But (j) and (jj) are the required conditions of the desymmetrization lemma in [9], thus u_i, $|i| = k$ are nontangentially bounded a.e. on E. Now Lemma 4 applies, and we can write $f = g+b$, $g \in \mathrm{Lip}(k, \boldsymbol{R}^n)$ and b vanishes on a set F of measure close to E. That $g \in t_k(x)$ a.e. $x \in \boldsymbol{R}^n$ can be seen in [2]. For the bad part b, we have $b(x+t) = O(|t|^k)$ a.e. $x \in F$, which again can be improved to $b(x+t) = o(|t|^k)$ a.e. $x \in F$, this time a basic argument on density points applies [9].

(5.3) The L^p counterpart in (5.2) is also true. The proof uses Theorem 2 [9], p. 248 in all its depth, i.e. the equivalences for non tangentially boundedness for conjugate harmonic functions. Could be of some interest to prove this result outside the framework of harmonic analysis as in (5.2).

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Basic sequences in a stable finite type power series space

by

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Abstract. A characterization is given of when a nuclear Fréchet space with basis is isomorphic to the subspace generated by a basic sequence in a stable finite type power series space. The characterization is in terms of an inequality very similar to the one obtained for basic sequences in (s) and a nuclearity condition. Several structural facts are obtained as applications of the main result.

In [7] and [8] we characterized, respectively, subspaces and quotient spaces (with basis) of the infinite type power series space (s). An interesting feature of this characterization is that it is done in terms of inequalities (type (d_3), (d_4) below) and the only difference between subspaces and quotient spaces is the sense of the inequality. Recently, Alpseymen [1] considered the case of a stable infinite type power series space $\Lambda_\infty(\alpha)$ and determined, for the characterization of subspaces, that the same inequality works with the additional requirement that the space be $\Lambda_{\boldsymbol{N}}(\alpha)$-nuclear in the sense of Ramanujan and Terzioglu [13].

In this paper we turn to finite type power series spaces. It turns out that subspaces with bases can again be characterized in terms of two kinds of conditions: an inequality and a stronger type of nuclearity. The inequality (type (d_5) below) is only slightly different from the one obtained for basic sequences in (s) [7] and the nuclearity condition is $\Lambda_1(\alpha)$-nuclearity as studied by Robinson [15].

Our results on subspaces and quotient spaces have been extended by Vogt and Wagner [16], [17] to eliminate the requirement of a basis. So far, this has only been done for subspaces and quotient spaces of (s).

We apply our main theorem to obtain several results about the structure of nuclear Köthe spaces. We are able to completely describe all power series subspaces of any stable power series space. We describe all $L_f(b, r)$ subspaces of a stable finite type power series space and obtain some new information in the absence of the stability assumption. Finally we obtain the interesting fact that the only type (d_2) subspace of a finite type power series space is a (finite type) power series space.

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Preliminaries. We denote by N the set of positive integers. Our topological vector spaces are always assumed to be nuclear Fréchet spaces. For the elementary theory of such spaces we refer to [12]. We will also make use of Kolmogorov diameters whose basic properties are described in [11]. By the term *subspace* we will understand a closed infinite dimensional subspace of a nuclear Fréchet space.

A sequence (x_n) in a nuclear Fréchet space E is a *basis* if for each $x \in E$ there is a unique sequence (t_n) of numbers such that $x = \sum t_n x_n$. The sequence (x_n) is a *basic sequence* in E if it is a basis for the closed subspace which it generates. If (x_n) is a basis in E and (p_n) is a strictly increasing sequence of positive integers with $p_0 = 0$, then a sequence (y_n) in E is a *block basic sequence* provided that each y_n is a non-zero linear combination of x_i, $p_{n-1} < i \leqslant p_n$. It is easy to check that a block basic sequence is, in fact, a basic sequence.

By the absolute basis theorem of Dynin and Mitiagin [9] the set of nuclear Fréchet spaces with basis is essentially the same as the set of Köthe spaces. A *Köthe space* $K(a)$ is determined by an infinite matrix $a = (a_n^k)$ which satisfies

$$0 < a_n^k < a_n^{k+1}, \qquad n, k \in N,$$

$$\forall\, k \in N\, \exists\, j \in N \ni \sum_n \frac{a_n^k}{a_n^j} < \infty.$$

Then we set

$$K(a) = \Big\{\xi = (\xi_n)\colon\ p_k(\xi) = \sum_n a_n^k |\xi_n| < \infty,\ k \in N\Big\}.$$

Using the fundamental system of norms (p_k) the space $K(a)$ becomes a nuclear Fréchet space in which the coordinate sequences form a basis. An alternative system of norms which is equivalent in the nuclear case is given by (g_k) where

$$g_k(\xi) = \sup_n a_n^k |\xi_n|, \qquad k \in N.$$

If (t_n) is a sequence of positive numbers and $K(a)$, $K(b)$ are Köthe spaces with $b_n^k = t_n a_n^k$, then we have an isomorphism $T\colon K(b) \to K(a)$ by $T(\xi) = (t_n \xi_n)$. We call T a *diagonal transform*.

