This shows that \( (\epsilon_k^d) \) and \( \sigma^{-w} \) define the same Köthe space so the proof is complete.

References

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A nondental set without the tree property

by

J. Bourgain

(Prague)

Abstract. The existence is shown of a bounded, closed, convex and nondental subset of \( X^* \), which does not contain a tree.

Introduction. Let \( X, || \) be a Banach space with dual \( X^* \). If \( x \in X \) and \( \epsilon > 0 \), then \( B(x, \epsilon) \) denotes the open ball with midpoint \( x \) and radius \( \epsilon \). For sets \( A \subset X \), let \( e(A) \) be the convex hull and \( \epsilon(A) \) the closed convex hull of \( A \). We will say that \( A \) is denetal if for all \( \epsilon > 0 \) there exists \( x \in A \) satisfying \( x \notin e(A \setminus B(x, \epsilon)) \). The Banach space \( X \) is said to be denetal if every nonempty, bounded subset of \( X \) is denetal. We say that \( X \) has the Radon–Nikodým property (RNP) provided for every measure space \( (\Omega, \Sigma, \mu) \) with \( \mu(\Omega) < \infty \) and every \( \mu \)-continuous measure \( \mu : \Sigma \to X \), there exists a Bochner integrable function \( f : \Omega \to X \) such that \( \mu(E) = \int_E f d\mu \) for every \( E \subset \Sigma \). The RNP of \( X \) is equivalent with the fact that any uniformly bounded \( X \)-valued martingale on a finite measure space is convergent a.e. (cf. [3], [18]). It is known that \( X \) is a denetal Banach space if and only if \( X \) has RNP. The reader will find the history of the equivalence between these two properties in the survey paper [6] of J. Diezelt and J. J. Uhl.

A set with the Radon–Nikodým property (RNP-set) is a bounded, closed and convex subset of \( X \) such that each of its nonempty subsets is denetal. For some remarkable properties of these sets, I refer the reader to [1], [2] and [15].

Definition 1. A bush \( B \) is a bounded subset of \( X \) such that for some \( \epsilon > 0 \) the property \( x \in B \) holds for all \( x \in B \).

A tree \( T \) is a bounded subset of \( X \) such that for some \( \epsilon > 0 \) we have that each point \( x \in T \) is the midpoint of 2 points \( y, z \in T \) with \( \|y - z\| \geq \epsilon \).

* Aspirant, N. F. W. O., Belgium.
Obviously every tree is a bush. It is also routine to check that every 
bush (resp. tree) contains a countable bush (resp. tree).

We say that a subset of $X$ possesses the bush (resp. tree) property if it contains a bush (resp. tree). The interest of these notions will be clear from the following 2 results:

**Proposition 1** (Hull-Morris, see [9]). All nondentable bounded, 
closed and convex subsets and therefore all Banach spaces failing RNP have the bush-property.

**Proposition 2** (Stogoll, see [14]). Every nondentable convex $w^*$-
compact subset of a conjugate space has the tree property. All duals without RNP have the tree property.

This leads to the natural questions:

**Problem 1**. Is it true that every Banach space failing RNP has the tree-property?

**Problem 2**. If a bounded, closed and convex set in a Banach space is not dentable, does it necessarily contain a tree?

Clearly an affirmative answer to Problem 2 would solve Problem 1 affirmatively. Unfortunately the answer to the second question is negative as will be shown in this paper and hence Problem 1 remains open.

If $A$ is a subset of $X$ and $Y$ a subspace of $X^*$, we agree to call $Y$ 
$\mathcal{A}$-norming provided $\iota^*: X^* \to Y^*$ maps $A$ viewed as a subset of $X^*$ isometrically on $\iota^*(A)$, where $\iota: Y \to X^*$ is the canonical imbedding. The following is related to the problems mentioned above.

**Proposition 3**. (1) If $C$ is a nondentable bounded, closed and convex 
subset of $X$ and $Y$ is a $C$-norming subspace of $X^*$, then either $C$ has the tree-property or $Y$ imbeds isomorphically in $Y^*$.

