. Fix $\alpha \geqslant \beta \geqslant -1/2$, $\alpha > -1/2$ and let $L^p = L^p_{(\alpha,\beta)}$, $1 \leqslant p < \infty$, denote the space of measurable functions $f(x)$ on $(-1,1)$ for which

$$\|f\|_p = \Big(\int_{-1}^{1} |f(x)|^p \, dm(x)\Big)^{1/p} < \infty,$$

where $dm(x) = dm_{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta dx$. Also let $R_n(x) = R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order (α,β), [16]. Each $f \in L^p$ has an expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} f^{\wedge}(n) h_n R_n(x),$$

where

$$f^{\wedge}(n) = \int_{-1}^{1} f(x) R_n(x) \, dm(x)$$

and

$$h_n = h_n^{(\alpha,\beta)} = \|R_n\|_2^{-2} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{2^{\alpha+\beta+1}\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)}.$$

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A sequence $m = \{m_n\}_{n=0}^{\infty} \in l^{\infty}$ is called a *multiplier of (strong) type* (p, q), notation $m \in M_p^q$, if for each $f \in L^p$ there exists a function $Mf \in L^q$ with

(1.1) $$Mf(x) \sim \sum_{n=0}^{\infty} m_n f\hat{\ }(n) h_n R_n(x), \qquad \|Mf\|_q \leqslant C\|f\|_p.$$

The smallest constant C independent of f for which (1.1) holds is called the *multiplier norm* of m and it is denoted by $\|m\|_{M_p^q}$.

As in our previous papers [10], [11], [12] we define the fractional difference operator Δ^{λ} by

$$\Delta^{\lambda} m_n = \sum_{j=n}^{\infty} A_{j-n}^{-\lambda-1} m_j, \qquad A_n^{\lambda} = \binom{n+\lambda}{n} = \frac{\Gamma(n+\lambda+1)}{\Gamma(n+1)\Gamma(\lambda+1)}$$

whenever the series converges. For $\gamma \geqslant 0$, $1 \leqslant q < \infty$, the *space* $\mathrm{wbv}_{q,\gamma}$ *of sequences of weak bounded variation of order* γ is defined by

$$\mathrm{wbv}_{q,\gamma} = \{m \in l^{\infty}\colon \|m\|_{q,\gamma;w} < \infty\}$$

where

$$\|m\|_{q,\gamma;w} = \|m\|_{\infty} + \sup_{k \geqslant 0}\Big(\sum_{2^{k}-1}^{2^{k+1}-1} n^{-1}|A_n^{\gamma}\Delta^{\gamma} m_n|^q\Big)^{1/q}.$$

Then our main results can be stated as follows.

THEOREM 1. *If* $\gamma > \alpha+1$, *then*

$$\mathrm{wbv}_{2,\gamma} \subset M_p^p, \qquad 1 < p < \infty.$$

THEOREM 2. (a) *If* $1 < p < \infty$ *and* $|1/p - 1/2| > 1/q$, *then*

$$\mathrm{wbv}_{q,\gamma} \subset M_p^p, \qquad \gamma > (2\alpha+2)|1/p - 1/2|.$$

(b) *If* $1 < p < \infty$, *then*

$$\mathrm{wbv}_{2,\gamma} \subset M_p^p, \qquad \gamma > (2\alpha+1)|1/p - 1/2| + 1/2.$$

Theorem 2 will be derived by interpolating between Theorem 1 and other known multiplier criteria. Note that part (b) of Theorem 2 implies that

(1.2) $$\mathrm{wbv}_{2,\gamma} \subset M_p^p, \qquad 1 < p < \infty$$

if $\gamma \geqslant \alpha+1$. This result is best possible in the sense that $\mathrm{wbv}_{2,\mu} \not\subset M_p^p$ when $\mu < \alpha+1$ and $p > 1$ is near 1. This can be seen by the counterexample of the Cesàro kernel when modifying the corresponding reasoning in [11], Section 8. In terms of our $\mathrm{wbv}_{2,\gamma}$ notation the Connett and Schwartz result in [7] reads:

(1.3) $$\mathrm{wbv}_{2,[\alpha+2]} \subset M_p^p, \qquad 1 < p < \infty, \ \alpha \geqslant \beta > -1/2,$$

where $[\alpha+2]$ is the integer part of $\alpha+2$; so (1.3) follows from (1.2). The case $\alpha > -1/2 = \beta$ is proved in [10].