The simplest and most useful examples of Köthe spaces are the power series spaces. An increasing sequence α is said to be a *nuclear exponent sequence of finite type* if $\lim \dfrac{\log n}{\alpha_n} = 0$ and of *infinite type* if $\sup \dfrac{\log n}{\alpha_n} < \infty$. In the former case we define the *finite type power series space* $\Lambda_1(\alpha)$ to be the Köthe space $K(a)$ with

$$a_n^k = e^{-\alpha_n/k}, \qquad n, k \in N$$

and in the latter case we define the *infinite type power series space* $\Lambda_\infty(\alpha)$ to be the Köthe space $K(a)$ with

$$a_n^k = e^{k\alpha_n}, \qquad n, k \in N.$$

We say that α is *stable* if $\sup \dfrac{\alpha_{2n}}{\alpha_n} < \infty$, and the power series space is *stable* if the exponent sequence is.

A more general class of Köthe spaces is given by the $L_f(b, r)$ spaces of Dragilev [3]. Let f be a non-decreasing, odd, non-constant function which is logarithmically convex on the real line. Let $b = (\beta_n)$ be a non-decreasing sequence of real numbers and let $-\infty < r \leqslant \infty$. We define $L_f(b, r)$ to be the Köthe space $K(a)$ with

$$a_n^k = e^{f(r_k \beta_n)}, \qquad n, k \in N$$

where (r_k) is any sequence which increases monotonically to r. In addition there is a condition on f and b which is equivalent to nuclearity and too complicated to be of interest here (see [3], p. 78). Obviously we obtain the power series spaces when f is the identity. It can be shown ([3], p. 78) that every $L_f(b, r)$ space is isomorphic either to one in which f is the identity or f satisfies the following condition:

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = \infty \qquad \text{for all } t > 1.$$

In this case we say that f is *rapidly increasing*.

Our characterization of basic sequences in $\Lambda_1(\alpha)$ will be in terms of a new basis property, (d_5). To justify this notation and also for use below we recall the definition of type (d_i), $i = 0, \ldots, 4$. A space is of *type* (d_i) if it has a basis of type (d_i). A basis (y_n) in a space is of *type* (d_i) if the space has a fundamental system of norms $(\|\cdot\|_k)$ such that

for type (d_0):

$$\frac{\|y_n\|_{k+1}}{\|y_n\|_k} \leqslant \frac{\|y_{n+1}\|_{k+1}}{\|y_{n+1}\|_k}, \qquad n, k \in N;$$

for type (d_1):

$$\exists\, k \ni \forall\, j\, \exists\, l \ni \frac{\|y_n\|_j}{\|y_n\|_k} \leqslant \frac{\|y_n\|_l}{\|y_n\|_j}, \qquad n \in N;$$

for type (d_2)

$$\forall k\, \exists\, j \ni \forall\, l\ \frac{\|y_n\|_l}{\|y_n\|_j} \leqslant \frac{\|y_n\|_j}{\|y_n\|_k}, \qquad n \in N;$$

for type (d_3):

$$\frac{\|y_n\|_{k+1}}{\|y_n\|_k} \leqslant \frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}}, \qquad n, k \in N;$$

for type (d_4):

$$\frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}} < \frac{\|y_n\|_{k+1}}{\|y_n\|_k}, \quad n, k \in \boldsymbol{N}.$$

The term *regular* is usually used for bases of type (d_0) and, unlike here, this property is generally included as an hypothesis in the definitions of types (d_1), (d_2). As given here, types (d_1) and (d_3) are equivalent but types (d_2) and (d_4) are not. For details and other properties of these types we refer to [2], [3], [7] and [8].

The second condition in our characterization is a type of nuclearity. Rather than recall the original definition, it suffices for our purpose to observe that in view of [15], Theorem 2.6, it follows that a locally convex space E is $\Lambda_1(\alpha)$-nuclear iff for each barreled nbd of 0, U, in E there exist a barreled nbd of 0, V, in E and $C > 0$ such that

$$d_n(V, U) \leqslant e^{-C\alpha_n}, \quad n \in \boldsymbol{N}$$

where d_n here refers to the nth Kolmogorov diameter.

Main results. The following lemma is a special case of Lemma 2 ([4], p. 261) along with equation (3) of its proof. We omit the details of the proof.

LEMMA. *Let (a_n^k) be an infinite matrix of positive numbers such that*

$$\frac{a_n^{k+1}}{a_n^k} < \frac{a_{n+1}^{k+1}}{a_{n+1}^k}, \quad n, k \in \boldsymbol{N}.$$

Given numbers $t_1, \ldots, t_p$ we define, for $k \in \boldsymbol{N}$,

$$q^k(t_1, \ldots, t_p) = \max\{q\colon \max_{1 \leqslant i \leqslant p} |t_i|\, a_i^k = |t_q|\, a_q^k\}.$$

Then if $0 < q^1 < \ldots < q^m \leqslant p$ are integers, it is possible to choose numbers $t_1, \ldots, t_p$ with $t_{q^l} \neq 0$ but otherwise arbitrary, $t_i = 0$ for $i \neq q^1, \ldots, q^m$ and

$$|t_{q^k}| \frac{a_{q^k}^{k+1}}{a_{q^{k+1}}^{k+1}} < |t_{q^{k+1}}| < |t_{q^k}| \frac{a_{q^k}^k}{a_{q^{k+1}}^k}, \quad k = 1, 2, \ldots, m-1.$$