(2) If $X$ is a separable Banach space failing RNP, then $X$ has the tree-property provided $\iota^*$ is not isomorphic to a quotient-space of $X$.

**Proof.** (1) If $\iota: Y \to X^*$ is the canonical embedding, then $\iota^*(C)$ is a nondentable subset of $Y^*$. From Theorem 1 of [2], the required result is obtained.

(2) This is an immediate consequence of (1) and [19].

**Counterexample to Problem 2.** We will first establish a result about the convergence of martingales in finite-dimensional spaces.

If $a$ is a positive real number, let $[a]$ denotes its integer part. For each integer $d$, $\mathcal{P}(d)$ will be the $d$-dimensional euclidean space.

**Lemma 1.** Let $d \in \mathbb{N}$ and $(\xi_k, \Sigma_k)$ be an $\mathcal{P}(d)$-valued martingale on a probability space $(\Omega, \Sigma, \mu)$, which is uniformly bounded by $M > 0$. Then for every $\varepsilon > 0$ there exists some $k \leq [M^d \varepsilon^{-1}] + 1$ satisfying $\|\xi_k - \xi_{k+1}\| < \varepsilon$.

**Proof.** Since for every coordinate $i = 1, \ldots, d$, $(\xi_k)_i$ is a real 
martingale, we obtain $\sum_{j=1}^{d} \xi_k^i \xi_{k+1}^i \mu d \xi_k = \|\xi_k\|^2$ and hence

$$\|\xi_k - \xi_{k+1}\| = \|\xi_k - \xi_{k+1}\| - \|\xi_k - \xi_k\|.$$ 

Therefore

$$\|\xi_{k+1} - \xi_k\| = \sum_{i=1}^{d} \|\xi_{k+1}^i - \xi_k^i\| = \|\xi_{k+1}^i - \xi_k^i\|. $$

Now suppose $n \in \mathbb{N}$ such that $\|\xi_{k+1} - \xi_k\| > \varepsilon$ for all $k = 1, \ldots, n$. Then we find

$$\sum_{k=1}^{n} \|\xi_{k+1} - \xi_k\| \leq \sum_{k=1}^{n} \|\xi_{k+1} - \xi_k\| = \|\xi_{k+1} - \xi_k\| \leq M^d \varepsilon^{-1}.$$ 

and hence $n \leq M^d \varepsilon^{-1}$. This completes the proof.

**Proposition 4.** Let $F$ be a real $d$-dimensional Banach space and let $(\xi_t, \Sigma_t)$ be an $\mathcal{P}(d)$-valued martingale on a probability space $(\Omega, \Sigma, \mu)$ which is uniformly bounded by $M > 0$. Then for every $\varepsilon > 0$ there exists some $k \leq [M^d \varepsilon^{-1}] + 1$ satisfying $\|\xi_{k+1} - \xi_k\| < \varepsilon$.

**Proof.** It is known that $F$ admits a biorthogonal sequence $(e_i, e_i^*)$ where $\|e_i\| = \|e_i^*\| = 1$. If $T: F \to F(d)$ is the operator defined by $Te_i = (e_i^*)^*(\xi_t)$, then it is easily verified that $\|T\| = \sqrt{d}$ and $\|T^{-1}\| = \sqrt{d}$. Since the $\mathcal{P}(d)$-valued martingale $(T\xi_t, \Sigma_t)$ is bounded by $\sqrt{d} M$, there is some $k \leq [M^d \varepsilon^{-1}] + 1$ so that $\|T\xi_{k+1} - T\xi_k\| < \varepsilon$ and thus $\|\xi_{k+1} - \xi_k\| < \varepsilon$.

For all $r, s \in \mathbb{N}$ with $r < s$, we let $B_{rs}$ be the Banach space $\mathbb{F}^r$. Define $B = \bigotimes_{r=1}^{s} B_{rs}$ as the $\mathbb{F}^s$-sum of the spaces $B_{rs}$. If $r \leq s$, then there is a natural projection $p_{rs}: B \to B_{rs}$. Take $\|e_i\| = \|e_i^*\| = \|e_i\| = \sup_{s \in \mathbb{F}^s} \|e_i^*\|$ for each $s \in B$.

