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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SOUTH CAROLINA

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# Divisible subspaces and problems of automatic continuity*

by

W. G. BADÉ (Berkeley, Calif.), P. C. CURTIS, Jr., (Los Angeles, Calif.) and K. B. LAURSEN (Copenhagen)

**Abstract.** This paper explores the role of divisible subspaces for linear operators in the theory of automatic continuity of homomorphisms between Banach algebras.

**§ 0. Introduction.** In 1972 Graham Allan [1] showed that the algebra of formal power series could be embedded in a commutative Banach algebra $B$ if and only if there exists an element $x$ in the radical of $B$ satisfying

$$\overline{Bx^{m+1}} \supseteq Bx^m \neq 0$$

for some positive integer $m$. He went on to observe that if this condition is satisfied, then there exists a discontinuous homomorphism $\theta$ from the disc algebra $A(\Delta)$ into the radical of $B$ with unit adjoined which maps the coordinate function $z$ onto $x$. To construct the mapping $\theta$, Allan first shows that if $x$ satisfies the above condition, then there exists a non-zero subspace $Z$ in rad $B$ such that for every $\lambda \in C$, the complex plane,

$$(x-\lambda)Z = Z.$$

The subspace $Z$, necessarily non-closed, is called *x-divisible*, and the existence of such a subspace is the essential tool in Allan's construction of $\theta$. Indeed in a later paper [29] M. Thomas observes that the existence of an $x$-divisible subspace $Z$ in a radical algebra $R$ is a necessary and sufficient condition for there to exist a discontinuous functional calculus for the element $x$. Since $\sigma(x) = \{0\}$, this is equivalent to the assertion that there exists a discontinuous homomorphism of the disc algebra into the algebra $R$ with unit adjoined which carries $z$ onto $x$.

In this paper we shall investigate the relation between the existence of divisible subspaces for elements $x$ in a commutative Banach algebra $B$ and the existence of discontinuous homomorphisms from certain commutative Banach algebras into B. Johnson and Sinclair [20] were the first to investigate the relationship between automatic continuity questions

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and the existence of divisible subspaces. The techniques employed here are extensions of those developed in their paper.

If $A = C(\Omega)$, and $\nu$ is a homomorphism of $A$ into $B$, the possible discontinuity of $\nu$ is completely determined by the divisible subspaces in $B$. Specifically, we prove in Section 2 that $\nu$ is discontinuous if and only if for some $f \in C(\Omega)$, $\nu(f)$ has a divisible subspace in $B$. Such a subspace necessarily lies in the radical $R$ of $B$. Furthermore if $\nu$ is discontinuous, there must exist elements $y \in R$ with divisible subspaces.

Examples of discontinuous homomorphisms of $C(\Omega)$ have been constructed by Dales [14] and Esterle [17] under the assumption of the continuum hypothesis. Other set theoretic models have been constructed by Woodin [31], [32] and Solovay [27] in which discontinuous homomorphisms of $C(\Omega)$ do not exist. On the other hand if $\nu$ is a discontinuous homomorphism of $A(\Delta)$ into $B$ which carries $z$ onto $b$, it is not known if $b$ must necessarily have a $b$-divisible subspace in the radical of $B$. However, this is the case for all known examples of such mappings. A discontinuous homomorphism $\nu$ need not map $A(\Delta)$ into the radical of $B$ with unit adjoined, and it may happen that $(\operatorname{rad} B)^2 = \{0\}$, so elements of the radical need not have divisible subspaces.

Section 1 contains the necessary aspects of the theory of divisible subspaces including the Mittag-Leffler Theorem [10] and the stability lemma [26], p. 11. These results are applied to continuity problems of algebra homomorphisms in Section 2. Two main results are the following.

THEOREM. *If $A$ is a Banach algebra with a bounded left approximate identity, $B$ is a commutative radical Banach algebra and $\nu$ is a non-zero homomorphism of $A$ into $B$, then for some $a \in A$, $\nu(a)$ has a divisible subspace.*

The continuity theorem for homomorphisms of $C(\Omega)$ mentioned above is a consequence of this result.

THEOREM. *Let $B$ be a commutative Banach algebra with scattered spectrum. A necessary and sufficient condition for every homomorphism from any Banach algebra $A$ into $B$ to be continuous is that $B$ have no non-trivial nilpotent elements and that for each $b \in B$ there exist no $b$-divisible subspaces in $B$.*

A locally compact Hausdorff space is *scattered* if it contains no compact perfect subsets. Furthermore a locally compact space is scattered if and only if each continuous function on it vanishing at infinity has countable range [23], [24]. Pełczyński and Semadeni in [23] call these spaces *dispersed*.

If the algebras $A$ are assumed to be separable and satisfy the condition that $\operatorname{co}(M^2) < \infty$ for each maximal modular ideal of codimension one, then this theorem remains true without the assumption about nilpotent elements. If the algebras $A$ are assumed to be Silov algebras, regular commutative semi-simple Banach algebras which are normal on their maximal ideal spaces, then the assumption that the maximal ideal space of $B$ is scattered may be dropped.

In Section 3 we study the continuity structure of a homomorphism from $A = C^n[0,1]$, the $n$-times continuously differentiable functions on $[0,1]$ into an arbitrary Banach algebra $B$. In [8] it was shown that if $\nu\colon C^n \to B$ is a homomorphism which is continuous on $C^k \subset C^n$ for some $k > n$ (in the $C^k$ norm), then $\nu$ is necessarily continuous on $C^{2n+1}$. We use the phrase eventual continuity to describe this situation. We prove that $\nu\colon C^n \to B$ is eventually continuous if and only if $\nu(z)$ has no non-trivial divisible subspace. It follows as a consequence that if $\nu\colon C^n \to B$ is continuous on $C^\infty$ with its usual Fréchet topology, then $\nu$ is continuous on $C^{2n+1}$.

Since a divisible subspace must lie in the radical of $B$, it follows that every homomorphism $\nu\colon C^n \to B$ is $C^{2n+1}$-continuous if $B$ has finite dimensional radical, a result proved also in [8]. We prove here the stronger result that if $\dim \operatorname{rad}(B) < \infty$, then $\nu$ is $C^{2n}$-continuous. It is not known whether every eventually continuous homomorphism of $C^n$ is continuous on $C^{2n}$. We close with a discussion of this problem.

We list now the basic concepts and facts needed in the sequel. For proofs we refer to Allan Sinclair's book [26], in particular Section 1.1.

0.1. DEFINITION. If $T\colon A \to B$ is a linear map and $A, B$ are Banach spaces, then the *separating space* of $T$, $\mathcal{S}(T)$ is defined by

$$\mathcal{S}(T) = \{y \in B \mid \text{there exists } \{x_n\} \subset A,\ x_n \to 0 \text{ for which } Tx_n \to y\}.$$

This space measures the discontinuity of $T$ because $\mathcal{S}(T) = \{0\}$ if and only if $T$ is continuous, by the closed graph theorem.

0.2. LEMMA. *Let $A$, $B$, $C$, $D$ be Banach spaces, $T\colon A \to B$ be continuous, $S\colon B \to C$ be linear and $R\colon C \to D$ be continuous. Then*

(i) *$\mathcal{S}(S)$ is a closed linear subspace of $C$;*

(ii) $\mathcal{S}(ST) \subseteq \mathcal{S}(S)$;

(iii) $(R\mathcal{S}(S))^- = \mathcal{S}(RS)$ ($^-$ denotes norm closure);

(iv) *$RS$ is continuous if and only if $R\mathcal{S}(S) = \{0\}$.*

0.3. DEFINITION. If $A$ is a Banach algebra, $B$ a Banach space and $T\colon A \to B$ a linear map, then the *continuity ideal* of $T$, $\mathcal{I}(T)$, is defined by

$$\mathcal{I}(T) = \{x \in A \mid y \to T(xy) \text{ is continuous}\}.$$

0.4. DEFINITION. Let $X$ be a Banach space and $T$ be a continuous linear operator on $X$. We say that a subspace $D \subseteq X$ is *$T$-divisible* if

$$(T-\lambda)D = D, \quad \text{for each } \lambda \in \boldsymbol{C}.$$

Among all such spaces $D$ there is a largest, the algebraic span of all $T$-divisible subspaces of $X$. A divisible subspace is never closed unless it is the zero subspace [20]. In most situations under study here, the space $X$ will be a commutative Banach algebra and $T$ will be multiplication by an element of the algebra.

0.5. Remark. We note that the largest $T$-divisible subspace $D$ of $X$ is characterized by $D$ being maximal with respect to $D=(T-\lambda)D$ for every $\lambda$ in the spectrum $\sigma(T)$ of $T$. To see this it suffices to show that $(T-\mu)D=D$ for every $\mu$ in the resolvent set $\varrho(T)$. Since $(T-\lambda)(T-\mu)^{-1}D$ $=(T-\mu)^{-1}(T-\lambda)D=(T-\mu)^{-1}D$ for any $\lambda\in\sigma(T)$, $\mu\in\varrho(T)$, it follows that $(T-\mu)^{-1}D\subseteq D$ by the maximality of $D$. Hence $D\subseteq(T-\mu)(T-\mu)^{-1}D$ $\subseteq(T-\mu)D\subseteq D$.

To study a discontinuous homomorphism $\nu$ one must also investigate the continuity properties of mappings $\nu_0(x)=\nu(a_0x)$ where $a_0$ is a fixed element in the domain of $\nu$. To do this it is convenient to introduce a somewhat wider class of mappings, the class $\mathscr{I}$ of generalized intertwining operators introduced in [7], § 2. We list here the basic facts about $\mathscr{I}$.

Let $A$ be a Banach algebra and $M$ be a left $A$-module with module action $\varrho\colon A\to\mathscr{B}(M)$, the bounded linear operators on $M$. We do not assume the homomorphism $\varrho$ to be continuous. For technical reasons we assume $M$ to be commutative, i.e.

$$\varrho(ab)m=\varrho(ba)m$$

for all $a,b\in A$, $m\in M$.

0.6. Definition. A linear map $S\colon A\to M$ is *of class* $\mathscr{I}$ if for every $a\in A$ the map $b\to S(ab)-\varrho(a)S(b)$ is continuous and if for all $a,b\in A$ we have $S(ab)=S(ba)$.

Let $\nu$ be a homomorphism of a Banach algebra $A$ into a commutative Banach algebra $B$ and assume $B=\overline{\nu(A)}$. Set $M=B$ and define $\varrho\colon A\to B$ $\to\mathscr{B}(B)$ by $\varrho(a)b=\nu(a)b$. If $a_0$ is fixed in $A$, and $S_0(x)=\nu(a_0x)$, then it is immediate [7] that $S_0$ is an operator of class $\mathscr{I}$. This class includes derivations, and other operators as well, but we shall be concerned here only with homomorphisms and "multiplied homomorphisms".

The following fact about maps of class $\mathscr{I}$ will be used repeatedly without specific references.

0.7. Lemma. *Let $A$ be a Banach algebra and $M$ be a left $A$-module with module action $\varrho$. If $S\colon A\to M$ is a map of class $\mathscr{I}$ with separating space $\mathscr{S}$ and continuity ideal $\mathscr{I}(S)$, then*

$$\mathscr{I}(S)=\{a\in A\mid\ \varrho(a)\mathscr{S}=\{0\}\}.$$

Proof. By Definition 0.3 $a\in\mathscr{I}(S)$ iff $b\to S(ab)$ is continuous, hence by definition of class $\mathscr{I}$, $a\in\mathscr{I}(S)$ iff $b\to\varrho(a)S(b)$ is continuous. The latter is the case iff $\varrho(a)\mathscr{S}=\{0\}$, by Lemma 0.2 (iv).



Let $B$ be a commutative Banach algebra and $I$ be an ideal in $B$. If $K\subseteq B$, we say $I$ is *totally reduced* by $K$ if there are no non-trivial divisible subspaces in $I$ for the operation of multiplication by elements of $K$, i.e. if $b\in K$, $D\subseteq I$ and $(b-\lambda)D=D$ for all $\lambda\in\boldsymbol{C}$, then $D=\{0\}$. We will say $I$ is totally reduced if $I$ is totally reduced by itself.

Some remarks are in order. The proofs are immediate.

(i) Let $R=\operatorname{rad}B$. In view of Lemma 1.4 the statements "$B$ is totally reduced" and "$R$ is totally reduced by $B$" are equivalent.

(ii) If $B$ is totally reduced and $B'$ is formed by adjoining an identity to $B$, then $B'$ is totally reduced.