Moreover, if such a choice is made, then

$$q^k(t_1, \ldots, t_p) = q^k, \quad k = 1, \ldots, m.$$

Our characterization of subspaces with basis of $\Lambda_1(\alpha)$ is in terms of a new basis property, (d_5). We will say that a basis (y_n) in a nuclear Fréchet space E is of *type* (d_5) if there is a fundamental system of norms $(\|\ \|_k)$ and a number $D \geqslant 1$ such that

$$\log \frac{\|y_n\|_{k+1}}{\|y_n\|_k} \leqslant D \log \frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}}, \quad n, k \in \boldsymbol{N}.$$

We will say that a nuclear Fréchet space is of *type* (d_5) if it has a basis of type (d_5).

THEOREM. *Let α be a stable nuclear exponent sequence of finite type. Then a nuclear Fréchet space E with a basis is isomorphic to a subspace of $\Lambda_1(\alpha)$ if and only if E is of type (d_5) and $\Lambda_1(\alpha)$-nuclear.*

Proof. First suppose that E has a basis and is isomorphic to a subspace of $\Lambda_1(\alpha)$. Without loss of generality we may assume that E is the space generated by a basic sequence (y_n) in $\Lambda_1(\alpha)$. Let (V_k) be the fundamental system of nbds of 0 for $\Lambda_1(\alpha)$ given by

$$V_k = \Big\{\xi = (\xi_n)\colon \sum_n |\xi_n|^2 \Big(\frac{k}{k+1}\Big)^{2\alpha_n} < 1\Big\}.$$

Then, given U we can find an index k and $R > 0$ and then choose V such that

$$RU \supset V_k \cap E \supset V_{k+1} \cap E \supset \frac{1}{R} V.$$

Using elementary properties of diameters and the fact that V_k, V_{k+1} are ellipsoids in $\Lambda_1(\alpha)$ we have

$$d_n(V, U) = R^2 d_n\Big(\frac{1}{R} V, RU\Big) \leqslant R^2 d_n(V_{k+1} \cap E, V_k \cap E)$$
$$\leqslant R^2 d_n(V_{k+1}, V_k) = R^2 \Big(\frac{k(k+2)}{(k+1)^2}\Big)^{\alpha_n}.$$

From the properties of α we have,

$$\log d_n(V, U) \leqslant \log R^2 - \alpha_n \log \frac{(k+1)^2}{k(k+1)} \leqslant -C\alpha_n$$

where C is appropriately chosen. Thus, E is $\Lambda_1(\alpha)$-nuclear.

Next, let $(\|\cdot\|_k)$ be the increasing fundamental system of norms for E given by

$$\|\xi\|_k = \sup_n |\xi_n| e^{-\alpha_n/k}, \quad k \in \boldsymbol{N},\ \xi = (\xi_n) \in E \subset \Lambda_1(\alpha).$$

We can write $y_n = (t_j^n)_j$ so that

$$\|y_n\|_k = |t^n_{q_n^k}|\, e^{-\alpha_{q_n^k}/k}$$

where q_n^k is the largest index at which the sup occurs. We then have

$$e^{\alpha_{q_n^k}/k(k+1)} = \frac{|t_{q_n^k}|\, e^{-\alpha_{q_n^k}/(k+1)}}{|t_{q_n^k}|\, e^{-\alpha_{q_n^k}/k}} \leqslant \frac{|t_{q_n^{k+1}}|\, e^{-\alpha_{q_n^{k+1}}/(k+1)}}{|t_{q_n^k}|\, e^{-\alpha_{q_n^k}/k}} \leqslant \frac{|t_{q_n^{k+1}}|\, e^{-\alpha_{q_n^{k+1}}/(k+1)}}{|t_{q_n^{k+1}}|\, e^{-\alpha_{q_n^{k+1}}/k}}$$
$$= e^{\alpha_{q_n^{k+1}}/k(k+1)}$$

and so,

$$\frac{\alpha_{q_n^k}}{k(k+1)} \leqslant \log\frac{\|y_n\|_{k+1}}{\|y_n\|_k} \leqslant \frac{\alpha_{q_n^{k+1}}}{k(k+1)},$$

and increasing k by 1 in the left hand inequality leads to

$$\frac{\alpha_{q_n^{k+1}}}{(k+1)(k+2)} \leqslant \log\frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}}.$$

Therefore we have

$$\log\frac{\|y_n\|_{k+1}}{\|y_n\|_k} \leqslant \frac{\alpha_{q_n^{k+1}}(k+2)}{k(k+1)(k+2)} \leqslant \frac{k+2}{k}\log\frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}} \leqslant 3\log\frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}}$$

which is the (d_5) condition.