The next section is devoted to the proof of the following result.

**Theorem 1.** Let $\eta: \mathbb{N} \to \mathbb{N}$ be any function which increases in both variables.

Then there exist a subspace $X$ of $B$, a sequence $(e_t)$ in $B$ and for each $s \in \mathbb{N}$ an operator $q_s: B \to B_s$ such that the following conditions are satisfied:

(1) $X = \text{span}(e_t; I)$;

(2) $\|e_t\| \leq r^{-1}$;

(3) $(e_t; I)$ is a bush in $X$;

(4) The restriction $q_s X$ of each operator $q_s$ has finite rank, which is denoted by $rk_s$;

(5) $\|p_{rs}\| \leq 3$;

(6) $s = \lim sup q_s(x)$ for all $x \in X$. 

...
Let \( r \leq s, s \leq n \leq \eta(r, n, \rho, \tau, \epsilon) \) and \( \lambda_1, \ldots, \lambda_n \geq 0 \), then the inequality
\[
\left\| \sum_{m=1}^{n} \lambda_m \phi_m \right\|_{\ell^{\infty}_{\rho, \tau, \epsilon}} \geq \frac{1}{r+1} \sum_{m=1}^{n} \lambda_m \|\phi_m - \phi_0\| \geq \frac{1}{r+1} \sum_{m=1}^{n} \lambda_m \|\phi_m - \phi_0\|'
\]
holds.

Using Theorem 1, a counterexample to Problem 2 will be obtained. More precisely:

**Lemma 2.** Define \( \eta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) by \( \eta(t, r) = 2^{2^t + 1} + 1 \) and let \( \ell_x = \{0, 1\} \omega \) and \( (\mu_{x,k}) \) be as in Theorem 1. Then \( G = \ell_x \) is a non-dentable subset of \( X \) without the tree-property.

**Proof.** Assume that \( \|\phi\| \leq r \) if \( x \in G \) and in particular \( G \) is contained in the unit ball of \( X \). The fact that \( G \) is not dentable is an immediate consequence of (3). It remains to show that \( G \) does not possess the tree-property. If we assume the converse, then there is an \( s > 0 \) and a system \((\eta_{x,s})_{x = 1, \ldots, k} \) in \( G \) such that \( \eta_{x,1} = \eta_{x,2} = \cdots = \eta_{x,s} = \eta_{x,s+1} = \cdots = \eta_{x,k} \) and \( \|\phi_{x,s} - \phi_{x,s+1}\| \geq s \).

For each \( k \) let \( \Sigma_k \) be the algebra of subsets of \( \{0, 1\} \) generated by the intervals \([(t-1)2^{-k}, t2^{-k}] \) and \( \Sigma_k = \Sigma_k \cap \ell_x \). Then \( (\Sigma_k, L_k) \) is a \( \mathcal{X} \)-valued martingale on the Lebesgue space \([0, 1], \Sigma_k, \mu_{x,k}\) and is uniformly bounded by \( 1 \). Hence by (4) and (5), for each \( s \), the martingale \((\phi_{x,s}, L_k) \) ranges in an \( \mathbb{R}^k \)-dimensional space and is uniformly bounded by \( 1 \). Since \( \eta_{x,s} \) is uniformly bounded by \( \eta(t) \), it is clear that \( \eta_{x,s} \) is uniformly bounded by \( \eta(t) \), for each \( s \), \( r \geq 1 \), and therefore \( \|\phi - \phi_{x,s}\| \geq \eta(t)/(t+1) \) for \( s \geq r \) and \( k \leq b \), then \( \gamma^2 \leq 2^{b+1} \leq \eta(r, n, \rho, \tau, \epsilon) \) and we deduce from (7)
\[
\|\phi - \phi_{x,s}\| \geq (\Sigma_{2^{-b}} - \Sigma_{2^{b-2}} - \Sigma_{2^{b-4}}) \|\phi_{x,s}\| \geq \frac{1}{r+1} \int \|\phi_{x,s}(t) - \phi_{x,s}(t)\|dt
\]
and the same with \( k \) replaced by \( k+1 \). This implies that
\[
2\|\phi_{x,s} - \phi_{x,s}\| \geq \frac{1}{r+1} \int \|\phi_{x,s}(t) - \phi_{x,s}(t)\|dt - \frac{1}{r+1} \int \|\phi_{x,s+1}(t) - \phi_{x,s}(t)\|dt
\]
For each \( s \) \( \geq r \) some \( k \leq b \), so that \( \|\phi - \phi_{x,s}\| \leq 1/r^2 \). It follows that \( \|\phi_{x,s} - \phi_{x,s}\| \geq 1/(r+1) \). From (6), we get the required contradiction.