Since, by [11], Lemma 1,

$$\mathrm{wbv}_{q,\gamma} \subset \mathrm{wbv}_{q,\mu}, \qquad 0 < \mu < \gamma, \ 1 \leqslant q < \infty,$$

to prove Theorem 1 it suffices to just prove it for the special case $\alpha+2 > \gamma > \alpha+1$. Also, in view of [11], Lemma 2, we may assume that $\{m_n\}$ has compact support and that $m_0 = 0$. In order to extend the Connett and Schwartz proof of (1.3) to get Theorem 1 we shall need the following fractional analogs of Proposition 4.1 and Lemmas 5.1–5.3 in [6].

PROPOSITION 1. *If* $\alpha+2 > \lambda > 1/2$ *and* $f(x) = \sum c_n h_n R_n(x)$, *where* $\{c_n\}$ *has compact support, then*

$$\int_{-1}^{1} (1-x)^{\lambda}(f(x))^2\, dm(x) \leqslant C \sum_{n=0}^{\infty} (\Delta^{\lambda} c_n)^2 h_n$$

with C *independent of* f.

LEMMA 1. *Let* $d\mu(\theta) = 2^{\alpha+\beta+2}\sin^{2\alpha+1}\theta\cos^{2\beta+1}\theta\, d\theta$ *and* $V_r(\theta) = W_r(\theta) - W_{r/2}(\theta)$, *where*

$$W_r(\theta) = \sum_{n=0}^{\infty} (1-r)^n h_n R_n(\cos 2\theta).$$

If $\alpha+2 > \gamma > \alpha+1$ *and* $m = \{m_n\}$ *has compact support, then*

$$\Big(\int_0^{\pi/2} (1-\cos\theta)^{\gamma}|MV_r(\theta)|^2\, d\mu(\theta)\Big)^{1/2} \leqslant C\|m\|_{2,\gamma;w} r^{\gamma-\alpha-1}$$

with C *independent of* m.

LEMMA 2. *If* $\alpha+2 > \gamma > \alpha+1$, $0 < \eta < \gamma - \alpha - 1$ *and* $\{m_n\}$ *has compact support, then*

$$\int_0^{\pi/2} |MV_r(\theta)|(\theta/r)^{\eta}\, d\mu(\theta) \leqslant C\|m\|_{2,\gamma;w}$$

with C *independent of* m.

LEMMA 3. *Suppose that the hypotheses of Lemma 2 hold and let*

$$a_i(\theta) = \big(M(W_{2^{-i-1}} - W_{2^{-i}})\big) * (W_{2^{-i-1}} + W_{2^{-i}})(\theta),$$

where the convolution $*$ *is defined as in* [7], [9]. *Then*

$$\int_E |a_i * \delta_{\varphi}(\theta) - a_i * \delta_{\varphi_0}(\theta)|\, d\mu(\theta) \leqslant C\min\Big\{\Big(\frac{r}{|\varphi-\varphi_0|}\Big)^{\eta}, \frac{|\varphi-\varphi_0|}{r}\Big\},$$

where δ_{φ} *is the unit mass concentrated at* φ *and*

$$E = \{\theta\colon |\theta-\varphi_0| > 2|\varphi-\varphi_0|\}.$$

Proposition 1, a weighted Parseval inequality, is proved in Section 2; it is the crux of the proof of our main result—in particular it is the key relation in the proof of Lemma 1, given in Section 3. Since the proofs of Lemmas 2 and 3 are repetitions of those of Lemmas 5.2 and 5.3 in [6], they will be omitted. Theorem 1 then follows from Proposition 1 and Lemmas 1–3 by the same argument as in Connett and Schwartz [7], pp. 241–248; see [10] for the case $\beta = -1/2$. After proving Theorem 2 in Section 4 we shall close with some additional remarks and a conjecture concerning multipliers.