(iii) If $B$ is semi-simple or has finite dimensional radical, then $B$ is totally reduced.

(iv) If $R$ is a radical Banach algebra and $R$ is nilpotent or satisfies $\bigcap_{n=0}^{\infty}r^nR=\{0\}$ for each $r\in R$, then $R$ is totally reduced. This follows from the observation that if $D$ is $r$ divisible, then $rD=D$, so

$$D=\bigcap_{n=1}^{\infty}r^nD\subseteq\bigcap_{n=1}^{\infty}r^nR=\{0\}.$$

We show by an example at the end of Section 2 that a radical Banach algebra can be totally reduced but $\bigcap_{n=1}^{\infty}r^nR\neq\{0\}$ for some $r\in R$.

(v) If $R$ is the radical of a Banach algebra $B$, then $R$ may be totally reduced but $B$ may not be. This situation arises in Dales' construction of a discontinuous homomorphism of the disc algebra [13]. We discuss the example in Section 2. However, if $\Phi_B$ is finite, and $R$ is totally reduced, then $B$ is also.

**§ 1.** Much of the usefulness of the notion of divisible subspaces derives from the basic result that is known as the Mittag–Leffler Theorem [10]. We quote here the version proved in [26].

1.1. Theorem. *Let $\{T_n\}$ be a commuting sequence of continuous linear operators on a Banach space $X$. Denote by $X_\infty$ the largest subspace of $X$ for which $T_nX_\infty=X_\infty$ for each $n$, and $X^\infty$ the largest closed subspace of $X$ for which $\overline{T_nX^\infty}=X^\infty$ for each $n$. Then $X^\infty=\overline{X}_\infty$.*

1.2. Remark. Although this result does not directly mention divisible subspaces, the connection is obtained as follows. Recall that if $T\in\mathscr{B}(X)$, the largest $T$-divisible subspace $D$ of $X$ is characterized by $D$ being maximal with respect to $D=(T-\lambda)D$ for every $\lambda$ in the spectrum $\sigma(T)$ of $T$. If $T$ has countable spectrum $\{\lambda_n\}$ and $W$ is the largest closed subspace of $X$ for which $\overline{(T-\lambda_n)W}=W$ for every $n$, then by the Mittag–Leffler Theorem $D$ is dense in $W$.

Another important consequence of this theorem is the following result. A more general formulation is contained in Corollary 1.9 of [26].

1.3. COROLLARY. *Let $X$ and $Y$ be Banach spaces and $T\colon X\to X$, $R\colon Y\to Y$ be continuous linear operators, with $R$ not a scalar multiple of the identity. Let $S\colon X\to Y$ be a linear operator for which $RS-ST\colon X\to Y$ is continuous.*

*If $\sigma(R|\mathscr{S}(S))$ is countable, there exists a non-constant polynomial $p$ whose roots lie in $\sigma(R|\mathscr{S}(S))$ such that $p(R)\mathscr{S}(S)\subseteq W(\mathscr{S}(S))$, where $W(\mathscr{S}(S))$ is the closure of the largest $R$-divisible subspace of the separating space $\mathscr{S}(S)$ of $S$.*

Proof. Since $RS-ST$ is continuous, $\overline{R\mathscr{S}(S)}=\mathscr{S}(RS)=\mathscr{S}(ST)\subseteq\mathscr{S}(S)$. By the Mittag-Leffler Theorem and the remark above it suffices to show the existence of a non-zero polynomial $p$ such that $p(R)\mathscr{S}(S)=(R-\gamma)(p(R)\mathscr{S}(S))^{-}$ for any $\gamma\in\sigma(R)$. If $\sigma(R)=\{\lambda_n\}$ arrange $\{\lambda_n\}$ in a sequence $\{\mu_m\}$ so that each $\lambda_m$ occurs infinitely often, let $T_m=T-\mu_m$, $R_m=R-\mu_m$ and use the Stability Lemma [26] to conclude that there exists a constant $M$ so that $m\geqslant M$ implies

$$[R_1\ldots R_m\mathscr{S}(S)]^{-}=[R_1\ldots R_M\mathscr{S}(S)]^{-}.$$

Let $p(R)=R_1\ldots R_M$ and let $\gamma\in\sigma(R)$. Then if $\gamma=\mu_m$ with $m>M$,

$$\begin{aligned}[(R-\gamma)(p(R)\mathscr{S}S)]^{-}&=[(R-\gamma)p(R)\mathscr{S}(S)]^{-}\\&=[R_mR_1\ldots R_M\mathscr{S}(S)]^{-}=[R_mR_1\ldots R_M\ldots R_{m-1}\mathscr{S}(S)]^{-}\\&=[R_1\ldots R_m\mathscr{S}(S)]^{-}=[R_1\ldots R_M\mathscr{S}(S)]^{-}\\&=[p(R)\mathscr{S}(S)]^{-}.\end{aligned}$$

In the case that $X$ is a commutative Banach algebra and $T$ is multiplication by an element of $X$, then divisible subspaces are ideals in the radical of $X$. The result is due to Marc Thomas and the proof is in [29].

1.4. LEMMA. *Let $B$ be a commutative Banach algebra, let $b\in B$ and let $D$ be the largest $b$-divisible subspace of $B$. Then $D$ is an ideal contained in the radical of $B$.*

1.5. LEMMA. *Let $A$ be a commutative semi-simple Banach algebra and $\nu\colon A\to B=\overline{\nu(A)}$ be an isomorphism with bounded inverse, i.e. there exists $M>0$ such that $\|\nu(x)\|\geqslant M\|x\|$ for every $x\in A$.*

*If $B$ is semi-simple, then $\nu$ is continuous. If $B$ is not semi-simple, then $\nu$ is discontinuous, and $\operatorname{rad}B=\mathscr{S}(\nu)$. Furthermore $B=\nu(A)\oplus\mathscr{S}(\nu)$.*

Proof. Let $R=\operatorname{rad}B$. We note that $\nu(A)\cap R=\{0\}$, because if $0\neq x\in A$, then

$$\|\nu(x^n)\|\geqslant M\|x^n\|\geqslant M\|\hat{x}^n\|_\infty=M\|\hat{x}\|_\infty^n,$$

where $\hat{x}\in C(\Phi_A)$ is the Gelfand transform. Hence the spectral radius of $\nu(x)$ is greater than $\|\hat{x}\|_\infty>0$, and therefore $\nu(x)\notin R$.



Next suppose $\{x_n\}\subset A$ and $\nu(x_n)\to b\in B=\overline{\nu(A)}$. Since $\|x_n-x_m\|\leqslant M^{-1}\|\nu(x_n-x_m)\|$, $\{x_n\}$ is a Cauchy sequence in $A$. If $x_n\to x_0$, then $x_n-x_0\to 0$ and $\nu(x_n-x_0)\to b-\nu(x_0)\in\mathscr{S}(\nu)$. Since $b=\nu(x_0)+(b-\nu(x_0))$, this shows that $\mathscr{S}(\nu)=R$ and $B=\overline{\nu(A)}=\nu(A)\oplus\mathscr{S}(\nu)$. Furthermore if $R=\{0\}$, then $\mathscr{S}=\{0\}$ and the first assertion follows.

**§ 2.** In this section we apply the results of Section 1 on divisible subspaces to obtain automatic continuity theorems for homomorphisms between Banach algebras. As an immediate consequence of Corollary 1.3 we have

2.1. THEOREM. *Let $A$ and $B$ be Banach algebras with $B$ commutative. Let $\nu\colon A\to B$ be a discontinuous homomorphism from $A$ into $B$. Let $a_0\in A$ be such that $\nu_0(x)=\nu(a_0)\nu(x)$ is discontinuous with continuity ideal $\mathscr{I}_0(\nu)$ and separating space $\mathscr{S}_0$. Suppose $\Phi_B$ is scattered and suppose $B$ is totally reduced by $\nu(A)$.*

*Then for every $a\in A$, not a complex multiple of the identity if $A$ is unital, there exists a non zero polynomial $p_a=p$, whose roots lie in the spectrum of $\nu(a)$ considered as a multiplication operator in $\mathscr{S}(\nu_0)$, such that $p(a)\in\mathscr{I}(\nu_0)$. The same conclusions hold for $\mathscr{I}(\nu)$ if $A$ is not unital.*

Proof. Referring to Corollary 1.3 let $X=A$, $Y=B$ and let $a\in A$. Let $T$ be left multiplication by $a$ and $R$ be multiplication by $\nu(a)$. Then with $S=\nu_0$ we have

$$b\to\nu(a)\nu_0(b)-\nu_0(ab)$$

continuous. Next we assert $\sigma(R|\mathscr{S}(\nu_0))$ is countable. To show this it is enough to show $\Phi_C$ is scattered, where $C=\overline{\nu(A)}$ in $B$. By adjoining a unit if necessary we may assume $B$ and $C$ to be unital. If $\varphi\in\partial C$, the Silov boundary of $C$, then by a well-known theorem of Silov, $\varphi$ extends to a homomorphism on $B$. Therefore if $i^*$ is the dual mapping of the injection of $C$ into $B$, then $i^*(\Phi_B)\supset\partial C$ and by [24], Theorem 1, $\partial C$ is scattered. Let $\tilde{C}$ be the uniform closure of $C$ on $\Phi_C$. Then $\Phi_{\tilde{C}}=\Phi_C$ and $\partial\tilde{C}=\partial C$. Consequently by [24], Theorem 4, $C$ is the algebra of all continuous functions on $\partial C$, and as a result $\Phi_C=\partial C$.

Now there exists by Corollary 1.3 a non zero polynomial $p$ such that

$$\nu(p(a))\mathscr{S}_0\subset W(\mathscr{S}_0)$$

the closure of the largest $\nu(a)$-divisible subspace of $\mathscr{S}(\nu)$. Since $B$ is totally reduced by $\nu(A)$, $W(\mathscr{S}_0)=\{0\}$ so $\nu(p(a))\mathscr{S}_0=\{0\}$, hence $p(a)\in\mathscr{I}(\nu_0)$ by Lemma 0.6. Replacing $\nu_0$ by $\nu$ in the above argument gives us the last assertion.

The next result, known as the prime ideal theorem, is from [7], Theorem 2.7. We have altered the phrasing to suit our present purposes, but the proof is essentially unchanged, so is omitted here.

2.2. Proposition (Prime Ideal Theorem). *Let $A$ and $B$ be Banach algebras. Assume $B$ to be commutative and $A$ to have an identity. Let $\nu\colon A\to B$ be a homomorphism with separating space $\mathcal{S}\neq\{0\}$ and continuity ideal $\mathcal{I}(\nu)$.*

*If $A/\mathcal{I}(\nu)$ is an integral domain, let $\nu_0=\nu$; if not, there exists $a_0\in A$ such that if $\nu_0(x)=\nu(a_0x)$, then $\nu_0$ is discontinuous, and if $\mathcal{S}_0$ and $\mathcal{I}(\nu_0)$ are the separating space and continuity ideal for $\nu_0$, respectively, then*

(i) *for every $a\in A$ $\big(\nu(a)\mathcal{S}_0\big)^-=\{0\}$ or $=\mathcal{S}_0$;*

(ii) *$\mathcal{I}(\nu_0)\supseteq\mathcal{I}(\nu)$ and $A/\mathcal{I}(\nu_0)$ is an integral domain.*

The result enables us to extract further information from 2.1.

2.3. Theorem. *Let $A$, $B$, and $\nu$ be as in Theorem 2.1. Then either $\mathcal{S}(\nu)^2=\{0\}$ or there exists $a_0\in A$ such that if $\nu_0(x)=\nu(a_0x)$, then $\nu_0$ is discontinuous and $\mathcal{S}(\nu_0)^2=\{0\}$.*

Proof. Suppose that $A/\mathcal{I}(\nu)$ is an integral domain. We assert $\mathcal{I}(\nu)$ is a maximal ideal of codimension one. Since $\mathcal{I}(\nu)$ is an ideal, $1\notin\overline{\mathcal{I}(\nu)}$. Therefore, there exists a continuous linear functional $a$ on $A$ such that $a\big(\mathcal{I}(\nu)\big)=0$ and $a(1)=1$. Let $a(a)=0$. By Theorem 2.1 there exists a non-zero polynomial $p$ such that $p(a)\in\mathcal{I}(\nu)$. Since $A/\mathcal{I}(\nu)$ is an integral domain, $a-\lambda\in\mathcal{I}(\nu)$ for some $\lambda\in\boldsymbol{C}$. But $0=a(a-\lambda)=-\lambda$, and $a\in\mathcal{I}(\nu)$. Since $\mathcal{I}(\nu)$ is closed and has codimension one, $\mathcal{S}(\nu)\subset\overline{\nu\big(\mathcal{I}(\nu)\big)}$. Therefore $\mathcal{S}(\nu)^2=\{0\}$. If $A/\mathcal{I}(\nu)$ is not an integral domain, then there exists $a_0$ such that $\nu_0(x)=\nu(a_0x)$ has a continuity ideal $\mathcal{I}(\nu_0)\supset\mathcal{I}(\nu)$ for which $A/\mathcal{I}(\nu_0)$ is an integral domain. Applying the same argument to $\nu_0$ we see that $\mathcal{I}(\nu_0)$ is maximal, and $\mathcal{S}(\nu_0)^2\subset\mathcal{S}(\nu)\cdot\mathcal{S}(\nu_0)=\{0\}$.