Turning to the converse we may assume without loss of generality that α is strictly increasing. Let $\sigma: N\times N \to N$ be the bijection given by $\sigma(j,n) = 2^{j-1}(2n-1)$. Let $(\|\cdot\|_k)$, D be such that the (d_5) condition holds for (y_n) and set $U_k = \{\sum_n t_n y_n \in E\colon \sum |t_n|\,\|y_n\|_k < 1\}$, $k\in N$. Applying the fact the E is $\Lambda_1(\alpha)$-nuclear to U_1 we have a nbd of 0, V, in E and a constant so the stated inequality holds. Choose k_0 so that U_{k_0} is contained in a multiple of V and assume that (y_n) has been permuted so that

$$d_n(U_{k_0}, U_1) = \frac{\|y_n\|_1}{\|y_n\|_{k_0}}, \quad n\in N.$$

Set $D_0 = D^{k_0}$ and choose $M > 1/D_0$ from the stability. It then follows that we can choose $C > 0$ so that

$$C\alpha_{2n-1} < CM\alpha_n \leqslant \log\frac{\|y_n\|_{k_0+1}}{\|y_n\|_1}, \quad n\in N.$$

We write $c_n^k = \|y_n\|_{k_0(k-1)+1}$ and it follows from the (d_5) condition that

$$\log\frac{c_n^{k+1}}{c_n^k} \leqslant D_0\log\frac{c_n^{k+2}}{c_n^{k+1}}, \quad n, k\in N.$$

Finally, set $r_k = C^{-1}(MD_0)^{k-1}$ so that we have a fundamental system of norms $(|\cdot|_k)$ for $\Lambda_1(\alpha)$ given by

$$|\xi|_k = \sup_n |\xi_n| e^{-\alpha_n/r_k}, \quad k\in N,\ \xi = (\xi_n)\in\Lambda_1(\alpha).$$

We begin our construction by selecting for each n a certain strictly increasing sequence of indices $(q_n^k)_k$. Set $q_n^1 = 1$; so we have

$$\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\alpha_{\sigma(q_n^1,n)} = \left(C - \frac{C}{MD_0}\right)\alpha_{2n-1} < C\alpha_{2n-1} < \log\frac{\|y_n\|_{k_0+1}}{\|y_n\|_1} = \log\frac{c_n^2}{c_n^1}.$$

Assume that $q_n^1 < q_n^2 < \ldots < q_n^k$ have been selected and let q_n^{k+1} be the smallest index such that

$$\log\frac{c_n^{k+1}}{c_n^k} \leqslant \left(\frac{1}{r_k}-\frac{1}{r_{k+1}}\right)\alpha_{\sigma(q_n^{k+1},n)}.$$

Assuming by induction that

$$\left(\frac{1}{r_k}-\frac{1}{r_{k+1}}\right)\alpha_{\sigma(q_n^k,n)} < \log\frac{c_n^{k+1}}{c_n^k}$$

it follows from the fact that α is strictly increasing that $q_n^k < q_n^{k+1}$. We then have,

$$\left(\frac{1}{r_{k+1}}-\frac{1}{r_{k+2}}\right)\alpha_{\sigma(q_n^{k+1},n)} = \left(\frac{1}{r_k}-\frac{1}{r_{k+1}}\right)\alpha_{\sigma(q_n^{k+1}-1,n)}\,\frac{\dfrac{1}{r_{k+1}}-\dfrac{1}{r_{k+2}}}{\dfrac{1}{r_k}-\dfrac{1}{r_{k+1}}}\,\frac{\alpha_{\sigma(q_n^{k+1},n)}}{\alpha_{\sigma(q_n^{k+1}-1,n)}}$$

$$< \left(\log\frac{c_n^{k+1}}{c_n^k}\right)\frac{r_kr_{k+2}-r_kr_{k+1}}{r_{k+1}r_{k+2}-r_kr_{k+2}}\,M$$

$$\leqslant \frac{(MD_0)^{2k}-(MD_0)^{2k-1}}{(MD_0)^{2k+1}-(MD_0)^{2k}}\,MD_0\log\frac{c_n^{k+2}}{c_n^{k+1}} = \log\frac{c_n^{k+2}}{c_n^{k+1}}.$$

We thus have constructed (q_n^k) to satisfy

$$\left(\frac{1}{r_k}-\frac{1}{r_{k+1}}\right)\alpha_{\sigma(q_n^k,n)} < \log\frac{c_n^{k+1}}{c_n^k} \leqslant \left(\frac{1}{r_k}-\frac{1}{r_{k+1}}\right)\alpha_{\sigma(q_n^{k+1},n)}, \quad n, k\in N.$$

Next define the sequence $(t_j^n)_j$ by writing

$$t^n_{\sigma(q_n^k,n)} = c_n^k e^{\alpha_{\sigma(q_n^k,n)}/r_k}, \quad n, k\in N$$

and $t_j^n = 0$ if j is not equal to some $\sigma(q_n^k, n)$. Then,

$$e^{\frac{1}{r_{k+1}}\left(\alpha_{\sigma(q_n^{k+1},n)}-\alpha_{\sigma(q_n^k,n)}\right)} \leqslant \frac{|t^n_{\sigma(q_n^{k+1},n)}|}{|t^n_{\sigma(q_n^k,n)}|} \leqslant e^{\frac{1}{r_k}\left(\alpha_{\sigma(q_n^{k+1},n)}-\alpha_{\sigma(q_n^k,n)}\right)}.$$

We may then apply the above lemma (after an appropriate truncation). Thus if we set $z_n = (t_j^n) \in \Lambda_1(\alpha)$ we have

$$|z_n|_k = |t^n_{\sigma(q_n^k,n)}|\, e^{-\alpha_{\sigma(q_n^k,n)}/r_k} = c_n^k, \quad n, k\in N$$

so that (z_n) is a block basic sequence in $\Lambda_1(\alpha)$ which generates a subspace isomorphic to E.