2. Proof of Proposition 1. Observe that if

$$f(x) \sim \sum_{n=0}^{\infty} d_n h_n^{(\alpha+\lambda,\beta)} R_n^{(\alpha+\lambda,\beta)}(x),$$

where

$$d_n = \int_{-1}^{1} f(x) R_n^{(\alpha+\lambda,\beta)}(x)(1-x)^{\alpha+\lambda}(1+x)^{\beta}\,dx, \tag{2.1}$$

then, by Parseval's relation,

$$\int_{-1}^{1} \big(f(x)\big)^2 (1-x)^{\alpha+\lambda}(1+x)^{\beta}\,dx = \sum_{n=0}^{\infty} d_n^2 h_n^{(\alpha+\lambda,\beta)}. \tag{2.2}$$

Thus, from (2.1) and the formula [2], (3.41)

$$(1-x)^{\lambda} R_n^{(\alpha+\lambda,\beta)}(x) = \sum_{k=n}^{\infty} a_k(n) R_k^{(\alpha,\beta)}(x), \qquad -1 < x < 1,\ \lambda > -\alpha/2-3/4,$$

where

$$a_k(n) = \frac{(2k+\alpha+\beta+1)\Gamma(n+k+\alpha+\beta+1)\Gamma(k-n-\lambda)\Gamma(\alpha+\lambda+1)2^{\lambda}}{(k-n)!\,\Gamma(n+k+\alpha+\beta+\lambda+2)\Gamma(-\lambda)\Gamma(\alpha+1)},$$

it follows that

$$d_n = \sum_{k=n}^{\infty} a_k(n) c_k. \tag{2.3}$$

Since $\{c_k\}$ has compact support, the sum (2.3) actually terminates and, for $\lambda > 0$, we have that $c_k = \Delta^{-\lambda}\Delta^{\lambda} c_k$. Hence

$$d_n = \sum_{k=n}^{\infty} a_k(n) \sum_{j=k}^{\infty} A_{j-k}^{\lambda-1} \Delta^{\lambda} c_j = \sum_{j=n}^{\infty} b_j(n) \Delta^{\lambda} c_j$$

with

$$\begin{aligned} b_j(n) &= \sum_{k=0}^{j-n} a_{n+k}(n) A_{j-n-k}^{\lambda-1} \\ &= \frac{2^{\lambda}\Gamma(2n+\alpha+\beta+2)\Gamma(\alpha+\lambda+1)\Gamma(\lambda+j-n)}{\Gamma(2n+\alpha+\beta+\lambda+2)\Gamma(\alpha+1)\Gamma(\lambda)(j-n)!} \times \\ &\quad \times {}_4F_3\left[\begin{matrix} 2n+\alpha+\beta+1, & n+(\alpha+\beta+3)/2, & -\lambda, & n-j; \\ n+(\alpha+\beta+1)/2, & 2n+\alpha+\beta+\lambda+2, & 1-\lambda+n-j \end{matrix}\right]. \end{aligned}$$

In order to obtain a sharp estimate for $b_j(n)$, we first use the transformation forumla [5], 4.5 (1) to find that for $k \geqslant 0$

$$b_{n+k}(n) = \frac{2^{\lambda}\Gamma(2n+\alpha+\beta+2)\Gamma(\alpha+\lambda+1)(2n+\alpha+\beta+\lambda+1)_k}{\Gamma(2n+\alpha+\beta+\lambda+2)\Gamma(\alpha+1)k!} \times$$

$$\times {}_4F_3\left[\begin{matrix} 2n+k+\alpha+\beta+\lambda+1, & n+(\alpha+\beta+2)/2, & n+\lambda+(\alpha+\beta+1)/2, & -k; \\ 2n+\alpha+\beta+\lambda+2, & n+(\alpha+\beta+\lambda+1)/2, & n+(\alpha+\beta+\lambda+2)/2 & \end{matrix}\right]$$

where $(a)_n = \Gamma(n+a)/\Gamma(a)$, and then use [4], (4.31) to show that this ${}_4F_3$ is equal to

$${}_3F_2\left[\begin{matrix} -k, & 2n+\alpha+\beta+2, & 2n+2\lambda+\alpha+\beta+1; \\ 2n+\alpha+\beta+\lambda+1, & 2n+\alpha+\beta+\lambda+2 & \end{matrix}\right].$$