**Proof of Theorem 1.** Let \( \mu \) denote the Lebesgue measure on \([0, 1] \). By induction we define sequences \((\xi_{n,k}) \) and \((\lambda_{n,k}) \) of positive integers, taking
\[
\xi_{n,k} = 1, \quad \lambda_{n+1} = \eta(\xi_{n+1}, \lambda_{n+1}, \ldots, \xi_{n}, \lambda_{n}) \quad \text{and} \quad \lambda_{n+1} = \mu_{x,n+1}.
\]
For each integer \( s \), let \( \mathcal{F} \) be the algebra of subsets of \([0, 1] \) generated by the intervals \( \{(i-1)\xi_{n+1}, i\xi_{n+1}\} \) where \( i = 1, \ldots, \xi_{n+1} \), which will be called \( s \)-primitive. We consider the algebra \( \mathcal{F} = \bigcup \mathcal{F} \). We say that an interval is \( s \)-primitive if it is \( s \)-primitive for some \( s \). Remark that \( 2 \) primitive intervals are either disjoint or comparable. For each integer \( s \), let \( \mathcal{F} \to \mathcal{F} \) be the mapping defined by \( \mathcal{F} \to \mathcal{F} \).

Let \( \mathcal{F}_{r,s} \) consist of the intervals which are \( r \)-primitive for some \( r \). Assume now \( \mathcal{F}_{r,s} \to \mathcal{F}_{r+1,s} \) obtained. \( \mathcal{F}_{r+1,s} \to \mathcal{F}_{r+1,s+1} \) will contain all sets of the form \( \bigcup_{k=1}^{n} (S_k \cup S_k) \) where \( n \leq \eta(\xi_{n+1}, r) \) and \( S_k \cup \bigcup_{k=1}^{n} (S_k \cup S_k) \) for each \( k \leq b \).

Take \( \mu_{x,s} = \eta(\xi_{n+1}, r) \) and \( \mu_{x,s} = \eta(\xi_{n+1}, 1) \). Then \( \mu_{x,s} \) increases when \( r \) increases and decreases when \( s \) increases. Moreover, we have

**Lemma 3.** If \( r \in \mathbb{N} \), then \( \lim \mu_{x,s} = 0 \).

**Lemma 4.** If \( B \in \mathcal{F}_{r,s} \), \( s \geq r \) and \( t \geq s \), then \( \mu(B) \leq \mu_{x,s} \) and \( \mu(B \setminus S) \leq \mu_{x,t} \).

**Proof.** (By induction on \( r \))

For \( r = 1 \), the statement is almost obvious. Assume now the property true for \( 1 = 1, \ldots, r \) and let \( B \in \mathcal{F}_{r,s} \).

Then \( B = \bigcup_{k} (S_k \cup S_k) \) where \( n \leq \eta(\xi_{n+1}, r) \) and \( S_k \cup \bigcup_{k=1}^{n} (S_k \cup S_k) \) for all \( k = 1, \ldots, r \).

Hence we show if \( n \leq s \leq r \), \( T \in \mathcal{F}_{r,s} \) and \( s+1 > r \), then \( m(T \setminus S) \leq \mu_{x,s} \).

We use the induction hypothesis and distinguish 2 cases:

(i) \( v \leq u \leq m(T \setminus S) \leq m(T) \leq \mu_{x,n} \leq \mu_{x,n+1} \leq \mu_{x,n+1} \);

(ii) \( u \geq v \geq m(T \setminus S) \leq \mu_{x,n+1} \leq \mu_{x,n+1} \).