This completes the proof of the theorem.

The two conditions which characterize basic sequences in $\Lambda_1(\alpha)$ in our theorem are probably what we should expect in general. The nuclearity condition expresses the idea that any subspace of $\Lambda_1(\alpha)$ must be "at least as strongly nuclear" as $\Lambda_1(\alpha)$. The (d_5) condition, which is, incidentally, independent of α, is an inequality that seems to be connected with the set of finite type power series spaces. Similarly, the (d_3) condition is related to infinite type power series spaces (see [7]). Eventually it may be possible to devise a mechanism for predicting inequalities which correspond to various classes of spaces.

Another simple observation to make is that this theorem gives more spaces for which "block basic sequences are enough". That is, we have the following general question:

If F is a nuclear Fréchet space with basis and E is a subspace of F and has a basis is there a block basic sequence in F which generates a subspace isomrophic to E?

According to our theorem, the answer is yes if F is a stable finite type power series space. In [1] and [7] the question is answered affirmatively for stable infinite type power series spaces.

Turning to more technical issues we point out that it is important to describe type (d_5) in terms which do not depend on the choice of basis or even on the system of norms. This is particularly useful if one is trying to show that a space is *not* of type (d_5). In view of a well-known result of Bessaga and Dragilev ([2], Lemma 2.0) and the absolute basis theorem of Dynin and Mitiagin it is not hard to show that in a nuclear Fréchet space of type (d_5), every basis is of type (d_5). Regarding the seminorms we have the following results.

PROPOSITION 1. *If a basis (y_n) in a nuclear Fréchet space E is of type (d_5) and $(|\cdot|_k)$ is any fundamental sequence of norms, then there is a sequence (k_j) of indices and a sequence (D_j) of positive constants such that*

$$\log\frac{|y_n|_{k_j+1}}{|y_n|_{k_j}}\leqslant D_j\log\frac{|y_n|_{k_j+2}}{|y_n|_{k_j+1}},\qquad n,j\in\mathbf{N}.$$

Proof. Suppose that (y_n) is of type (d_5) and $(|\cdot|_k)$ is a fundamental system of norms. Let $(\|\cdot\|_k)$ be the system of norms and D the constant given by the (d_5)-condition. Then using the nuclearity we can select by induction subsequences of indices (k_j) and (l_j) such that for each j there exists n_j such that

$$|y_n|_{k_j+1}\leqslant\|y_n\|_{l_j+1}\leqslant\|y_n\|_{2l_j+1-l_{j-1}}\leqslant|y_n|_{k_j+2},\qquad n\geqslant n_j.$$

Then for $n\geqslant n_j$,

$$\frac{|y_n|_{k_j+1}}{|y_n|_{k_j}}\leqslant\frac{\|y_n\|_{l_j+1}}{\|y_n\|_{l_{j-1}}}\leqslant\left(\frac{\|y_n\|_{2l_j+1-l_{j-1}}}{\|y_n\|_{l_j+1}}\right)^{D^{l_{j+1}-l_{j-1}}}\leqslant\left(\frac{|y_n|_{k_j+2}}{|y_n|_{k_j+1}}\right)^{D^{l_{j+1}-l_{j-1}}}$$

and it suffices to take D_j sufficiently larger than $D^{l_{j+1}-l_{j-1}}$ so that the inequality holds for all n. This completes the proof.

We are now able to prove the converse of this result with the additional assumption that the space E is $\Lambda_1(\alpha)$-nuclear.

PROPOSITION 2. *Let α be a stable nuclear exponent sequence of finite type and let E be a $\Lambda_1(\alpha)$-nuclear Fréchet space.*

Suppose further that E has a basis (y_n) and a fundamental system of norms $(\|\cdot\|_k)$ for which there exists a sequence (D_k) of positive constants such that

$$\log\frac{\|y_n\|_{k+1}}{\|y_n\|_k}\leqslant D_k\log\frac{\|y_n\|_{k+2}}{\|y_n\|_{k+1}},\qquad n,k\in\mathbf{N}.$$

Then E is of type (d_5).