Application of the transformation formula

$${}_3F_2\left[\begin{matrix} -k, & a, & b \\ c, & n+a+b-c-k & \end{matrix}\right] = \frac{(c-a-n+1)_k(c-b)_k}{(c)_k(c-a-b-n+1)_k}\,{}_3F_2\left[\begin{matrix} -k, & b, & 1-n; \\ 1+b-c-k, & 1+c-a-n & \end{matrix}\right]$$

which is a consequence of the Thomae relations in [5], Chap. 3, yields

$$\begin{aligned} b_{n+k}(n) &= \frac{2^{\lambda}\Gamma(2n+\alpha+\beta+2)\Gamma(\alpha+\lambda+1)(\lambda)_k(\lambda-1)_k}{\Gamma(2n+k+\alpha+\beta+\lambda+2)\Gamma(\alpha+1)k!}\,{}_3F_2\left[\begin{matrix} -k, 2n+\alpha+\beta+2, 1-k; \\ 2-\lambda-k, \quad 1-\lambda-k \end{matrix}\right]. \end{aligned} \tag{2.4}$$

Clearly $b_n(n) \leqslant Cn^{-\lambda}$. Since

$$\frac{(-k)_j(3-2\lambda-k)_j}{(2-\lambda-k)_j(1-\lambda-k)_j} = \frac{\Gamma(k+1)\Gamma(k+2\lambda-2)\Gamma(k-j+\lambda-1)\Gamma(k-j+\lambda)}{\Gamma(k+\lambda-1)\Gamma(k+\lambda)\Gamma(k-j+1)\Gamma(k-j+2\lambda-2)}$$

is uniformly bounded for $k-1 \geqslant j \geqslant 0$ and each of the terms in the ${}_3F_2$ series in (2.4) is nonnegative when $\lambda > 1/2$, this ${}_3F_2$ is bounded by a constant multiple of

$${}_2F_1\left[\begin{matrix} 2n+\alpha+\beta+2, & 1-k; \\ 3-2\lambda-k & \end{matrix}\right] = \frac{(2n+\alpha+\beta+2\lambda+1)_{k-1}}{(2\lambda-1)_{k-1}}$$

and so

$$b_{n+k}(n) \leqslant Cn^{1-2\lambda}(2n+k)^{\lambda-2}, \qquad k \geqslant 1.$$

Here and below n^a is to be replaced by 1 when $n=0$. Now

$$|d_n| \leqslant \sum_{j=n}^{\infty} |b_j(n)\,\Delta^\lambda c_j|$$

$$\leqslant Cn^{-\lambda}|\Delta^\lambda c_n| + C\sum_{j=n+1}^{\infty} n^{1-2\lambda}(n+j)^{\lambda-2}|\Delta^\lambda c_j|$$

and, from (2.2) and the fact that $h_n^{(\alpha,\beta)} \sim n^{2\alpha+1}$,

(2.5) $$\int_{-1}^{1} (f(x))^2(1-x)^{\alpha+\lambda}(1+x)^\beta\,dx$$

$$\leqslant C\sum_{n=0}^{\infty}(n^{-\lambda}\Delta^\lambda c_n)^2 n^{2\alpha+2\lambda+1} + C\sum_{n=0}^{\infty}\Big(\sum_{j=n}^{\infty}(n+j)^{\lambda-2}|\Delta^\lambda c_j|\Big)^2 n^{2\alpha-2\lambda+3}.$$

The first sum above is clearly dominated by

$$\sum_{n=0}^{\infty}(\Delta^\lambda c_n)^2 h_n^{(\alpha,\beta)}.$$

Since the second sum in (2.5) is

$$\leqslant C\sum_{n=0}^{\infty}\Big(\sum_{j=n}^{\infty} j^{\lambda-2}|\Delta^\lambda c_j|\Big)^2 n^{2\alpha-2\lambda+3}$$

$$\leqslant C\sum_{n=0}^{\infty} n^{2\alpha-2\lambda+3}(n^{\lambda-1}\Delta^\lambda c_n)^2$$

$$\leqslant C\sum_{n=0}^{\infty}|\Delta^\lambda c_n|^2 h_n^{(\alpha,\beta)}$$

for $\alpha+2>\lambda>1/2$ by [13], Theorem 346, the proof is complete.