Immediate continuity results are available from Theorem 2.3 for the class of semi-prime Banach algebras. Recall that a Banach algebra is *semi-prime* if it contains no two-sided nilpotent ideals. A commutative Banach algebra is semi-prime if and only if it contains no nilpotent elements.

A class of examples of radical semi-prime commutative Banach algebras is provided by the weighted $L^1$-algebras on $\boldsymbol{R}_+$, usually denoted by $L^1(\boldsymbol{R}_+,a)$, where $a\colon\boldsymbol{R}_+\to\boldsymbol{R}_+$ satisfies $a(t_1+t_2)\leqslant a(t_1)a(t_2)$, $t_1,t_2\in\boldsymbol{R}_+$. The multiplication is by convolution and

$$\|f\|=\int_{\boldsymbol{R}_+}|f(t)|\,a(t)\,dt$$

provided $|f|a$ is Lebesgue-integrable. If $a(t)$ is rapidly decreasing, say $a(nt)^{1/n}\to 0$ as $n\to\infty$, then $L^1(\boldsymbol{R}_+,a)$ is radical. Moreover, the Titchmarsh Theorem [28], p. 324, shows that $L^1(\boldsymbol{R}_+,a)$ contains no nilpotent elements. More elaborate examples of radical, semi-prime Banach algebras may be found in [12].



2.4. Theorem. *Let $B$ be a commutative Banach algebra with scattered maximal ideal space. A necessary and sufficient condition that every homomorphism from any Banach algebra into $B$ is continuous is that $B$ is totally reduced and semi-prime.*

Proof. First assume there exists $b\in B$ with countable spectrum $\sigma(b)$ having a divisible subspace in $B$. Let $\{V_i\}$ be a sequence of compact neighborhoods of $\sigma(b)$ such that $\bigcap V_i=\sigma(b)$. Let $\mathcal{O}$ be the algebra of germs of analytic functions on $\sigma(b)$, and assume that the inductive limit topology in $\mathcal{O}$ is defined by the sequence of Banach algebras $A(V_i)$ of those continuous functions on $V_i$ holomorphic on $\operatorname{int}V_i$. There exists by [29], Theorem 2.11, a homomorphism $\nu$ of $\mathcal{O}$ into $B$ such that $\nu(z)=b$ and $\nu$ is discontinuous with respect to the inductive limit topology in $\mathcal{O}$. On the other hand, $\nu$ induces a homomorphism $\nu_i\colon A(V_i)\to B$, $\nu_i(z)=b$. Furthermore, by [16], p. 433–434, $\nu_i$ must be discontinuous for some $i$.

On the other hand if $B$ has nilpotent elements, then it is easy to verify that there exists a discontinuous homomorphism from the algebra of $C^1$ functions on $[0,1]$ vanishing at 0 into $B$. Conversely if $B$ is totally reduced and $\nu\colon A\to B$ is a discontinuous homomorphism from a Banach algebra $A$, then we may infer from Theorem 2.3 that $\mathcal{S}(\nu)$ contains non zero elements which are nilpotent of order two. This contradicts the fact that $B$ is semi-prime.

An intriguing conjecture is that the assumption that $\Phi_B$ is scattered may be dropped in the above result. Some evidence in this direction is furnished by Theorem 2.13.

Automatic continuity results may be obtained if the assumption that the range algebra is semi-prime is replaced by a restriction on the maximal ideals of $A$ of codimension one.

2.5. Theorem. *Let $A$ and $B$ be unital Banach algebras, with $B$ commutative. Suppose that $A$ is separable, and $M^2$ has finite codimension for every maximal ideal $M$ of codimension one. Let $\nu$ be a homomorphism of $A$ into $B$ and assume that $\overline{\nu(A)}$ is totally reduced by $\nu(A)$. If $\Phi_B$ is scattered, then $\nu$ is continuous.*

Proof. If $\nu$ is discontinuous, then the proof of 2.3 shows that either $\mathcal{I}(\nu)$ or $\mathcal{I}(\nu_0)$ is a maximal ideal of codimension one. This contradicts [7], Theorem 2.8.

In certain cases the only possible homomorphism turns out to be the zero homomorphism. We need first the following generalization of a theorem of G. Allan [2].

2.6. Proposition. *Let $B$ be a commutative Banach algebra with a bounded approximate identity. Then there exist a non-zero element $b\in B$ and a non-zero subspace $D$ of $B$ for which*

$$bD=D.$$

*If $B$ is in addition a radical algebra, then there exists a divisible subspace for $b$.*

Proof. In view of Remark 0.5 we need only prove the first assertion. Let $\{a_n\} \subset B$ be a sequence converging to zero, and let $C$ be the smallest closed ideal containing $\{a_n\}$. Since $B$ has a bounded approximate identity, $C = \overline{\mathrm{sp}\{ca_n: c \in B\}}$. By the Cohen factorization theorem for sequences and its proof [9], p. 62–63, there exist $b \in B$, $\{b_n\} \subset B$ such that $\{b_n\}$ converges to zero and $a_n = bb_n$ for each $n$. Furthermore for each $\varepsilon > 0$, there exists $c \in B$ such that $\sup_n \|b_n - ca_n\| \leqslant \varepsilon$. Therefore $\{b_n\} \subset C$, and consequently $ca_n = cbb_n = bcb_n \in bC$. Hence $C \subset \overline{bC} \subset C$. The existence of a subspace $D$ satisfying $D = bD$, now follows by Theorem 1.1 and Remark 1.2.

2.7. THEOREM. *Let $A$ be a Banach algebra with a bounded left approximate identity and let $B$ be a commutative radical algebra. If $\nu$ is a homomorphism from $A$ into $B$ such that $B$ is totally reduced by $\nu(A)$, then $\nu = 0$.*

Proof. We assert first that $\nu$ is continuous. If not, then by Theorem 2.1 for each $a \in A$ there is an integer $n$ such that $a^n \in \mathscr{I}(\nu)$. Consequently by Proposition 2.2 we may choose an $a_0 \in A$ such that if $\nu_0(x) = \nu(a_0 x)$, then $\nu_0$ is discontinuous and $\mathscr{I}(\nu_0) = A$. However, since $A$ has an approximate identity $\mathscr{I}(\nu_0)^2 = A^2 = A$ is closed and hence, by [7], Lemma 2.5, $\nu_0$ must be continuous. This contradiction shows that $\nu$ is continuous.

Consequently the kernel of $\nu$ is closed, and assuming $\nu \not\equiv 0$ we may factor out $\ker\nu$ and assume $\nu$ to be one to one. Since $A$ has a bounded approximate identity, so will the commutative Banach algebra $A_1 = A/\ker\nu$. By Proposition 2.6 there exists a non-zero element $\tilde{a} \in A_1$, and a non trivial $\tilde{a}$-divisible subspace $D \subset A_1$. Since $\nu$ is one to one on $A_1$, $\nu(D)$ is a $\nu(\tilde{a})$-divisible subspace of $B$. This contradiction completes the proof.

We note that there exist continuous homomorphisms $\nu$ such that $\nu(A)$ fails to totally reduce $B = \overline{\nu(A)}$. A simple example illustrating this remark is provided by the disc algebra $A(\Delta)$: Let $A_1 = \{f \in A(\Delta): f(1) = 0\}$ and let $K$ be any closed primary ideal in $A(\Delta)$ with hull $= \{1\}$ such that $K \neq A_1$. If $B = A_1/K$, then $B$ is radical and the canonical homomorphism of $A_1$ onto $B$ is the desired example. Thomas' Theorem 2.11 in [29] can be used to construct discontinuous homomorphisms of $A_1$ into $B$.

When $A = C(\Omega)$, the algebra of continuous complex valued functions on the compact Hausdorff space $\Omega$, Theorem 2.7 combined with a theorem of Foias and Vasilescu [19] yields the following result.

2.8. THEOREM. *A homomorphism $\nu: C(\Omega) \to B$ is continuous if and only if no $\nu(a) \in B$ has a non-trivial divisible subspace.*

Proof. If $\nu: C(\Omega) \to B$ is a homomorphism and no $\nu(a)$ has a non-trivial divisible subspace in $B$, then there are no divisible subspaces for $\nu(a)$ in $\overline{\nu(C(\Omega))}$, so we may assume $B = \overline{\nu(C(\Omega))}$. By [4], Theorem 4.3, $\nu = \mu + \lambda$ where $\lambda: C(\Omega) \to \mathrm{rad}\, B = R$, and $\mu(a)R = \{0\}$ for every $a \in M(F)$, the ideal of functions vanishing on the singularity set $F$ of $\nu$. Furthermore, $\lambda$ is a homomorphism on $M(F)$. Since $\mu(a)$ annihilates $R$, no $\lambda(a)$ has a non-trivial divisible subspace, for $a \in M(F)$; hence by Theorem 2.7 $\lambda \equiv 0$, i.e. $\nu$ is continuous.

For the converse, suppose $\nu: C(\Omega) \to B$ is continuous and suppose $a \in C(\Omega)$; let $A$ be the closed algebra generated by $a$ and its complex conjugate $a^*$, viewed as $C(\sigma(a))$. In the terminology of [19] we see that $\nu(a)$ is a generalized scalar operator with a spectral distribution of order 0, hence by [19], Theorem 3.2, $\bigcap_{\lambda \in \mathbf{C}} (\nu(a) - \lambda)^2 B = \{0\}$; thus $\nu(a)$ can have no non-trivial divisible subspace. This completes the proof.

If $\nu$ is a homomorphism from a Banach algebra $A$ to $B$ which is discontinuous, we may lift $\nu$ to a homomorphism $\theta$ from $A$ to $\mathscr{B}(\mathscr{S}(\nu))$, the bounded operators on $\mathscr{S}(\nu)$, by setting $\theta(f)s = \nu(f)s$, $s \in \mathscr{S}(\nu)$. This homomorphism is useful in the study of $\nu$. We give some of its properties in the following lemma.

2.9. LEMMA. *Let $A$ and $B$ be commutative unital Banach algebras and let $\nu: A \to B$ be a discontinuous unital homomorphism satisfying $B = \overline{\nu(A)}$. Let $a_0 \in A$ and assume $\nu_0(x) = \nu(a_0 x)$ is discontinuous with separating space $\mathscr{S}_0$ and continuity ideal $\mathscr{I}(\nu_0)$. Define $\theta: A \to \mathscr{B}(\mathscr{S}_0)$ by*

$$\theta(x)s = \nu(x)s, \quad s \in \mathscr{S}_0, \ x \in A,$$

*and set $\mathscr{D} = \overline{\theta(A)}$.*

(a) *Let $\theta^*$ be the adjoint map from $\Phi_{\mathscr{D}}$ to $\Phi_A$ defined by*

$$\theta^*(\psi)x = \psi(\theta(x)), \quad x \in A, \ \psi \in \Phi_{\mathscr{D}}.$$

*Then $\theta^*$ maps $\Phi_{\mathscr{D}}$ homeomorphically (for the weak star topologies) into hull $(\mathscr{I}(\nu_0))$.*

(b) *Let $H: B \to \mathscr{B}(\mathscr{S}_0)$ be defined by*

$$H(b)(s) = b \cdot s, \quad b \in B, \ s \in \mathscr{S}_0.$$

*Then $\mathscr{S}(\theta) = \overline{H(\mathscr{S}(\nu))}$ and in particular $\theta$ is continuous if and only if $\mathscr{S}(\nu) \cdot \mathscr{S}_0 = \{0\}$. In this case $\mathscr{I}(\nu_0) = \ker\theta$ is closed.*

(c) *If $a \in A$ and $\mathscr{E}$ is a $\theta(a)$-divisible subspace of $\mathscr{D}$, i.e. $(\theta(a) - \gamma)\mathscr{E} = \mathscr{E}$ for all $\gamma \in \mathbf{C}$, then*

$$V = \mathrm{span}\{Ts \mid T \in \mathscr{E}, \ s \in \mathscr{S}_0\}$$

*is $\nu(a)$-divisible in $B$.*

Proof. (a) Since $\theta(A)$ is dense in $\mathscr{D}$, $\theta^*$ is one-to-one. If $\psi_\alpha(T) \to \psi_0(T)$ for every $T \in \mathscr{D}$, then $\theta^*(\psi_\alpha)(f) = \psi_\alpha(\theta(f)) \to \psi_0(\theta(f)) = \theta^*(\psi_0)(f)$ for all $f \in A$. Thus $\theta^*$ is a homeomorphism onto its range. Let $\varphi = \theta^*(\psi)$, where $\psi \in \Phi_{\mathscr{D}}$. Since $\theta(f) = 0$ for $f \in \mathscr{I}(\nu_0)$, we have $\theta^*(\psi)(f) = \psi(\theta(f)) = 0$, so $\varphi \in \operatorname{hull}(\mathscr{I}(\nu))$.