Proof. The result is established if we show that E is isomorphic to a subspace of $\Lambda_1(\alpha)$. We do this by repeating the argument in the proof of the theorem with a different choice of the sequence (r_k). An analysis of that proof shows that the only properties of this sequence that are required are:

$$\operatorname*{Lim}_k r_k=\infty,\qquad \frac{1}{r_1}-\frac{1}{r_2}\leqslant C,\qquad \frac{r_k}{r_{k+2}}\cdot\frac{(r_{k+2}-r_{k+1})}{(r_{k+1}-r_k)}\leqslant\frac{1}{MD_k}.$$

Moreover, if these properties hold then we can complete the construction using the inequality in the statement of this proposition rather than the (d_5)-condition. To obtain these properties we merely select an increasing sequence (r_k) such that the first two conditions hold and also,

$$\frac{r_k}{r_{k+1}-r_k}\leqslant\frac{1}{MD_k},\qquad k\in\mathbf{N}\ \Big(M=\sup_n\frac{\alpha_{2n}}{\alpha_n}\Big).$$

But this relation implies the third property since the left hand side of the inequality increases in r_{k+2} (with r_k, r_{k+1} fixed) up to $\dfrac{r_k}{r_{k+1}-r_k}$ so it is dominated by that quantity and the proposition is proved.

Applications. Our first application is to an analysis of $L_f(b,r)$ subspaces of $\Lambda_1(\alpha)$. If α is stable, the information is now complete.

PROPOSITION 3. *Let α be a stable nuclear exponent sequence of finite type. If f is identity, then $L_f(b,r)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ iff $\sup_n\dfrac{\alpha_n}{\beta_n}<\infty$. If f is rapidly increasing and $r\leqslant 0$, then $L_f(b,r)$ is not isomorphic to a subspace of $\Lambda_1(\alpha)$. If f is rapidly increasing and $0<r\leqslant\infty$,*

then $L_f(b, r)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$ iff there exists a positive constant $A < r$ such that

$$\sup_n \frac{\alpha_n}{f(A\beta_n)} < \infty.$$

Proof. The first statement is an easy consequence of the theorem. It is also implicitly contained in the results of [4] and [6]. The second statement follows from the result of Zahariuta [18] that in this case, every operator from $L_f(b, r)$ to $\Lambda_1(\alpha)$ is compact.

For the last statement, we first observe that it is known [3] that if $0 < r < \infty$, then $L_f(b, r)$ is of type (d_1) and hence (d_3) so it is (d_5). If $A < r$ and $C\alpha_n \leqslant f(A\beta_n)$, $n \in N$, then given $k < r$, let $B = \max(A, k) < j < r$. From the definition of rapidly increasing we have

$$f(j\beta_n) \geqslant 2f(B\beta_n)$$

for n sufficiently large so

$$f(j\beta_n) - f(k\beta_n) \geqslant 2f(B\beta_n) - f(k\beta_n) \geqslant f(A\beta_n)$$

and therefore

$$e^{c\alpha_n} \leqslant e^{f(j\beta_n) - f(k\beta_n)}.$$

The theorem can then be applied to conclude that $L_f(b, r)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$. The converse is clear so the proof is complete.

The last statement in Proposition 3 can also be proved using the characterization of Alpseymen [1] along with some recent results of T. Terzioglu.

If α is not stable, there is much less known. Using a variation of our general method (cf. [5], proof of Theorem 1) we can obtain one new result.

PROPOSITION 4. *Every finite type power series space has a subspace isomorphic to some $L_f(b, \infty)$ space, f rapidly increasing.*

Proof. Without loss of generality we may assume that a given finite type power series space $\Lambda_1(\alpha)$ satisfies

$$\frac{\alpha_{n+1}}{\alpha_{n+2}} \leqslant \frac{\alpha_n}{\alpha_{n+1}} \leqslant \frac{1}{n(n+1)}, \quad n \in N.$$

Let $0 = p_0 \leqslant p_{n-1} < p_n$, $n \in N$ be a sequence of integers with $\operatorname{Lim}(p_n - p_{n-1}) = \infty$. Let $(\|\cdot\|_k)$ be the fundamental system of norms for $\Lambda_1(\alpha)$ given by $\|\xi\| = \sup_n |\xi_n| e^{-\alpha_n/k}$, $\xi = (\xi_n) \in \Lambda_1(\alpha)$. Consider the Köthe space $K(c)$ where $c_n^1 = 1$ and

$$c_n^k = e^{\frac{1}{k(k-1)}\alpha_{p_{n-1}+k}}, \quad k > 1, \ n \in N.$$

We will embed $K(c)$ as a subspace of $\Lambda_1(\alpha)$ generated by the block basic sequence (y_n) where $y_n = (t_j^n)_j$, $t_j^n = 0$ unless $p_{n-1} < j \leqslant p_n$ and

$$t_{p_{n-1}+k}^n = c_n^k e^{\frac{1}{k}\alpha_{p_{n-1}+k}}, \quad 1 \leqslant k \leqslant p_n - p_{n-1}.$$

To establish the embedding we apply our lemma to show that

$$q^k(t_{p_{n-1}+1}^n, \ldots, t_{p_n}^n) = p_{n-1} + k, \quad 1 \leqslant k \leqslant p_n - p_{n-1}.$$

To do this we must show that

$$e^{\frac{1}{k(k+1)}\alpha_{p_{n-1}+k}} \leqslant \frac{c_n^{k+1}}{c_n^k} \leqslant e^{\frac{1}{k(k+1)}\alpha_{p_{n-1}+k+1}}, \quad 1 \leqslant k \leqslant p_n - p_{n-1}.$$

This is clear for $k = 1$ and for $k > 1$ it reduces to

$$\frac{\alpha_{p_{n-1}+k}}{k+1} \leqslant \frac{\alpha_{p_{n-1}+k+1}}{k+1} - \frac{\alpha_{p_{n-1}+k}}{k-1} \leqslant \frac{\alpha_{p_{n-1}+k+1}}{k+1}.$$

The right hand inequality is clear and the left hand reduces to

$$\frac{\alpha_{p_{n-1}+k}}{\alpha_{p_{n-1}+k+1}} \leqslant \frac{k-1}{2k}$$

which holds because of our initial assumption.