3. Proof of Lemma 1. Let $\alpha+2>\gamma>\alpha+1$. It suffices to consider only the case when γ is not an integer since the integer case has already been proved in [6], [7]. By Proposition 1

(3.1) $$\Big(\int_0^{\pi/2} (1-\cos\theta)^\gamma |MV_r(\theta)|^2\,d\mu(\theta)\Big)^{1/2} \leqslant C\Big(\sum_{k=0}^{\infty}|\Delta^\gamma m_k c_k|^2 h_k\Big)^{1/2},$$

where $c_k = (1-r/2)^k-(1-r)^k$.

Let us first consider the case $\gamma<1$. By series manipulations,

(3.2) $$\Delta^\gamma m_k c_k = m_k\Delta^\gamma c_k + c_k\Delta^\gamma m_k + R_k$$

with

$$R_k = \sum_{n=k+1}^{\infty} A_{n-k}^{-\gamma-1}(m_n-m_k)(c_n-c_k).$$

If we denote the right-hand side of (3.1) by S, then it follows from (3.2) that

(3.3) $$S \leqslant C\Big(\sum_{k=0}^{\infty}|m_k\Delta^\gamma c_k|^2 k^{2\alpha+1}\Big)^{1/2} + C\Big(\sum_{k=0}^{\infty}|c_k\Delta^\gamma m_k|^2 k^{2\alpha+1}\Big)^{1/2} +$$

$$+C\Big(\sum_{k=0}^{\infty}\Big|\sum_{n=k+1}^{\infty} A_{n-k}^{-\gamma-1}(m_n-m_k)\sum_{j=k}^{n-1}\Delta c_j\Big|^2 k^{2\alpha+1}\Big)^{1/2}$$

$$= S_1+S_2+S_3.$$

Since

(3.4) $$\Delta^\gamma(1-r)^k = r^\gamma(1-r)^k$$

and

$$\sum n^\lambda(1-r)^n \leqslant Cr^{-\lambda-1}, \qquad \lambda>-1,$$

we have

$$S_1 \leqslant C\|m\|_\infty\Big(\sum_{k=0}^{\infty} r^{2\gamma}(1-r/2)^{2k}k^{2\alpha+1}\Big)^{1/2}$$

$$\leqslant C\|m\|_\infty r^{\gamma-\alpha-1}.$$

Also, since

(3.5) $$c_k = k\int_{1-r}^{1-r/2} x^{k-1}\,dx \leqslant k(1-r/2)^{k-1}r/2$$

and $c_0=0$,

$$S_2 \leqslant C\sum_{l=1}^{\infty}\Big(\sum_{k=2^{l-1}}^{2^l-1}|c_k|^2|\Delta^\gamma m_k|^2 k^{2\alpha+1}\Big)^{1/2}$$

$$\leqslant C\sum_{l=1}^{\infty}2^{l(\alpha+1/2)}\Big(\sum_{k=2^{l-1}}^{2^l-1}|k(1-r/2)^{k-1}r\Delta^\gamma m_k|^2\Big)^{1/2}$$

$$\leqslant Cr\|m\|_{2,\gamma;w}\sum_{l=1}^{\infty}(2^l)^{\alpha-\gamma+2}(1-r/2)^{2^{l-1}}$$

$$\leqslant C\|m\|_{2,\gamma;w}r^{\gamma-\alpha-1}$$

by [6], bottom of p. 67. Clearly, $|\Delta c_j| \leqslant Cr(1-r/2)^j$ and so

$$S_3 \leqslant Cr\|m\|_\infty \Big(\sum_{k=0}^{\infty}\Big|\sum_{n=k+1}^{\infty}(n-k)^{-\gamma-1}\sum_{j=k+1}^{n}(1-r/2)^j\Big|^2 k^{2\alpha+1}\Big)^{1/2}$$
$$= Cr\|m\|_\infty \Big(\sum_{k=0}^{\infty}\Big|\sum_{j=k+1}^{\infty}(1-r/2)^j\sum_{n=j}^{\infty}(n-k)^{-\gamma-1}\Big|^2 k^{2\alpha+1}\Big)^{1/2}$$
$$\leqslant Cr\|m\|_\infty \Big(\sum_{k=0}^{\infty}(1-r/2)^{2k}\Big|\sum_{j=1}^{\infty}j^{-\gamma}(1-r/2)^j\Big|^2 k^{2\alpha+1}\Big)^{1/2}$$
$$\leqslant C\|m\|_\infty r^{\gamma-\alpha-1}$$

as in [6], p. 67.