(b) Since $\theta = H \circ \nu$, this is an immediate consequence of Lemma 0.2 (iii).

(c) Let $\lambda \in C$. If $b \in V$, $b = \sum_{i=1}^{k} T_i s_i$, where $T_i \in \mathscr{E}$, $s_i \in \mathscr{S}_0$. By assumption on $\mathscr{E}$ there exist $U_i \in \mathscr{E}$ such that $T_i = (\theta(a) - \lambda) U_i$, and hence

$$b = (\theta(a) - \lambda) \sum_{i=1}^{k} U_i s_i = (\nu(a) - \lambda) \cdot v$$

where $v = \sum_{i=1}^{k} U_i s_i \in V$. This shows $V \subseteq (\nu(a) - \lambda) V$. On the other hand if $u = \sum_{i=1}^{k} T_i s_i \in V$, then

$$(\nu(a) - \lambda) \cdot u = \sum_{i=1}^{k} T_i (\theta(a) - \lambda)(s_i) \in V,$$

since $T_i$ and $\theta(a)$ commute.

Next we use these results to show that for certain commutative Banach algebras $A$, if $\nu$ is a discontinuous homomorphism then the hull $h(\mathscr{I}(\nu))$ of $\mathscr{I}(\nu)$ must contain a perfect set.

2.10. Theorem. *Let $A$ be a commutative, separable, unital Banach algebra and suppose* $\operatorname{co}(M^2) < \infty$ *for each maximal ideal $M$. Let $\nu$ be a homomorphism of $A$ into a Banach algebra $B$ and assume $B = \overline{\nu(A)}$. If $B$ is totally reduced by $\nu(A)$ and $\nu$ is discontinuous, then $h(\mathscr{I}(\nu))$ contains a perfect set.*

Proof. We first need a preliminary

2.11. Lemma. *Under the assumptions of the theorem if $a_0 \in A$, $\nu_0(x) = \nu(a_0 x)$ is discontinuous and $h(\mathscr{I}(\nu_0))$ is scattered, then $\theta$ is continuous and $\mathscr{S}(\nu_0)^2 = 0$.*

Proof. Apply Theorem 2.5 to the homomorphism $\theta\colon A \to \mathscr{B}(\mathscr{S}_0)$ of Lemma 2.9. By 2.9 (c) $\mathscr{D} = \theta(A)^-$ is totally reduced by $\theta(A)$ and by 2.9 (a) $\Phi_{\mathscr{D}}$ is scattered. Thus $\theta$ is continuous so $\mathscr{S}(\nu_0)^2 = 0$ by 2.9 (b).

To prove the theorem assume $h(\mathscr{I}(\nu))$ is scattered. Then by the lemma, $\theta$ is continuous, and consequently $\mathscr{I}(\nu)$ is closed. Let $a \to \bar{a}$ be the canonical map of $A$ onto $A/\mathscr{I}(\nu)$. If $\bar{a} - \lambda \cdot 1$ is invertible in $A/\mathscr{I}(\nu)$, then $\nu(a) - \lambda \cdot 1 | \mathscr{S}(\nu)$ is a one-to-one mapping of $\mathscr{S}(\nu)$ onto itself. Therefore for $a \in A$ $\sigma(\nu(a)|\mathscr{S}(\nu)) \subset \sigma(\bar{a})$ which is countable. By 2.1 if $\nu(a) \neq \lambda \cdot 1$, $\nu(a)$ is algebraic over $\mathscr{I}(\nu)$. If $\mathscr{I}(\nu)$ is prime, then the proof of 2.3 shows that $\mathscr{I}(\nu)$ is maximal which is impossible by [7], Theorem 2.8. If $\mathscr{I}(\nu)$ is not prime, choose $a_0$ so that $\nu_0(x) = \nu(a_0 x)$ is discontinuous with prime continuity ideal $\mathscr{I}(\nu_0)$. But $\mathscr{I}(\nu_0) \supset \mathscr{I}(\nu)$, so $h(\mathscr{I}(\nu_0))$ is scattered, and by the lemma, $\mathscr{I}(\nu_0)$ is closed. Repeating the above argument for $\mathscr{I}(\nu_0)$ in place of $\mathscr{I}(\nu)$, we get that $\mathscr{I}(\nu_0)$ is maximal which is again a contradiction.



For Silov algebras, Theorem 2.10 yields the following automatic continuity corollary.

2.12. Corollary. *Let $A$ be a separable Silov algebra such that $M^2$ has finite codimension for each maximal ideal $M$. Let $\nu$ be a homomorphism of $A$ into the commutative Banach algebra $B$. If $B$ is totally reduced, then $\nu$ is continuous.*

Proof. If $\nu$ were discontinuous, then $h(\mathscr{I}(\nu))$ would be finite by [25], Theorem 2.2. This contradicts 2.10.

Some algebras to which Theorem 2.10 applies are the Silov algebras $L_1(G)$ with adjoined unit, where $G$ is a locally compact abelian group, and $AC[0,1]$, the algebra of functions absolutely continuous on $[0,1]$ with norm $\|f\| = \|f\|_\infty + \int_0^1 |f'(t)|\,dt$. In these algebras maximal ideals have bounded approximate identities, so $M^2 = M$ for each maximal ideal.

For Silov algebras the condition $\operatorname{co}M^2 < \infty$ may be replaced by a condition that the range of $\nu$ be semi-prime.

2.13. Theorem. *Let $\nu$ be a homomorphism of the Silov algebra $A$ into the commutative Banach algebra $B$. If $B$ is totally reduced and semi-prime, then $\nu$ is continuous.*

Proof. Suppose $\nu$ is discontinuous and assume $a_0 \in A$ is chosen so that $\nu_0(x) = \nu(a_0 x)$ is discontinuous. If $\mathscr{I}(\nu)$, and $\mathscr{I}(\nu_0)$ are the respective continuity ideals, $\mathscr{I}(\nu) \subset \mathscr{I}(\nu_0)$. Therefore $h(\mathscr{I}(\nu_0)) \subset h(\mathscr{I}(\nu))$ and the latter is finite. Therefore by 2.9 (a), $\sigma(\nu(a)|\mathscr{S})$, and $\sigma(\nu_0(a)|\mathscr{S}_0)$ are finite for each $a \in A$, where $\mathscr{S}$ and $\mathscr{S}_0$ are the respective separating spaces. Therefore we may apply the arguments of 2.1 and 2.3 and conclude that there exists an $a_0 \in A$ such that $\nu_0$ is discontinuous and $\mathscr{S}_0^2 = \{0\}$. Since there are no non zero nilpotents in $B$, $\mathscr{S}_0 = 0$, and $\nu_0$ is continuous, which is a contradiction.

2.14. Remarks. Recall [26], Remark 8.1, that if $A$ is a Banach algebra and $M$ is a Banach $A$-bimodule in which $\|am\|, \|ma\| \leqslant \|a\|\,\|m\|$ for all $a \in A$ and $m \in M$, then $B = A \oplus M$ with norm $\|(a, m)\| = \|a\| + \|m\|$ and product $(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2)$ is a Banach algebra. If $A$ has a unit 1, then $(1, 0)$ is a unit for $B$. Evidently $\{0\} \oplus M$ becomes an ideal which is nilpotent of order two, and if $A$ is semi-simple, then $\operatorname{rad}(B) = \{0\} \oplus M$.

We shall use this general construction to present two relevant examples: let $T\colon M \to M$ be a continuous linear operator and let $(A, \|\cdot\|)$ be a Banach algebra of power series with the property that if $(a_i)_{i=0}^{\infty} \in A$, then $\sum_{i=0}^{\infty} |a_i|\,\|T^i\| < \infty$, and $\|\sum_{i=0}^{\infty} a_i T^i\| \leqslant \|(a_i)\|$. Almost by definition $M$

is a Banach $A$-module via the module action

$$(a_i)m = m(a_i) = \sum_{i=1}^{\infty} a_i T^i m.$$

(i) We use this method first to give an example of a commutative Banach algebra $B$ with radical $R$ such that $R$ is nilpotent of order two and has an element $a \in B$ with a non-trivial divisible subspace. With the above notation let $M = C([0,1])$, let $T\colon M \to M$ be the Volterra operator

$$(Tm)(t) = \int_0^t m(x)\,dx$$

and let $A = \{(a_i) \mid \sum_{i=0}^{\infty} |a_i| < \infty\}$ with $\|(a_i)\| = \sum_{i=0}^{\infty} |a_i|$. Since $T$ is quasinilpotent we have $\|\sum_{i=0}^{\infty} a_i T^i\| \leqslant \sum_{i=0}^{\infty} |a_i|\, \|T^i\| \leqslant \sum_{i=0}^{\infty} |a_i|$ for every $(a_i) \in A$. Let $B = A \oplus M$. As pointed out in [13], where this example originated, the set

$$M_0 = \{m \in C^{\infty}([0,1]) \mid m^{(p)}(0) = 0,\ p = 0, 1, \ldots\}$$

is $T$-divisible. If $z\colon t \to t$ denotes the identity function, then obviously $z \in A$; if we let $a = (z, 0) \in B$ and $D = \{0\} \oplus M_0$, then $(a-\lambda)D = ((z, 0) - \lambda(1, 0))D = (z-\lambda, 0)(\{0\} \oplus M_0) = \{0\} \oplus (T-\lambda)M_0 = \{0\} \oplus M_0 = D$. So the algebra $B$ is certainly not totally reduced. However, $R = \operatorname{rad} B = \{0\} \oplus M$ is totally reduced since it is nilpotent.

Thus while we know in general that a Banach algebra $B$ can have a totally reduced radical without being itself reduced, it is not hard to see that if $\Phi_B$ is finite the two properties are equivalent. This follows easily from the fact that if $e$ is an idempotent in $B$ and $D$ is $b$-divisible, then $eD$ is $eb$ divisible. We do not know whether the properties are equivalent under the assumption that $\Phi_B$ is countable.

(ii) We next give an example of a commutative radical Banach algebra $R$ which is totally reduced, but has an element $a \in R$ for which $\bigcap_{n=1}^{\infty} a^n R \neq \{0\}$. Our example is based on Marc Thomas' (unpublished) example of a quasinilpotent operator $T$ on a Banach space $M$ which has no non-trivial divisible subspace, but for which $\bigcap_{n=1}^{\infty} T^n M \neq \{0\}$. We wish to thank M. Thomas for permission to use his example here.

Let $\tau$ be the countable tree illustrated below: the trunk of $\tau$ contains countably many points, there are countably many branches, the $n$th branch consisting of $n$ points, all branches joining the trunk at the point $x_0$.



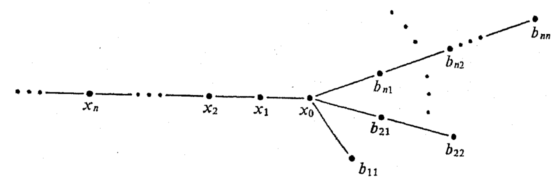


Fig. 1

We let $M$ be the weighted $l^1$ space on $\tau$ with weights chosen so that if $T$ is the leftward shift on $\tau$ (with addition at $x_0$), then $T$ becomes quasinilpotent.