It then follows from our lemma that

$$\|y_n\|_k = t_{p_{n-1}+k}^n e^{-\alpha_{p_{n-1}+k}/k} = c_n^k, \quad 1 \leqslant k \leqslant p_n - p_{n-1}$$

so that $K(c)$ is isomorphic to a subspace of $\Lambda_1(\alpha)$. On the other hand, it follows from our initial assumption that

$$e^{\alpha_{p_{n-1}+k}} \leqslant e^{\frac{1}{k(k+1)}\alpha_{p_{n-1}+k+1}} \leqslant e^{\alpha_{p_{n-1}+k+1}}, \quad n, k \in N,$$

so that $K(c) = K(a)$ where $a_n^k = e^{\alpha_{p_{n-1}+k}}$. But in [5] it was pointed out that $K(a)$ is an $L_f(b, r)$ space so the proof is complete.

Our next application is to obtain a somewhat surprising restriction on subspaces of a finite type power series space. Any $L_f(b, r)$ space, $r \leqslant 0$, is of type (d_2), and we have seen above that in this case if f is rapidly increasing, then $L_f(b, r)$ is not a subspace of $\Lambda_1(\alpha)$ but if f is the identity, then it may be. We strengthen this result by showing that this is the only way that a (d_2) space can be a subspace of $\Lambda_1(\alpha)$, even if α is not stable. A weaker version of this result was previously obtained by Ramanujan and Terzioglu [14].

PROPOSITION 5. *If E is a Köthe space which is of type (d_2) and is isomorphic to a subspace of a finite type power series space, then E is isomorphic to a finite type power series space.*

Proof. Using a standard argument with the (d_2) condition we may assume that E is isomorphic to $K(b)$ and

$$b_n^k b_n^l \leqslant [b_n^{k+1}]^2, \qquad n, k, l \in \boldsymbol{N}.$$

It is easy to see that this relation remains true if k runs through any subsequences of $\boldsymbol{N}$. Since E is isomorphic to a subspace of a finite type power series space, it is isomorphic to a subspace of one in which the exponent sequence is stable. By our theorem, E is of type (d_5) and by Proposition 1 we may assume that E is isomorphic to $K(c)$ and we have a sequence of positive constants (D_k) with

$$\frac{c_n^l}{c_n^{k+1}} \leqslant \frac{c_n^{k+1}}{c_n^k} \leqslant \left(\frac{c_n^{k+2}}{c_n^{k+1}}\right)^{D_k}, \qquad n, k, l \in \boldsymbol{N}.$$

Thus we have quantities D_{nk}, $n, k \in \boldsymbol{N}$ with $0 < D_{nk} \leqslant D_k$ and

$$\frac{c_n^{k+1}}{c_n^k} = \left(\frac{c_n^{k+2}}{c_n^{k+1}}\right)^{D_{nk}}, \qquad n, k \in \boldsymbol{N}$$

so that

$$c_n^k = c_n^2 \left(\frac{c_n^2}{c_n^1}\right)^{F_{nk}}, \qquad n \in \boldsymbol{N} \text{ and } k > 2$$

where

$$F_{nk} = \frac{1}{D_{n1}} + \frac{1}{D_{n1}D_{n2}} + \ldots + \frac{1}{D_{n1}\ldots D_{n,k-2}}.$$

Thus by a diagonal transform we may conclude that E is isomorphic to $K(a)$ where $\log a_n^k = F_{nk} \log \beta_n$ and we have

$$1 < \beta_n,$$
$$0 < F_{nk} < F_{n,k+1},$$
$$\frac{F_{n,k+1} - F_{nk}}{F_{n,k+2} - F_{n,k+1}} = D_{nk} \leqslant D_k,$$
$$F_{nl} - F_{n,k+1} \leqslant F_{n,k+1} - F_{nk}, \qquad n \in \boldsymbol{N}.$$

Hence if we write $F_n = \sup_k F_{nk}$, it follows from the above relations that

$$F_n = \sup_k F_{nk} = \lim_l F_{nl} \leqslant 2F_{n4} - F_{n3} < \infty, \qquad n \in \boldsymbol{N}.$$

We may then write

$$\gamma_n = \beta_n^{F_n}, \quad s_{nk} = 1 - \frac{F_{nk}}{F_n},$$

so that

$$a_n^k = \gamma_n^{-s_{nk}} \gamma_n, \qquad n, k \in \boldsymbol{N}.$$

Therefore, by another diagonal transform we may conclude that E is isomorphic to $K(d)$ where $d_n^k = \gamma_n^{-s_{nk}}$ and

$$1 < \gamma_n, \quad s_{n,k+1} < s_{nk}, \quad \lim_k s_{nk} = 0, \qquad n, k \in \boldsymbol{N},$$
$$\frac{s_{nk} - s_{n,k+1}}{s_{n,k+1} - s_{n,k+2}} = D_{nk} \leqslant D_k, \qquad n, k \in \boldsymbol{N},$$
$$\frac{s_{n,k+1} - s_{nl}}{s_{nk} - s_{n,k+1}} \leqslant 1, \qquad n, k, l \in \boldsymbol{N}.$$