Now consider the case $\gamma > 1$. From Peyerimhoff [15], p.3,

$$\Delta^\gamma m_k c_k = m_k \Delta^\gamma c_k + \sum_{j=0}^{[\gamma]}\binom{\gamma}{j}\Delta^j c_k \Delta^{\gamma-j} m_{k+j} + R_k$$

with

$$R_k = (-1)^{[\gamma]}\sum_{n=k+1+[\gamma]}^{\infty} A_{n-k}^{-\gamma-1}(m_n - m_k)\sum_{j=k+1}^{n-[\gamma]} A_{n-[\gamma]-j}^{[\gamma]-1}(\Delta^{[\gamma]}c_j - \Delta^{[\gamma]}c_k).$$

So, applying Minkowski's inequality and using a notation like that in (3.3), we have

$$S \leqslant S_1 + \sum_{j=0}^{[\gamma]} S_{2,j} + S_3 .$$

Again

$$S_1 \leqslant C\|m\|_\infty \Big(\sum_{k=0}^{\infty} r^{2\gamma}(1-r/2)^{2k}k^{2\alpha+1}\Big)^{1/2} \leqslant C\|m\|_\infty r^{\gamma-\alpha-1}.$$

If $j \geqslant 1$, then $\Delta^j c_k \leqslant Cr^j(1-r/2)^k$ by (3.4) and so

$$S_{2,j} \leqslant Cr^j \sum_{l=0}^{\infty}\Big(\sum_{k=2^l-1}^{2^{l+1}-2}(1-r/2)^{2k}|(k+j)^{\gamma-j}\Delta^{\gamma-j}m_{k+j}|^2 k^{2j-2\gamma+2\alpha+1}\Big)^{1/2}$$
$$\leqslant Cr^j \sum_{l=0}^{\infty}(1-r/2)^{2^l-1}2^{l(\alpha+1-\gamma+j)}\|m\|_{2,\gamma-j;w}$$
$$\leqslant C\|m\|_{2,\gamma;w} r^{\gamma-\alpha-1}$$

as before. Similarly the case $j = 0$ follows by using (3.5). Hence it remains to estimate the remainder term S_3. We have, employing two changes in order of summation,

$$|R_k| \leqslant C\sum_{n=k+1+[\gamma]}^{\infty}(n-k)^{-\gamma-1}\|m\|_\infty \sum_{j=k+1}^{n-[\gamma]}(n-[\gamma]-j)^{[\gamma]-1}\sum_{i=k}^{j-1}|\Delta^{[\gamma+1]}c_i|$$
$$\leqslant Cr^{[\gamma]+1}\|m\|_\infty \sum_{n=k+1+[\gamma]}^{\infty}(n-k)^{-\gamma-1}\sum_{j=k+1}^{n-[\gamma]}(n-[\gamma]-j)^{[\gamma]-1}\sum_{i=k+1}^{j}(1-r/2)^i$$
$$\leqslant Cr^{[\gamma]+1}\|m\|_\infty \sum_{n=k+1+[\gamma]}^{\infty}(n-k)^{-\gamma-1}\sum_{i=k+1}^{n-[\gamma]}(1-r/2)^i\sum_{j=i}^{n-[\gamma]}(n-[\gamma]-j)^{[\gamma]-1}$$
$$\leqslant Cr^{[\gamma]+1}\|m\|_\infty \sum_{n=k+1+[\gamma]}^{\infty}(n-k)^{-\gamma-1}\sum_{i=k+1}^{n-[\gamma]}(n-[\gamma]-i)^{[\gamma]}(1-r/2)^i$$
$$= Cr^{[\gamma]+1}\|m\|_\infty \sum_{i=k+1}^{\infty}(1-r/2)^i\sum_{n=i+[\gamma]}^{\infty}(n-k)^{-\gamma-1}(n-[\gamma]-i)^{[\gamma]}$$
$$\leqslant Cr^{[\gamma]+1}\|m\|_\infty \sum_{i=k+1}^{\infty}(1-r/2)^i\sum_{n=i+[\gamma]}^{\infty}(n-k)^{[\gamma]-\gamma-1}$$
$$\leqslant Cr^{[\gamma]+1}\|m\|_\infty \sum_{i=k+1}^{\infty}(i+[\gamma]-k)^{[\gamma]-\gamma}(1-r/2)^i$$
$$\leqslant C\|m\|_\infty(1-r/2)^k r^\gamma$$