From this it follows that \( m(S) \leq \eta(\xi_{n+1}, r) \mu_{x,s} = \mu_{x,t} \). Let further \( t \geq s \). It is clear that
\[
S \setminus S = \bigcup_{k=1}^{n} (S_k \cup S_k \setminus (S_k \cup S_k)) \]
But since \( \cup_{i=1}^{n} S_{i} = S \), it follows that \( S \setminus S_{i} = S_{i+1} \setminus S_{i} \), and hence

\[
m(S \setminus S_{i}) = \eta(\Omega_{n+1}, r) \mu_{n+1} \leq \eta(\Omega_{1}, r) \mu_{1} = \mu_{n+1} \mu_{n+1}.
\]

So the proof is complete.

**Lemma 5.** If \( S \in \mathcal{G}_{r,e} \), then \( \mu_{e} = 0 \).

**Proof.** By Lemma 4, \( m(S) \leq \mu_{e} \leq \eta(\Omega_{1}, r) \mu_{1} \), the measure of the \( e \)-primitive intervals.

From Lemma 3 and Lemma 4, we obtain

**Lemma 6.** If \( r \in \mathbb{N} \), then \( \lim_{n \to \infty} m(S_{n}) = 0 \).

It is clear that for \( e > 0 \) the family \( \mathcal{G}_{r,e} \) is infinite and therefore can be identified with \( \mathbb{N} \). For each primitive interval \( I \), we introduce an element

\[
e_{I} = \frac{m(I)}{m(I)} e_{I}
\]

in \( \mathcal{G}_{r,e} \). Let \( X = \text{span}(e_{I} ; I \text{ primitive}) \).

**Lemma 7.** \( (e_{I}) \) is a basis in \( X \).

**Proof.** If \( I \) is an \( e \)-primitive interval, then \( I \) is the disjoint union of \( n = e_{e+1} \) \((e+1)\)-primitive intervals \( I_{1}, \ldots, I_{n} \). It is easily verified that

\[
e_{I} = \frac{m(I_{1})}{m(I)} e_{I_{1}} + \cdots + \frac{m(I_{n})}{m(I)} e_{I_{n}} = \frac{1}{n} e_{I_{1}} + \cdots + \frac{1}{n} e_{I_{n}}.
\]

For each \( n = 1, \ldots, e \) we have that

\[
||e_{I} - e_{I_{n}}|| = \frac{m(I_{1})}{m(I)} ||e_{I_{1}} - e_{I_{n}}|| = \frac{1}{n} \geq \frac{1}{e}.
\]

For each \( e \leq e_{I} \), we define \( q_{I} : 0 \to B \) by

\[
q_{I}(x) = \frac{1}{e} e_{I} + \frac{1}{e} x - e_{I}
\]

Obviously \( q_{I} \) is a linear operator and \( ||q_{I}|| \leq 3 \).

**Lemma 8.** The rank \( r_{k} \) of the restriction \( q_{I} \) is at most \( O_{(1)} \).

**Proof.** In fact \( q_{I}(X) = \text{span}(e_{I} ; I \text{ primitive}) \). To see this, let \( J \) be a \( k \)-primitive interval. If \( u < t \), then using Lemma 7, we obtain \( e_{J} \in q_{I}(e_{J}) \) and hence \( q_{I}(e_{J}) \in \text{span}(q_{I}(e_{J}); I \text{ primitive}) \).

If \( u \geq t \), then \( J \) is contained in some \( e \)-primitive interval \( I \). For \( e \in \mathbb{N} \), we find

\[
q_{I}(e_{J}) = \frac{m(I \cap S)}{m(I)} e_{I} + \frac{m(S \setminus S_{J})}{m(I)} e_{I}
\]

and

\[
q_{I}(e_{J}) = \frac{m(S \setminus S_{J})}{m(I)} e_{I}
\]

showing that \( q_{I}(e_{J}) = q_{I}(e_{J}) \).