We will complete the proof by showing that $K(d) = \Lambda_1(\alpha)$ with $\alpha_n = \log \gamma_n^{s_{n3}}$. First we show that $\inf_n \frac{s_{n,k+1}}{s_{nk}} > 0$, $k \geqslant 3$. If this were not so for some k, then we would have an infinite $N_0 \subset \boldsymbol{N}$ with $\lim_{n \in N_0} \frac{s_{n,k+1}}{s_{nk}} = 0$, so

$$0 < \frac{1}{D_k} \leqslant \frac{s_{n,k+1} - s_{n,k+2}}{s_{nk} - s_{n,k+1}} \leqslant \frac{s_{n,k+1}}{s_{nk} - s_{n,k+1}} = \frac{s_{n,k+1}}{s_{nk}} \frac{1}{1 - \frac{s_{n,k+1}}{s_{nk}}}$$

which goes to 0 as n goes to ∞ in N_0 and this is a contradiction. Thus we have,

$$\inf_n \frac{s_{nk}}{s_{n3}} > 0 \quad \text{for} \quad k \geqslant 3.$$

Next we observe that since $\lim_k s_{nk} = 0$ we have

$$\frac{s_{n,k+1}}{s_{nk} - s_{n,k+1}} \leqslant 1$$

or

$$s_{n,k+1} \leqslant \tfrac{1}{2} s_{nk}, \qquad n \in \boldsymbol{N} \text{ and } k \geqslant 3.$$

Hence,

$$\lim_k \sup_n \frac{s_{nk}}{s_{n3}} \leqslant \lim_k 2^{-k+3} = 0.$$

These two conclusions give us, on the one hand that for each $l \geqslant 3$ we have $k \in \boldsymbol{N}$ with $\frac{s_{nl}}{s_{n3}} \geqslant \frac{1}{k}$, $n \in \boldsymbol{N}$ so that

$$d_n^l = \gamma_n^{-s_{nl}} \leqslant \gamma_n^{-s_{n3}/k} = e^{-\alpha_n/k}, \qquad n \in \boldsymbol{N},$$

and, on the other hand, for each $k \in \boldsymbol{N}$ we have $j \in \boldsymbol{N}$ with $\sup_n \frac{s_{nj}}{s_{n3}} \leqslant \frac{1}{k}$,

$n \in N$ so that

$$e^{-a_n/k} = \gamma_n^{-s_{n3}/k} \leqslant \gamma_n^{-s_{nj}} = d_n^j, \quad n \in N.$$

This shows that (d_n^k) and $e^{-a_n/k}$ define the same Köthe space so the proof is complete.

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A nondentable set without the tree property

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Abstract. The existence is shown of a bounded, closed, convex and nondentable subset of l^∞, which does not contain a tree.

Introduction. Let X, $\|\ \|$ be a Banach space with dual X^*. If $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon)$ denotes the open ball with midpoint x and radius ε. For sets $A \subset X$, let $c(A)$ be the convex hull and $\bar{c}(A)$ the closed convex hull of A. We will say that A is *dentable* if for all $\varepsilon > 0$ there exists $x \in A$ satisfying $x \notin \bar{c}(A \setminus B(x, \varepsilon))$. The Banach space X is said to be *dentable* if every nonempty, bounded subset of X is dentable. We say that X has the *Radon–Nikodým property* (RNP) provided for every measure space (Ω, Σ, μ) with $\mu(\Omega) < \infty$ and every μ-continuous measure F: $\Sigma \to X$ of finite variation, there exists a Bochner integrable function f: $\Omega \to X$ such that $F(E) = \int_E f\,d\mu$ for every $E \in \Sigma$. The RNP of X is equivalent with the fact that any uniformly bounded X-valued martingale on a finite measure space is convergent a.e. (cf. [5], [18]). It is known that X is a dentable Banach space if and only if X has RNP. The reader will find the history of the equivalence between those two properties in the survey paper [6] of J. Diestel and J. J. Uhl.

A set with the Radon–Nikodým property (RNP-set) is a bounded, closed and convex subset of X such that each of its nonempty subsets is dentable. For some remarkable properties of these sets, I refer the reader to [1], [2] and [15].

DEFINITION 1. A *bush* B is a bounded subset of X such that for some $\varepsilon > 0$ the property $x \in c(B \setminus B(x, \varepsilon))$ holds for all $x \in B$.

A *tree* $\mathscr{T}$ is a bounded subset of X such that for some $\varepsilon > 0$ we have that each point $x \in \mathscr{T}$ is the midpoint of 2 points $y \in \mathscr{T}$, $z \in \mathscr{T}$ with $\|y - z\| \geqslant \varepsilon$.

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