and so, finally,

$$S_3 \leqslant C\Big(\sum_{k=0}^{\infty}|R_k|^2 k^{2\alpha+1}\Big)^{1/2}$$
$$\leqslant C\|m\|_\infty r^\gamma\Big(\sum_{k=0}^{\infty}(1-r/2)^{2k}k^{2\alpha+1}\Big)^{1/2} \leqslant C\|m\|_\infty r^{\gamma-\alpha-1}.$$

4. Proof of Theorem 2. Part (a) follows by interpolating between Theorem 1 and $l^\infty = M_2^2$, using the same methods as in Connett and Schwartz [8] and Gasper and Trebels [11], Sec. 8.

For part (b) first observe that, by [11], Theorem 5(e), $\mathrm{wbv}_{2,\gamma_0} \subset l^\infty$, $\gamma_0 > 1/2$. Part (b) then follows by interpolating between

$$\mathrm{wbv}_{2,\gamma_0} \subset M_2^2$$

and (by Theorem 1)

$$\mathrm{wbv}_{2,\gamma_1} \subset M_p^p$$

with $\gamma_1 > \alpha+1$ and $p > 1$ near 1.

Remarks (i) Analogous results may be derived for M_p^q-multipliers for Jacobi expansions and for Hankel transforms with $1 < p < q < \infty$, see [11] and [12].

(ii) As already mentioned in the Introduction Theorem 1 improves the multiplier criteria of Hörmander type derived in Connett and Schwartz [7] for Jacobi expansions (and even extends Theorem 9.2 in [8] for ultraspherical multipliers to all $\lambda > 0$). In a letter to one of the authors Connett and Schwartz claimed that they have extended their proof in [8] to give (1.2) for $\gamma \geqslant \alpha+1$ in the special case $\alpha \geqslant 0$. Also let us note that on account of Theorem 5 (c) in [11], Theorem 1 contains the Marcinkiewicz type multiplier criteria in [10].

(iii) As in Connett and Schwartz [8], p. 80, part (a) of Theorem 2 gives the correct multiplier range of the sequence $\{e^{ik^\nu}k^{-\gamma\nu}\}$ when $\alpha = \beta > -1/2$ and $|1/p - 1/2| > 1/q$.

(iv) For $\gamma < \alpha+1$ part (b) of Theorem 2 can be slightly improved by using the lower end-point result

$$\mathrm{wbv}_{2,\gamma_0} \subset M_p^p, \quad \gamma_0 > 1/2, \ 1 \leqslant \frac{2\alpha+2}{\alpha+1+\gamma_0/2} < p < \frac{2\alpha+2}{\alpha+1-\gamma_0/2} \leqslant \infty$$

contained in [11], Theorem 6 (b). There are strong reasons to conjecture that for $1/2 < \gamma_0 < 1$ the latter result can be improved to

$$\mathrm{wbv}_{2,\gamma_0} \subset M_p^p, \quad 1 \leqslant \frac{2\alpha+2}{\alpha+3/2} < p < \frac{2\alpha+2}{\alpha+1/2} \leqslant \infty.$$

Here the main difficulty is to obtain multiplier criteria of Hörmander type ($q = 2$) for weighted cosine expansions. For the Marcinkiewicz ($q = 1$) case, see Askey and Wainger [3] and Askey [1], p. 589.

Added in proof. The latter conjecture is established among other results in G. Gasper and W. Trebels, Math. Ann. 242 (1979), pp. 225–240.

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