**Lemma 9.** \( x = \lim_{n \to \infty} q_{I}(x) \) for each \( x \in \mathbb{N} \).

**Proof.** Of course we can take \( x = e_{J} \) where \( I \) is primitive. If \( r \in \mathbb{N} \), \( e \geq r \) and \( S \in \mathcal{G}_{r,e} \), then

\[
(x - q_{I}(x)) = \frac{1}{m(I)} m(I \cap S)
\]

Choose \( e > 0 \) and \( r \in \mathbb{N} \) with \( \delta_{r} = \frac{1}{e} \). Take then \( t_{r} = t_{r} \) such that \( m(T) \leq \delta_{r} \) whenever \( r \leq t_{r} \), \( t_{r} \geq t_{r} \) and \( T \in \mathcal{G}_{r,e} \), which is possible by Lemma 6. We claim that \( \|x - q_{I}(x)\| \leq \delta_{r} \) if \( r \geq r_{e} \). Let us then \( r \in \mathbb{N} \), \( e \geq r \) and \( S \in \mathcal{G}_{r,e} \). If \( r > r_{e} \), then \( (x - q_{I}(x)) \leq \frac{1}{e} \delta_{r} \leq \delta \). If \( r \leq r_{e} \), then \( r + 1 \leq r_{e} \) and \( m(S \setminus S_{J}) \leq m(S_{J}) \), since \( S \setminus S_{J} \in \mathcal{G}_{r_{e}+1,e} \). Therefore also \( \|x - q_{I}(x)\| \leq \delta \).

It remains to verify condition (7) of the theorem. By Lemma 8, it will be enough to prove

**Lemma 10.** If \( r \leq s, x \leq \eta(\Omega_{r}, r), x_{1}, \ldots, x_{n} \in q_{I}(x) \) (I primitive) and \( x_{1}, \ldots, x_{n} \geq \delta_{e} \), then

\[
\sum_{m=1}^{n} \delta_{m} := \frac{1}{m(I)} m(I \cap S_{m})
\]

Choose \( r \geq s \) and \( r \in \mathbb{N} \) with \( \delta_{r} = \frac{1}{e} \). Take then \( t_{r} = t_{r} \) such that \( m(T) \leq \delta_{r} \) whenever \( r \leq t_{r} \), \( t_{r} \geq t_{r} \) and \( T \in \mathcal{G}_{r,e} \), which is possible by Lemma 6. We claim that \( \|x - q_{I}(x)\| \leq \delta_{r} \) if \( r \geq r_{e} \). Let us then \( r \in \mathbb{N} \), \( e \geq r \) and \( S \in \mathcal{G}_{r,e} \). If \( r > r_{e} \), then \( (x - q_{I}(x)) \leq \frac{1}{e} \delta_{r} \leq \delta \). If \( r \leq r_{e} \), then \( r + 1 \leq r_{e} \) and \( m(S \setminus S_{J}) \leq m(S_{J}) \), since \( S \setminus S_{J} \in \mathcal{G}_{r_{e}+1,e} \). Therefore also \( \|x - q_{I}(x)\| \leq \delta \).

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\[
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\]

Choose \( r \geq s \) and \( r \in \mathbb{N} \) with \( \delta_{r} = \frac{1}{e} \). Take then \( t_{r} = t_{r} \) such that \( m(T) \leq \delta_{r} \) whenever \( r \leq t_{r} \), \( t_{r} \geq t_{r} \) and \( T \in \mathcal{G}_{r,e} \), which is possible by Lemma 6. We claim that \( \|x - q_{I}(x)\| \leq \delta_{r} \) if \( r \geq r_{e} \). Let us then \( r \in \mathbb{N} \), \( e \geq r \) and \( S \in \mathcal{G}_{r,e} \). If \( r > r_{e} \), then \( (x - q_{I}(x)) \leq \frac{1}{e} \delta_{r} \leq \delta \). If \( r \leq r_{e} \), then \( r + 1 \leq r_{e} \) and \( m(S \setminus S_{J}) \leq m(S_{J}) \), since \( S \setminus S_{J} \in \mathcal{G}_{r_{e}+1,e} \). Therefore also \( \|x - q_{I}(x)\| \leq \delta \).
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d\text{References}