Now $T$ has no divisible subspace, because if $m \in \operatorname{range} T$, then $m$ vanishes at the end of each branch, hence if $D \subset M$ satisfies $T^n D = D$ for each $n = 1, 2, \ldots$, then $m \in D$ means $m$ is zero on each branch, hence at $x_0$, hence everywhere, i.e. $D = \{0\}$. On the other hand, $\bigcap_{n=1}^{\infty} T^n(l^1) \neq \{0\}$, as the characteristic function of $\{x_0\}$ satisfies $1_{\{x_0\}} = T^n(1_{\{b_{nn}\}})$ for each $n = 1, 2, \ldots$

Next, let $A$ be the algebra of power series

$$A = \Big\{(a_i)_{i=1}^{\infty} \Big| \sum_{i=1}^{\infty} |a_i|\,\|T^i\| < \infty\Big\}$$

and use the defining sum as the norm on $A$. Since $\|T^i\| \leqslant \|T^k\|\,\|T^{i-k}\|$ for $k = 1, 2, \ldots, i-1$ and $i = 1, 2, \ldots, A$ is a Banach algebra; also, since $\|T^{ni}\|^{1/i} \to 0$ as $i \to \infty$ for each $n = 1, 2, \ldots$, it is elementary to check that $A$ is a radical algebra. Let $R = A \oplus M$ be the Banach algebra defined at the beginning of these remarks. Because $A \to A \oplus \{0\}$ is an isometric embedding of the radical algebra $A$ and because $\{0\} \oplus M$ is nilpotent it is clear that $R$ is a radical algebra.

For later use we remark that if $f \in A$, then $\bigcap_{n=1}^{\infty} f^n A = \{0\}$, because if $b = f^n a$, then $b = (\underbrace{0, 0, \ldots, 0}_{n+1}, *, *, \ldots)$.

About $R$ we claim the following:

(i) *there exists* $(f_0, m_0) \in R$ *such that* $\bigcap_{n=1}^{\infty} (f_0, m_0)^n R \neq \{0\}$, *and*

(ii) $R$ *is totally reduced.*

Proof of (i). We know that $\bigcap_{n=1}^{\infty} T^n(l_1) \neq \{0\}$, so let $0 \neq m_0 \in \bigcap_{n=1}^{\infty} T^n(l_1)$; write $m_0 = T^n m_n$ for $n = 1, 2, \ldots$ Let $f_0 = (1, 0, 0, 0, \ldots) \in A$ and note that $(f_0, 0)^n(0, m_n) = (0, m_0)$, because

$$(f_0, 0)^n(0, m_n) = (f_0^n, 0)(0, m_n) = (0, T^n m_n) = (0, m_0).$$

Thus $(0, m_0) \in \bigcap_{n=1}^{\infty} (f_0, 0)^n R$.

Proof of (ii). Suppose $\tilde{D}$ is a divisible subspace in $R$ for some $(f_0, m_0) \in R$, i.e. $(f_0, m_0)\tilde{D} = \tilde{D}$. If $D_1 = \{f \in A | \ (f, m) \in \tilde{D}$ for some $m\}$, then it is easy to see that $D_1$ is $f_0$-divisible in $A$. By the previous remark $D_1 = \{0\}$ and hence $\tilde{D} = 0 \oplus D$. Thus $(f_0, m_0)\tilde{D} = \tilde{D} \Leftrightarrow (f_0, m_0)(0 \oplus D) = 0 \oplus D \Leftrightarrow \big(0, f_0(T)D\big) = 0 \oplus D \Leftrightarrow f_0(T)D = D$. We are now reduced to showing that $f_0(T)$ has no divisible subspace $D$ in $l^1$. But since $f_0(T) = \sum_{i=1}^{\infty} a_i T^i$ we see that if $f_0(T)D = D$, then each $m \in D$ vanishes on each branch of $\tau$, and hence $m$ vanishes everywhere. To see this suppose $a_n$ is the first coefficient of $\sum_{i=1}^{\infty} a_i T^i$ which is not zero. Then $D = f_0(T)D \subseteq \text{range}\, T^n(l^1)$, i.e. every element of $D$ vanishes on the last $n$ points of each branch of $\tau$. Since $n \geqslant 1$ repetition of this argument shows that $D = \{0\}$, hence $R$ is totally reduced.

**§ 3.** In this section we examine the role of divisible subspaces in the study of continuity properties of homomorphisms of Banach algebras of differentiable functions. Let $n$ be a non-negative integer. We denote by $C^n$ the Banach algebra $C^n([0, 1])$ of $n$-times continuously differentiable functions defined on the unit interval $[0, 1]$, with the norm

$$\|f\|_n = \max_{t \in [0,1]} \sum_{j=0}^{n} \frac{|f^{(j)}(t)|}{j!}.$$

In [8], Theorem 2.5, it was shown that if $\nu\colon C^n \to B$ is a homomorphism of $C^n$ into a Banach algebra, and the restriction of $\nu$ to $C^k$ is continuous with respect to the $C^k$-norm for some $k > n$, then $\nu$ is necessarily continuous on $C^k$ for all $k \geqslant 2n+1$. We use the phrases *eventual continuity* and *$C^{2n+1}$-continuity* to describe this situation. Analogously, if $\nu$ is discontinuous on every $C^k$, $k \geqslant n$ (equivalently, $\nu$ is discontinuous on $C^k$ for some $k \geqslant 2n+1$) we say that $\nu$ is *permanently discontinuous*. A basic result of [8], Theorem 2.6, was that if $\nu\colon C^n \to B$ and $\dim \text{rad}(B) < \infty$, then $\nu$ is $C^{2n+1}$-continuous.

In this section we obtain in Theorem 3.3 a set of necessary and sufficient conditions for a homomorphism $\nu\colon C^n \to B = \overline{\nu(C^n)}$ to be $C^{2n+1}$-continuous. Most important of these is the condition that the operator of multiplication on $B$ by the image $\nu(z)$ of the generator $z(t) \equiv t$ shall have no non-trivial divisible subspaces. We obtain as a consequence of this result the fact that if $\nu$ is a homomorphism of $C^n$ into a Banach algebra with totally reduced radical, then $\nu$ is $C^{2n+1}$-continuous, strengthening the theorem mentioned above. The same theorem is also strengthened in another direction: If $\nu\colon C^n \to B$ and $\dim \text{rad}(B) < \infty$, then $\nu$ is $C^{2n}$-continuous. As pointed out in [8], the value $2n$ is best possible.



It remains an open question whether $C^{2n+1}$-continuity for a homomorphism $\nu\colon C^n \to B$ always implies $C^{2n}$-continuity, and further whether $C^{2n}$-continuity implies a formally stronger set of equivalent conditions established in Theorem 3.11.(1)

Let $\nu\colon C^n \to B$ be a discontinuous homomorphism. Since $C^n$ is a Silov algebra, the continuity ideal $\mathscr{I}(\nu)$ has finite hull and contains the ideal $J(F)$ of all functions vanishing in neighborhoods of $F = \text{hull}\,(\mathscr{I}(\nu))$. (See [4], [8] and [25].) As in [4] we will call the hull of $\mathscr{I}(\nu)$ the *singularity set* of $\nu$. If $\gamma_0$ is any point of $F$ and $a_0$ is any function in $C^n$ which equals one in a neighborhood of $\gamma_0$ and vanishes in a neighborhood of the remaining points of $F$, then the multiplied homomorphism $\nu_0(f) = \nu(a)\nu(f)$, $f \in C^n$, is also discontinuous with separating space $\mathscr{S}(\nu_0) = \overline{\nu(a_0)\mathscr{S}(\nu)}$ and continuity ideal $\mathscr{I}(\nu_0) = \{f |\ a_0 f \in \mathscr{I}(\nu)\}$. Consequently hull $\mathscr{I}(\nu_0) = \{\gamma_0\}$. These observations allow us to reduce the study of arbitrary homomorphisms of $C^n$ to that of multiplied homomorphisms whose singularity sets have only one point via the next lemma.

3.1. Lemma. *Let $n \geqslant 1$ and let $\nu\colon C^n \to B$ be a homomorphism with separating space $\mathscr{S}(\nu)$ and continuity ideal $\mathscr{I}(\nu)$. Let $F = \{\gamma_1, \ldots, \gamma_r\}$ be the singularity set for $\nu$ and let $a_i$ be functions in $C^n$ such that*

(i) *$a_i a_j = 0$ if $i \neq j$, $i, j = 1, \ldots, r$;*

(ii) *$a_i \equiv 1$ in a neighborhood of $\gamma_i$ and $\equiv 0$ in a neighborhood of $\{\gamma_j | j \neq i\}$ for $i = 1, \ldots, r$.*

*Define $a_0 = 1 - \sum_{i=1}^{r} a_i$ and $\nu_i(f) = \nu(a_i f)$, $f \in C^n$, $i = 0, \ldots, r$. Then*

(iii) *$\mathscr{S}(\nu) = \mathscr{S}(\nu_1) \oplus \ldots \oplus \mathscr{S}(\nu_r)$;*

(iv) *$\mathscr{I}(\nu) = \bigcap_{i=1}^{r} \mathscr{I}(\nu_i)$.*

Proof. Since $a_0$ vanishes in a neighborhood of $F$, it follows that $a_0 \in \mathscr{I}(\nu)$. Thus $\nu_0$ is continuous. Clearly $\mathscr{S}(\nu_i) = \big(\nu(a_i)\mathscr{S}(\nu)\big)^- \subseteq \mathscr{S}(\nu)$, so $\sum_{i=1}^{r} \mathscr{S}(\nu_i) \subseteq \mathscr{S}(\nu)$. Conversely, if $s \in \mathscr{S}(\nu)$, then

$$s = \nu(1)s = \sum_{i=1}^{r} \nu(a_i)s \in \sum_{i=1}^{r} \mathscr{S}(\nu_i),$$

since $a_0 \in \mathscr{I}(\nu)$. Since for $s \in \mathscr{S}(\nu_i)$, $\nu(a_i)s = s$, $\nu(a_j)s = 0$, $j \neq i$, $i = 1, \ldots, r$, the sum is direct.

(1) Added in proof: cef. footnote (3), p. 183.

To show that $\mathscr{I}(\nu) = \bigcap_{i=1}^{r} \mathscr{I}(\nu_i)$ we note that if $a \in \mathscr{I}(\nu)$, then $a_i a \in \mathscr{I}(\nu)$, so $f \to \nu(a_i af) = \nu_i(af)$ is continuous for $i = 1, \ldots, r$, i.e. $\mathscr{I}(\nu) \subseteq \bigcap_{i=1}^{r} \mathscr{I}(\nu_i)$. If $f \to \nu(a_i af)$ is continuous for $i = 1, \ldots, r$ then, since $a_0 \in \mathscr{I}(\nu)$, we obtain the continuity of $f \to \nu(af) = \sum_{i=0}^{r} \nu(a_i af)$. Hence $a \in \mathscr{I}(\nu)$.

We will also need a notion from operator theory. Let $n$ be a non-negative integer. An operator $T$ with real spectrum on a Banach space $X$ is called a *$C^n$-operator* [21] if there is a closed interval $I \supseteq \sigma(T)$ and a continuous homomorphism $\mu\colon C^n(I) \to \mathscr{B}(X)$ with $\mu(z) = T$. These operators coincide with the generalized scalar operators with real spectrum having a spectral distribution of order $n$ which are discussed in [11], Chapter 5. Improving an earlier result of Vrbova [30], Foias and Vasilescu [19] prove that if $T$ is a $C^n$-operator, then $\bigcap_{\lambda \in C} (T-\lambda I)^{n+2} X = \{0\}$. Hence $T$ has no non-trivial divisible subspaces. As a consequence we have:

3.2. Lemma. *If $\mu\colon C^n \to B = \overline{\mu(C^n)}$, $n \geqslant 0$, is a continuous homomorphism, the operator of multiplication on $B$ by $\mu(z)$ has no non-trivial divisible subspaces.*

3.3. Theorem. *Let $n \geqslant 1$ and $\nu\colon C^n \to B = \overline{\nu(C^n)}$ be a homomorphism with singularity set $F = \{\gamma_1, \ldots, \gamma_r\}$. The following statements are equivalent:*

(i) *$\nu$ is $C^{2n+1}$-continuous;*

(ii) *The continuity ideal $\mathscr{I}(\nu)$ contains the polynomial $v = \prod_{i=1}^{r} (z-\gamma_i)^{n+1}$;*

(iii) *The operator on $B$ of multiplication by $\nu(z)$ has no non-trivial divisible subspaces;*

(iv) *$\mathscr{S}(\nu)^3 = \{0\}$;*

(v) *$\mathscr{S}(\nu)^k = \{0\}$ for some $k \geqslant 3$.*

Proof. The equivalence of (i) and (ii) is implicit in the arguments of [8]. In the proof of [8], Theorem 2.5, it is shown that $C^k$-continuity of $\nu_i$ for some $k > n$ implies that $\mathscr{I}(\nu_i)$ contains a power of $(z-\gamma_i)$. Arguing as in the proof of [8], Corollary 2.3, we get $(z-\gamma_i)^{n+1} \in \mathscr{I}(\nu_i)$, $i = 1, \ldots, r$. Then it follows from Lemma 3.1 that $v \in \mathscr{I}(\nu)$. The proof of [8], Corollary 2.4 shows that (ii) $\to$ (i).(2)

To prove that (ii) $\to$ (iii), suppose that $v \in \mathscr{I}(\nu)$ and that $D$ is a $\nu(z)$-divisible subspace of $B$. Define $B_1 = \overline{\nu(v)B}$ and $\varphi\colon C^n \to \mathscr{B}(B_1)$ by

$$\varphi(f)(b) = \nu(f)\cdot b, \qquad b \in B_1,\ f \in C^n.$$

(2) In [8], page 266, line 25, read "$\mathscr{I}(\nu)$" for "$M_{n,n,t_0}$".



We show $\varphi$ is a continuous homomorphism by showing $\mathscr{S}(\varphi) = \{0\}$. If $R \in \mathscr{S}(\varphi)$, there is a sequence $f_n \Rightarrow 0$ in $C^n$ such that $\varphi(f_n) \Rightarrow R$ in $\mathscr{B}(B_1)$. However, if $b = \nu(v)b' \in \nu(v)B$, $R(v) = \lim \nu(f_n)\nu(v)b' = 0$, as $v \in \mathscr{I}(\nu)$. Thus $R = 0$. Since $\nu(v)D = D$, we have $D \subseteq B_1$. Also

$$\big(\varphi(z)-\gamma I\big)D = \big(\nu(z)-\gamma\big)D = D, \quad \gamma \in C,$$

so $D$ is $\varphi(z)$-divisible. Lemma 3.2 shows $D = \{0\}$.

We prove next that (iii) $\to$ (i). Let $\nu^{(2n)} = \nu|C^{2n}$, where $C^{2n}$ has its natural norm. By [8], Corollary 1.1, $\nu^{(2n)}$ is bounded on the subalgebra $C^{2n} \cap J(F)$. Thus we have

$$\nu^{(2n)}(\lambda) = \mu(f)+\lambda(f), \quad f \in M_{2n,2n}(F),$$

where $\mu$ is the unique $C^{2n}$-continuous extension of $\nu^{(2n)}$ to $M_{2n,2n}(F)$, $\mu$ and $\lambda$ are homomorphisms, and $\lambda \neq 0$ if and only if $\nu$ is discontinuous for the $C^{2n}$-norm. If $\nu$ is $C^{2n}$-continuous we are done. Therefore suppose $\lambda \neq 0$. Then

$$\mathscr{S}(\lambda) = \overline{\lambda\big(M_{2n,2n}(F)\big)} = \mathscr{S}(\nu^{(2n)}) \subseteq \mathscr{S}(\nu),$$

and

$$\mu\big(M_{2n,2n}(F)\big)\mathscr{S}(\lambda) = \{0\}.$$

(See the arguments of [4], page 601.)

Let $q = \prod_{i=1}^{r} (z-\gamma_i)^{2n+1}$, where $F = \{\gamma_1, \ldots, \gamma_r\}$ is the singularity set for $\nu$. We apply Corollary 1.3 to the mappings $S = \lambda$, $Tf = q\cdot f$, $f \in M_{2n,2n}(F)$, and $Rs = \lambda(q)\cdot s$, $s \in \mathscr{S}(\lambda)$. It follows that there exists a non-zero polynomial $p$ whose roots lie in $\sigma(R)$ such that

$$p\big(\lambda(q)\big)\mathscr{S}(\lambda) \subseteq W\big(\mathscr{S}(\lambda)\big),$$

where $W\big(\mathscr{S}(\lambda)\big)$ is the closure of the largest $\lambda(q)$-divisible subspace $D$ in $\mathscr{S}(\lambda)$. Since $\sigma(R) = \{0\}$, $p = z^N$ for some positive integer $N$.

We next show that $W\big(\mathscr{S}(\lambda)\big) = \{0\}$. For each $\gamma \in C$,

$$D = \big(\lambda(q)-\gamma\big)D = \big(\nu(q)-\mu(q)-\gamma\big)D = \big(\nu(q)-\gamma\big)D,$$

so $D$ is $\nu(q)$-divisible. Let $D_1 \supseteq D$ be the largest $\nu(q)$-divisible subspace in $B$. We show that $D_1$ must be $\nu(z)$-divisible, and hence $\{0\}$ by assumption. If $\gamma \in C$, we can select $\gamma_0$ so that $z-\gamma$ is a factor of $q-\gamma_0$. Since by Lemma 1.4 $D_1$ is an ideal in $B$, it suffices to show $D_1 \subseteq \big(\nu(z)-\gamma\big)D_1$. If $d_1 \in D_1$, select $d_2 \in D_1$ so that $d_1 = \big(\nu(q)-\gamma_0\big)d_2$. Then

$$d_1 = \big(\nu(q)-\gamma_0\big)d_2 = \big(\nu(z)-\gamma\big)\nu(q_1)d_2$$

for some polynomial $q_1$, and hence $d_1 \in \big(\nu(z)-\gamma\big)D_1$. We conclude that $W\big(\mathscr{S}(\lambda)\big) = \{0\}$, so the polynomial $q^N$ belongs to $\mathscr{I}(\nu^{(2n)}) = \mathscr{I}(\lambda)$. From the implication (ii) $\to$ (i) applied to $\nu^{(2n)}$, we find that $\nu$ is $C^{4n+1}$-continuous, and hence is $C^{2n+1}$-continuous by [8], Theorem 2.5. This completes the proof that (iii) → (i).

To prove that (i) → (iv) let $F = \{\gamma_1, \ldots, \gamma_r\}$ be the singularity set for $\nu$ and write

$$\nu = \nu_0 + \sum_{i=1}^{r} \nu_i$$

as in Lemma 3.1. Clearly $\nu$ is $C^{2n+1}$-continuous if and only if each $\nu_i$ is. It suffices to show $\mathcal{S}(\nu_i)^3 = \{0\}$, $i = 1, \ldots, r$, because then

$$\mathcal{S}(\nu)^3 = \sum_{i=1}^{r} \nu(a_i)^3 \mathcal{S}(\nu)^3 = \sum_{i=1}^{r} \big(\nu(a_i)\mathcal{S}(\nu)\big)^3 \subseteq \sum_{i=1}^{r} \mathcal{S}(\nu_i)^3 = \{0\}.$$

(Here we have used $a_i a_j = 0$, $i \neq j$, and $1 - \sum_{i=1}^{r} a_i \in \mathcal{I}(\nu)$.) Now $\nu_i$ has singularity set $\{\gamma_i\}$ and $M_{n,n}(\gamma_i)^2 \subseteq \mathcal{I}(\nu_i)$ by [8], Theorem 2.1, and the fact $(z-\gamma_i)M_{n,n}(\gamma_i) = |z-\gamma_i| M_{n,n}(\gamma_i)$. If $s_1, s_2, s_3 \in \mathcal{S}(\nu_i) \subseteq \mathcal{S}(\nu)$, there exist sequences $\{f_m\}$, $\{g_m\}$ in $M_{n,n}(\gamma_i)$ such that $f_m \to 0$, $g_m \to 0$, and $\nu(f_m) \to s_1$, $\nu(g_m) \to s_2$. But $\nu(f_m g_m)\mathcal{S}(\nu_i) = 0$, so

$$s_1 s_2 s_3 = \lim_{m\to\infty} \nu(f_m g_m) s_3 = 0.$$

It is trivial that (iv) → (v).

Finally we prove that (v) → (i). Let $\nu\colon C^n \to B$ be a homomorphism satisfying $\mathcal{S}(\nu)^k = \{0\}$ for some $k \geqslant 3$. Consider the splitting $\nu^{(2n)} = \mu + \lambda$. The proof that (iii) → (i) shows that either the polynomial $q^N$ is in $\mathcal{I}(\nu^{(2n)})$, and hence $\nu$ is $C^{2n+1}$-continuous, or else there is a non-trivial $\lambda(q)$-divisible subspace $D$ in $\mathcal{S}(\nu)$. Since $\lambda(q) \in \mathcal{S}(\nu)$ and $\mathcal{S}(\nu)$ is nilpotent, $D = \{0\}$, so this alternative does not occur. This completes the proof of the equivalences.

A slight modification of the proof that (v) → (i) yields

3.4. Corollary. *Let $n \geqslant 1$ and $\nu\colon C^n \to B = \overline{\nu(C^n)}$ be a homomorphism. If* rad *$B$ is totally reduced, then $\nu$ is $C^{2n+1}$-continuous. If $B$ is in addition semi-prime, then $\nu$ is continuous.*

3.5. Corollary. *If $\nu\colon C([0,1]) \to B = \overline{\nu(C[0,1])}$ is a homomorphism, then $\nu$ is continuous on $C^1$ if and only if the operator on $B$ of multiplication by $\nu(z)$ has no non-trivial divisible subspace.*

Proof. In the proof of Theorem 3.3 the implications (i) → (ii) → (iii) are valid when $n = 0$. To prove (iii) → (i) we apply the previous argument to $\nu^{(1)} = \nu|C^1$ to get that $\nu^{(3)}$ is continuous. We now apply [8], Theorem 2.5, to get that $\nu^{(1)}$ is continuous.

One should note that the absence of non-trivial divisible subspaces for $\nu(z)$ does not imply that $\nu$ is continuous on $C[0,1]$ (cf. Theorem 2.8), for assuming the continuum hypothesis there exists a discontinuous homomorphism $\nu$ on $C[0,1]$ such that $\nu$ vanishes on the principal ideal $zC[0,1]$. In fact Esterle shows in [18] that $\nu$ may vanish on a prime ideal of $C[0,1]$ containing $zC[0,1]$.



3.6. Corollary. *Let $\nu\colon C^n \to B$ be a homomorphism with separating space $\mathcal{S}(\nu)$. Then*

$$\overline{\mathcal{S}(\nu)^k} = \overline{\mathcal{S}(\nu)^3} \quad \textit{for all } k \geqslant 3.$$

Proof. We may assume that $B = \overline{\nu(C^n)}$. Then $\mathcal{S}(\nu) \supseteq \overline{\mathcal{S}(\nu)^2} \supseteq \overline{\mathcal{S}(\nu)^3} \supseteq \ldots \supseteq \overline{\mathcal{S}(\nu)^k}$ is a descending chain of ideals. Let $k > 3$ and consider $B/\overline{\mathcal{S}(\nu)^k}$. If $P\colon B \to B/\overline{\mathcal{S}(\nu)^k}$ is the natural map, then $P\nu$ is a homomoprhism with $\overline{\mathcal{S}(P\nu)^k} = \overline{P\mathcal{S}(\nu)^k} = \{0\}$. Hence $\mathcal{S}(P\nu)^3 = \{0\}$ by Theorem 3.3, i.e. $P\overline{\mathcal{S}(\nu)^3} = \{0\}$ or $\overline{\mathcal{S}(\nu)^3} = \overline{\mathcal{S}(\nu)^k}$.

The same type of argument can be used to prove an analogous stability result.

3.7. Corollary. *Let $\nu\colon C^n \to B$ be a homomorphism and let $\mathcal{S}_k = \mathcal{S}(\nu|C^k)$ for every $k \geqslant n$. Then there exists $N$ such that $\mathcal{S}_k = \overline{\mathcal{S}_k^2} = \mathcal{S}_N$ for every $k \geqslant N$.*

Proof. If $\nu$ is eventually continuous, then the result is trivial as $\mathcal{S}_k = \{0\}$ for every $k \geqslant 2n+1$. If $\nu$ is permanently discontinuous, there exists an integer $N$ so that if $k \geqslant N$, then $\mathcal{S}_k = \mathcal{S}_N$, by [22], Proposition 2.1. Suppose $\overline{\mathcal{S}_N^2} \neq \mathcal{S}_N$ and consider the natural map $S\colon \overline{\nu(C^N)} \to \overline{\nu(C^N)}/\overline{\mathcal{S}_N^2}$. The composite homomorphism $\varphi = S\circ\nu|C^N$ has a separating space $\mathcal{S}(\varphi) = \overline{S(\mathcal{S}_N)} \neq \{0\}$ and $\mathcal{S}(\varphi)^2 = S(\overline{\mathcal{S}_N^2}) = \{0\}$. This shows that $\varphi$ is eventually continuous (Theorem 3.3), hence $\mathcal{S}(\varphi|C^{2N+1}) = \{0\}$ or $\mathcal{S}_{2N+1} \subseteq \overline{\mathcal{S}_N^2}$. But since $\mathcal{S}_{2N+1} = \mathcal{S}_N$, this contradicts the assumption that $\overline{\mathcal{S}_N^2} \neq \mathcal{S}_N$.

We next consider the problem whether eventual continuity for a homomoprhism $\nu$ of $C^n$ implies $C^{2n}$-continuity and related questions. For this discussion we will need some further properties of ideals in $C^n$.

Let $F = \{\gamma_1, \ldots, \gamma_r\}$ be a finite set in $[0,1]$ and $p = \prod_{i=1}^{r} (z-\gamma_i)^n$. Define the algebra

$$A_n(F) = \Big\{ f \in C^n([0,1] \sim F) \;\Big|\; \prod_{i=1}^{r} (t-\gamma_i)^j g^{(j)}(t) \to 0 \text{ as } t \to \gamma_i,\ i = 1, \ldots, r;\ j = 0, \ldots, n \Big\}.$$

We give $A_n(F)$ the norm

$$\|f\|_{A_n(F)} = \sum_{j=0}^{n} \frac{1}{j!} \sup\Big\{ \Big|\prod_{i=1}^{r} (t-\gamma_i)^j g^{(j)}(t)\Big| \;\Big|\; t \in [0,1] \sim F \Big\}.$$

3.8. Lemma. *With the given norm $A_n(F)$ is a Banach algebra with a bounded approximate identity. The map $T(g) = pg$, $g \in A_n(F)$ is a linear homeomorphism of $A_n(F)$ onto $M_{n,n}(F)$.*

Proof. The proof that $A_n(F)$ is a Banach algebra is straightforward and we omit it. If $F = \{\gamma_1, \ldots, \gamma_r\}$, then $A_n(\gamma_i) \subseteq A_n(F)$, $i = 1, \ldots, r$, and the natural imbedding is continuous. Here $A_n(\gamma_i)$ is the algebra for the case $F = \{\gamma_i\}$. The proof given in [6], Theorem 2.1, for the case $F = \{0\}$ shows that $A_n(\gamma_i)$ has a bounded approximate identity, and we obtain a bounded approximate identity for $A_n(F)$ by taking products of the bounded approximate identities for the $A_n(\gamma_i)$. Now

$$M_{n,n}(F) = \bigcap_{i=1}^{r} M_{n,n}(\gamma_i) = \bigcap_{i=1}^{r} (z-\gamma_i)^n A_n(\gamma_i).$$

An element of the last set can be written

$$c = (z-\gamma_1)^n a_1 = (z-\gamma_2)^n a_2 = \ldots = (z-\gamma_r)^n a_r$$

where $a_i \in A_n(\gamma_i)$. We have

$$a_2 = (z-\gamma_1)^n b_1$$

where $b_1 = (z-\gamma_2)^{-n} a_1 \in A_n(\{\gamma_1, \gamma_2\})$, so $c = (z-\gamma_1)^n (z-\gamma_2)^n b_1$. Repeating this argument we get that $c = pb_r$, where $b_r \in A_n(F)$. Thus we have shown that $M_{n,n}(F) \subseteq pA_n(F)$. A detailed computation shows the reverse inclusion, and that $T$ carries $A_n(F)$ continuously into $M_{n,n}(F)$. The fact that $T$ is a homeomorphism follows easily from the closed graph theorem.

3.9. Corollary. *Let $F = \{\gamma_1, \ldots, \gamma_r\}$ be a finite subset of* $[0, 1]$ *and let $p = \prod_{i=1}^{r} (z-\gamma_i)^n$. Then*

$$M_{n,n}(F)^2 = \bigcap_{i=1}^{r} M_{n,n}(\gamma_i)^2 = pM_{n,n}(F).$$

Proof. We note that

$$M_{n,n}(F)^2 \subseteq \bigcap_{i=1}^{r} M_{n,n}(\gamma_i)^2 = \bigcap_{i=1}^{r} (z-\gamma_i)^n M_{n,n}(\gamma_i).$$

An argument like the one given in Lemma 3.8 shows that

$$\bigcap_{i=1}^{r} (z-\gamma_i)^n M_{n,n}(\gamma_i) = p\bigcap_{i=1}^{r} M_{n,n}(\gamma_i) = pM_{n,n}(F).$$

Finally by the Cohen factorization theorem we have $A_n(F) = A_n(F)^2$, so by Lemma 3.8

$$pM_{n,n}(F) = p^2 A_n(F) = p^2 A_n(F)^2 = M_{n,n}(F)^2.$$

The importance of this result lies in the fact that $M_{n,n}(F)^2$ becomes a Banach space, when it is given the graph norm

$$\|f\|_p = \|f\|_n + \|f/p\|_n, \quad f \in M_{n,n}(F)^2$$



for the map $T(g) = pg$, $g \in M_{n,n}(F)$ which carries $M_{n,n}(F)$ onto $M_{n,n}(F)^2$. This norm is clearly also a Banach algebra norm.

We note if $a \in A_n(F)$, then the map $g \to ag$ defines a continuous operator on $M_{n,n}(F)$: since the map $a \to pa$: $A_n(F) \to M_{n,n}(F)$ is bicontinuous,

$$\|ag\| \leqslant C_1 \|ag/p\|_{A_n(F)} \leqslant C_1 \|a\|_{A_n(F)} \|g/p\|_{A_n(F)} \leqslant C\|a\|_{A_n(F)} \|g\|, \quad g \in M_{n,n}(F).$$

Thus $A_n(F)$ lies in the multiplier algebra of $M_{n,n}(F)$. We note next that $A_n(F)$ leaves invariant continuity ideals.

3.10. Lemma. *Let $\nu\colon C^n \to B$ be a homomorphism and let $a_0 \in C^n$. Suppose that the multiplied homomorphism $\nu_0(f) = \nu(a_0 f)$, $f \in C^n$ is discontinuous and let $F = \operatorname{hull}(\mathscr{I}(\nu_0))$. Then $A_n(F)(\mathscr{I}(\nu_0) \cap M_{n,n}(F)) \subseteq \mathscr{I}(\nu_0) \cap \cap M_{n,n}(F)$.*

Proof. Since $M_{n,n}(F)$ is closed and has finite codimension in $C^n$, $f \in \mathscr{I}(\nu_0)$ if and only if there is a constant $K_f$ such that $\|\nu_0(fg)\| \leqslant K_f \|g\|$ for all $g \in M_{n,n}(F)$. Let $a \in A_n(F)$ and $f \in \mathscr{I}(\nu_0) \cap M_{n,n}(F)$. Then

$$\|\nu_0(afg)\| = \|\nu_0(fag)\| \leqslant K_f \|ag\| \leqslant K_f C \|a\|_{A_n(F)} \|g\|, \quad g \in M_{n,n}(F).$$

3.11. Theorem. *Let $n \geqslant 1$ and $\nu\colon C^n \to B$ be a discontinuous homomorphism with singularity set $F = \{\gamma_1, \ldots, \gamma_r\}$. Consider the following statements*

(a) *$\mathscr{S}(\nu)$ is finite dimensional;*

(b) *$\mathscr{I}(\nu)$ has finite codimension;*

(c) *$\mathscr{I}(\nu)$ is closed and contains $M_{n,n-1}(F)$;*

(d) *$\mathscr{S}(\nu)^2 = \{0\}$;*

(e) *$p(z) = \prod_{i=1}^{r} (z-\gamma_i)^n \in \mathscr{I}(\nu)$;*

(f) *$\nu$ is continuous on $M_{n,n}(F)^2 = pM_{n,n}(F)$ for the graph norm $\|f\|_p = \|f\| + \|f/p\|$;*

(g) *$\nu$ is $C^{2n}$-continuous.*

*We have the following implications:*

$$\text{(a)} \Rightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(e)} \Leftrightarrow \text{(f)} \Rightarrow \text{(g)}.$$

Proof. (a) $\Rightarrow$ (b). Since $\mathscr{I}(\nu) = \{f \in C^n \mid \nu(f)\mathscr{S}(\nu) = \{0\}\}$, it follows that $C^n/\mathscr{I}(\nu)$ is faithfully represented on $\mathscr{S}(\nu)$ by $\theta(f + \mathscr{I}(\nu))(s) = \nu(f)s$, $f \in C^n$, $s \in \mathscr{S}(\nu)$. Hence if $\mathscr{S}(\nu)$ is finite dimensional, the same is true for $C^n/\mathscr{I}(\nu)$.

(b) $\Leftrightarrow$ (c). We split $\nu = \sum_{i=0}^{r} \nu_i$ as in Lemma 3.1. If $\mathscr{I}(\nu)$ is of finite codimension, so is each of the ideals $\mathscr{I}(\nu_i)$. If we show that each $\mathscr{I}(\nu_i)$ is closed, then the fact that $\mathscr{I}(\nu)$ is closed follows from Lemma 3.1. So let $\nu_i$

be a multiplied homomorphism with one point singularity set which for convenience we take to be $\{0\}$. Suppose $\mathscr{I}(\nu_i)$ has codimension $k$ and let $A_n = A_n(\{0\})$. We shall prove that $z^n A_n \subseteq \mathscr{I}(\nu_i)$, so $M_{n,n} = z^n A_n \subseteq \mathscr{I}(\nu_i)$. To prove that $z^n A_n \subset \mathscr{I}(\nu_i)$ we claim that it is sufficient to show that $z^n a^{k+1} \in \mathscr{I}(\nu_i)$ for every $a \in A_n$. To see this we refer to a general version of Cohen's factorization theorem [3]: if $b \in A_n$, then there exists $a \in A_n$ and $\{c_j\} \subset A_n$ such that $b = a^j c_j$ for $j = 1, 2, \ldots$ Thus $z^n b = z^n a^{k+1} c_{k+1} \in \mathscr{I}(\nu_i)$ by Lemma 3.10.

So let $a \in A_n$ and consider $\{z^n a, z^n a^2, \ldots, z^n a^{k+1}\} \subset M_{n,n}$. Since $\mathscr{I}(\nu_i)$ is of codimension $k$ there exist scalars $\alpha_1, \ldots, \alpha_{k+1}$ such that

$$c \equiv \alpha_1 z^n a + \ldots + \alpha_{k+1} z^n a^{k+1} \in \mathscr{I}(\nu_i).$$

Let $q$ be the first index for which $\alpha_q \neq 0$ and take $\alpha_q = 1$. Then $q \leqslant k$ and

$$c = z^n a^q (1 + \alpha_{q+1} a + \ldots + \alpha_{k+1} a^{k+1-q}) \equiv z^n a^q (1+b).$$

Since $b$ vanishes at 0 there is a neighborhood $N$ of 0 such that $1+b \neq 0$ on $N$. Let $e_0 \in C^n$ be supported on $N$ and be identically 1 in some neighborhood $N_1$ of 0. The function $k = ae_0(1+b)^{-1}$ clearly belongs to $A_n$. On $N_1$ we have $ck = z^n a^{q+1}$ and hence $ck - z^n a^{q+1}$ vanishes near 0, and thus belongs to $\mathscr{I}(\nu_i)$. Since $ck \in \mathscr{I}(\nu_i)$, we obtain $z^n a^{q+1} \in \mathscr{I}(\nu_i)$, hence $z^n a^{k+1} \in \mathscr{I}(\nu_i)$ by Lemma 3.10. Thus we have shown that $\mathscr{I}(\nu_i)$ is closed, so $\mathscr{I}(\nu_i) \supseteq M_{n,n}(\gamma_i)$. But it follows then that $\mathscr{I}(\nu_i) \supseteq M_{n,n-1}(\gamma_i)$. The argument to prove this is exactly as in the proof of [8], Lemma 1.7. Consequently $\mathscr{I}(\nu) \supseteq M_{n,n-1}(F)$. Thus (b) $\Rightarrow$ (c). If $\mathscr{I}(\nu)$ is closed, then $\mathscr{I}(\nu) \supseteq M_{n,n}(F) = J(F)^-$, so (c) $\Rightarrow$ (b).

(c) $\Leftrightarrow$ (d). If $s_1, s_2 \in \mathscr{S}(\nu)$ we can choose $\{f_m\} \subseteq M_{n,n}(F) \subseteq \mathscr{I}(\nu)$ with $\nu(f_m) \to s_1$. Thus $0 = \nu(f_m)s_2 \to s_1 s_2$ as $m \to \infty$, showing (c) $\Rightarrow$ (d). Now suppose $\mathscr{S}(\nu)^2 = \{0\}$ and let $\theta\colon C^n \to \mathscr{B}(\mathscr{S}(\nu))$ be defined by $\theta(f)(s) = \nu(f)s$, $f \in C^n$, $s \in \mathscr{S}(\nu)$. By Lemma 2.9, $\theta$ is continuous so $\ker(\theta) = \mathscr{I}(\nu)$ is closed. Thus (d) $\Rightarrow$ (c).

(c) $\Leftrightarrow$ (e). The function $p(z) = \prod_{i=1}^{r} (z-\gamma_i)^n$ belongs to $M_{n,n-1}(F)$, so (c) $\Rightarrow$ (e). Assuming (e) we have that $(z-\gamma_i)^n \in \mathscr{I}(\nu_i)$, since $C^n$ is a Silov algebra. Hence it follows from Lemma 3.10 that $M_{n,n}(\gamma_i) = (z-\gamma_i)^n A_n(\gamma_i) \subseteq \mathscr{I}(\nu_i)$. Consequently $\mathscr{I}(\nu_i)$ is closed for every $i$, so (e) $\Rightarrow$ (c).

(e) $\Leftrightarrow$ (f). If $\nu$ is bounded on $M_{n,n}(F)^2$ for the graph norm we have

$$\|\nu(pg)\| \leqslant K[\|g\| + \|pg\|] \leqslant K'\|g\|, \quad g \in M_{n,n}(F),$$

so $p \in \mathscr{I}(\nu)$. Conversely, if $p \in \mathscr{I}(\nu)$, there is a constant $C$ such that $\|\nu(pg)\| \leqslant C\|g\| \leqslant C[\|g\| + \|pg\|]$, $g \in C^n$.

The implication (e) $\Rightarrow$ (g) follows from [8], Corollary 1.14, applied to each of the maps $\nu_i$.



3.12. Remarks. (i) In [5], p. 375, there is given an example of a module derivation $D\colon M_{n,0} \to \mathscr{M}$ which is discontinuous and for which $\mathscr{S}(D)$ is infinite dimensional. Using the standard method of obtaining a homomorphism from a derivation (as described in [26], Remark 8.1) we obtain a homomorphism $\nu\colon C^n \to C^n \oplus \mathscr{M}$ for which $\mathscr{S}(\nu)$ is infinite dimensional, but $\mathscr{S}(\nu)^2 = \{0\}$, thus proving that in Theorem 3.11 (a) is strictly stronger than (b).

(ii) The implication (a) $\to$ (g) provides us with a new proof of the result of Dales and McClure [15] that higher point derivations on $C^n$ are always $C^{2n}$-continuous. (See the discussion in [8], Section 2.)

(iii) It is a curious consequence of Theorem 3.11 that if $F$ is a finite subset of $[0, 1]$, the closed ideal $M_{n,n}(F)$ is never the continuity ideal of a discontinuous homomorphism of $C^n$.

(iv) Suppose $B$ is finite dimensional and $\nu\colon C^n \to B$ is a homomorphism with singularity set $F$. Then $\ker(\nu)$ has finite codimension. It follows that $\ker(\nu) \supseteq J(F)$. To see this let $g, h \in J(F)$ and suppose $h$ is identically one on the support of $g$. There exists a polynomial $P$ without constant term such that $P(h) \in \ker(\nu)$ and $P(h)g = g$. Thus $g \in \ker(\nu)$. Now we have

$$\overline{J(F)} = M_{n,n}(F) \subseteq \overline{\ker(\nu)} \subseteq \nu^{-1}(\mathscr{S}(\nu)).$$

Since $\mathscr{I}(\nu) \supseteq M_{n,n}(F)$, we have

$$\nu(M_{n,n}(F)^2) \subseteq \nu(\mathscr{I}(\nu))\mathscr{S}(\nu) = \{0\}.$$

This answers a question raised in [8], p. 266.

3.13. Remark. Theorems 3.3 and 3.11 raise the following questions.

Question 1. Does there exist a $C^{2n+1}$-continuous homomorphism $\nu\colon C^n \to B$ which is not $C^{2n}$-continuous? (3)

Question 2. Does there exist a $C^{2n}$-continuous homomorphism $\nu\colon C^n \to B$ which is not continuous on $M_{n,n}(F)^2$ for the graph norm?

To place these questions in proper perspective we suppose for simplicity that $\nu$ is a homomorphism with singularity set $\{0\}$, and that $\overline{\mathscr{I}(\nu)} = M_{n,n}$. By [8], Theorem 1.9, $\nu$ is continuous for the graph norm on $M_{n,n} \cdot \mathscr{I}(\nu)$, i.e. there is a constant $K$ such that

$$\|\nu(f)\| \leqslant K[\|f\| + \|f/z^n\|], \quad f \in M_{n,n} \cdot \mathscr{I}(\nu).$$

Thus we may write $\nu = \mu + \lambda$, where $\mu$ is continuous on $M_{n,n}^2$ for the graph norm,

$$\lambda\colon M_{n,n}^2 \to \mathrm{rad}(B) \quad \text{and} \quad \lambda(M_{n,n} \cdot \mathscr{I}(\nu)) = 0.$$

(3) Added in proof. In the following paper *Eventual continuity in Banach algebras of differentiable functions*, H. G. Dales gives a negative solution to Question 1, assuming the continuum hypothesis. He shows there is a sense in which one has $C^{2n+\varepsilon}$ continuity for every $\varepsilon > 0$.

Now

$$M_{2n+1,2n+1} \subseteq M_{2n,2n} = M_{n,n}^2 \cap C^{2n}.$$

Also

$$M_{2n+1,2n+1} = z^{2n+1}A_{2n+1} \subseteq z^{n+1}(z^n A_n) = z^{n+1}M_{n,n}.$$

Suppose that $\nu$ is eventually continuous. Then $z^{n+1} \in \mathscr{I}(\nu)$ and hence $M_{2n+1,2n+1} \subseteq M_{n,n} \cdot \mathscr{I}(\nu)$. Thus necessarily $\lambda(M_{2n+1,2n+1}) = 0$, showing that $\nu$ is always continuous for the graph norm on $M_{2n+1,2n+1}$. We now show the following are equivalent:

(i) $\nu$ is $C^{2n}$-continuous;

(ii) $\lambda(M_{2n,2n}) \equiv 0$;

(iii) $\nu$ is continuous on $M_{2n,2n}$ for the graph norm.

Supposing (i) we get that $\lambda$ is $C^{2n}$-continuous on $M_{2n,2n}$, since $\mu$, being continuous for the graph norm, must be $C^{2n}$-continuous. Now (ii) follows, since we can approximate each function in $M_{2n,2n}$ by functions in $M_{2n+1,2n+1}$ for the $C^{2n}$-norm. Clearly (ii) ⇒ (iii) ⇒ (i). Thus for the answer to Question 1 to be affirmative we must have $\nu = \mu + \lambda$ with $\lambda(M_{2n,2n}) \not\equiv 0$ and for Question 2 we need $\lambda(M_{2n,2n}) \equiv 0$ but $\lambda(M_{n,n}^2) \not\equiv 0$.

As shown in [8], Section 2, one may use discontinuous higher order point derivations to construct discontinuous homomorphisms of $C^n$ into finite dimensional Banach algebras. In [6] there is given a more complex example of an isomorphism of $C^1$ which is discontinuous on every dense subalgebra of $C^1$. This example has one-dimensional separating space.

3.14. Remark. The only examples known to us of permanently discontinuous homomorphisms of $C^n$ are those that arise as the restrictions to $C^n$ of discontinuous homomorphisms of $C^0 = C[0,1]$. Such homomorphisms have been constructed by Dales [14] and Esterle [17]. Their difficult constructions are made possible by the assumption of the continuum hypothesis. It would be very interesting to know if a permanently discontinuous homomorphism of $C^n$ can be constructed without this assumption.

Let $C^\infty = C^\infty([0,1])$ be the space of all complex functions which are infinitely differentiable on the closed interval $[0,1]$. We give $C^\infty$ the Fréchet topology determined by the norms

$$\|f\|_m = \max_{t\in[0,1]} \sum_{j=0}^{m} \frac{|f^{(j)}(t)|}{j!}, \quad m = 1, 2, \ldots$$

3.15. Theorem. *Let $\nu$: $C^n \to B$ be a homomorphism whose restriction to $C^\infty$ is continuous. Then $\nu$ is $C^{2n+1}$-continuous.*

Proof. By hypothesis there exists a constant $K$ and an integer $k \geqslant n$ such that

$$\|\nu(f)\| \leqslant K\|f\|_k, \quad f \in C^\infty.$$



Thus there exists a continuous homomorphism $\mu$: $C^k \to B$ such that $\mu(f) = \nu(f)$, $f \in C^\infty$. The operator $T$ of multiplication by $\nu(z)$ $(= \mu(z))$ on $B_k = \overline{\nu(C^k)}$ is then a $C^k$-operator. By Lemma 3.2 and Theorem 3.3 (iii), $\nu$ is $C^{2k+1}$-continuous, and hence $C^{2n+1}$-continuous.

In the same circle of ideas we have:

3.16. Theorem. *Let $I$ be a closed interval and $\mu$: $C^n(I) \to B$ be a continuous homomorphism. Suppose $\nu$: $C^n(I) \to B$ is any other homomorphism for which $\nu(z) = \mu(z)$. Then*

$$[\nu(f) - \mu(f)]^3 = 0, \quad \text{all } f \in C^n(I).$$

Proof. We may suppose $\nu$ is discontinuous and write $\nu = \mu + \lambda$. Then $\mathscr{S}(\nu) = \mathscr{S}(\lambda)$. We assert that $\mathscr{S}(\nu) = \overline{\lambda(C^n(I))}$. Clearly $\mathscr{S}(\nu) \subseteq \overline{\lambda(C^n(I))}$. If $f \in C^n(I)$ and $\{p_k\}$ is a sequence of polynomials with $p_k \to f$, then

$$\nu(f - p_k) = \mu(f - p_k) + \lambda(f) \to \lambda(f)$$

since $\mu$ is continuous and $\lambda$ vanishes on polynomials. Thus $\lambda(f) \in \mathscr{S}(\nu)$. Now as in the last proof we conclude that the operator of multiplication by $\nu(z)$ has no non-trivial divisible subspaces, so $\nu$ is $C^{2n+1}$-continuous. Thus $\lambda(f)^3 \in \mathscr{S}(\nu)^3 = \{0\}$ for all $f \in C^n(I)$, by Theorem 3.3.

3.17. Corollary. *The natural operational calculus for a $C^n$-operator is unique, up to the addition of nilpotents of order three.*

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## Multiplier criteria of Hörmander type for Jacobi expansions

by

GEORGE GASPER* (Evanston, Ill.) and WALTER TREBELS (Darmstadt)

**Abstract.** It is shown how the multiplier criteria of Hörmander type derived in Connett and Schwartz [7] for Jacobi expansions by the use of finite differences can be substantially improved by using fractional differences. The main result, stated n Theorem 1, is in a certain sense best possible.

**1. Introduction.** In this paper we show how the multiplier criteria of Hörmander type [14] derived in Connett and Schwartz [7] for Jacobi expansions by the use of finite differences can be substantially improved by using fractional differences.

To state our results we shall employ the following notation which, for the convenience of the reader, is essentially that in [7]. Fix $\alpha \geqslant \beta \geqslant -1/2$, $\alpha > -1/2$ and let $L^p = L^p_{(\alpha,\beta)}$, $1 \leqslant p < \infty$, denote the space of measurable functions $f(x)$ on $(-1,1)$ for which

$$\|f\|_p = \Big(\int_{-1}^{1} |f(x)|^p\, dm(x)\Big)^{1/p} < \infty,$$

where $dm(x) = dm_{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta dx$. Also let $R_n(x) = R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of order $(\alpha,\beta)$, [16]. Each $f \in L^p$ has an expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} f\hat{\ }(n) h_n R_n(x),$$

where

$$f\hat{\ }(n) = \int_{-1}^{1} f(x) R_n(x)\, dm(x)$$

and

$$h_n = h_n^{(\alpha,\beta)} = \|R_n\|_2^{-2} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{2^{\alpha+\beta+1}\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)}.$$

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