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Dentability and finite-dimensional decompositions

by

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Abstract. It is shown that a Banach space possesses the Radon–Nikodým property if and only if every subspace with a finite-dimensional Schauder decomposition has the Radon–Nikodým property.

Introduction. Let $X, \| \ \|$ be a Banach space with dual X^*. If $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon)$ denotes the open ball with midpoint x and radius ε. For sets $A \subset X$, let $c(A)$ be the convex hull and $\bar{c}(A)$ the closed convex hull of A. We will say that A is *dentable* if for all $\varepsilon > 0$ there exists $x \in A$ satisfying $x \notin \bar{c}(A \setminus B(x, \varepsilon))$. The Banach space X is said to be *dentable* if every nonempty, bounded subset of X is dentable. We say that X has the Radon–Nikodým property (RNP) provided for every measure space (Ω, Σ, μ) with $\mu(\Omega) < \infty$ and every μ-continuous measure $F\colon \Sigma \to X$ of finite variation, there exists a Bochner integrable function $f\colon \Omega \to X$ such that $F(E) = \int_E f\,d\mu$ for every $E \in \Sigma$. The RNP of X is equivalent to the fact that any uniformly bounded X-valued martingale on a finite measure space is convergent a.e. (cf. [5], [21]).

It is known that X is a dentable Banach space if and only if X has RNP. For the history of the equivalence between those two properties, I refer the reader to the J. Diestel and J. J. Uhl survey paper [6].

Recall that $(P_n, M_n)_n$ is a finite-dimensional Schauder decomposition for the Banach space $\mathscr{X}$ iff each P_n is a continuous linear projection of $\mathscr{X}$ onto the finite-dimensional M_n, $P_n P_m = 0$ if $n \neq m$ and $x = \sum_n P_n(x)$ for each $x \in \mathscr{X}$. The partial sum operators S_n are defined by $S_n = \sum_{i=1}^{n} P_i$. Since $(S_n)_n$ is pointwise convergent to the identity operator, it is uniformly bounded. We denote by $G(M_n; n)$ the number $\sup_n \|S_n\|$, which is called the Grynblum constant of the decomposition. Our main result is the following:

THEOREM 1. *Assume X without RNP. Then for each $\lambda > 1$ there exist a subspace $\mathscr{X}$ of X, a uniformly bounded $\mathscr{X}$-valued martingale $(\xi_n)_n$ on $[0, 1]$*

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and a sequence $(S_n\colon \mathscr{X}\to\mathscr{X})_n$ *of finite rank projections, such that:*

(1) $x = \lim_n S_n(x)$, *for each* $x\in\mathscr{X}$,

(2) $\|S_n\|\leqslant\lambda$,

(3) $S_m S_n = S_n S_m = S_m$ *if* $m\leqslant n$,

(4) $S_n\xi_{n+1} = \xi_n$,

(5) $(\xi_n)_n$ *is nowhere convergent.*

Theorem 1 if of course a refinement of the existence of non-convergent martingales in Banach spaces without RNP. Various authors pointed out that the RNP is separably determined (cf. [20], [10], [13], [15]). Theorem 1 yields the following improvement.

THEOREM 2. *If X fails RNP, then for every $\lambda>1$ there exists a subspace $\mathscr{X}_\lambda$ of X without RNP and with a finite-dimensional Schauder decomposition with Grynblum constant at most λ.*

Indeed, $\mathscr{X}$ fails RNP. If we take $P_1 = S_1, P_{n+1} = S_{n+1}-S_n$, then $(P_n, P_n\mathscr{X})_n$ is a finite-dimensional Schauder decomposition of $\mathscr{X}$ with $G(P_n\mathscr{X}; n)\leqslant\lambda$.

Theorem 2 is related to the following question :

PROBLEM 1. *If every subspace of X possessing a Schauder basis possesses the RNP, then need X also possess it?*

To the best of my knowledge, this problem is still open.

Preliminary geometric lemmas. In this section, C will be a fixed nonempty, bounded, closed and convex subset of X.

We first introduce some terminology.

DEFINITION 1. If $x^*\in X^*$, define $M(x^*, C) = \sup x^*(C)$. For each $\alpha>0$, let $S(x^*,\alpha, C) = \{x\in C;\ x^*(x)\geqslant M(x^*, C)-\alpha\}$ and $\mathring{S}(x^*,\alpha,C) = \{x\in C;\ x^*(x)> M(x^*, C)-\alpha\}$. We will call $S(x^*,\alpha, C)$ a *slice* and $\mathring{S}(x^*,\alpha,C)$ an *open slice.* If S is a slice, $\mathring{S}$ will denote the corresponding open slice.

The reader will easily verify the following property.

LEMMA 1. *If $S(x^*,\alpha,C)$ is a slice, then there exist $\varepsilon>0$ and $\beta>0$ such that $\|x^*-y^*\|<\varepsilon$ implies $S(y^*,\beta,C)\subset\mathring{S}(x^*,\alpha,C)$.*

DEFINITION 2. Let $S = S(x^*,\alpha,C)$ be a slice. Define

$$V(S) = \{x\in C;\ \exists y^*\in x^* \text{ with } y^*(x) = M(y^*,C) > M(y^*, C\setminus\mathring{S})\}.$$

If further $n\in\mathbf{N}, x_1^*,\dots,x_n^*\in X^*$ and $\varepsilon>0$, let

$$V(S, x_1^*,\dots,x_n^*,\varepsilon) = \{x\in C;\ \exists y^*\in x^*, \exists\beta>0 \text{ with } y^*(x) = M(y^*,C),\ S(y^*,\beta,C)\subset\mathring{S} \text{ and } o(x_k^*|\ S(y^*,\beta,C))<\varepsilon\ (1\leqslant k\leqslant n)\}.$$

("o" means oscillation).

Using Lemma 1 and the Bishop–Phelps density result on the supporting functionals [1], we obtain immediately

LEMMA 2. *If S is a slice, then $V(S) = \emptyset$.*

DEFINITION 3. Suppose $S = S(x^*,\alpha,C)$ a slice. Let $\tilde{C}$ be the w^*-closure of C in X^{**} and $\operatorname{ex}(\tilde{C})$ the extreme points of $\tilde{C}$. Define

$$E(S) = \{x^{**}\in\operatorname{ex}(\tilde{C});\ x^{**}(x^*) > M(x, C^*)-\alpha\}.$$

It follows from the Krein–Milman theorem that $E(S)$ is nonempty.

LEMMA 3. *If S is a slice, $x_1^*,\dots,x_n^*\in X^*$ and $\varepsilon>0$, then there is a slice T such that:*

(1) $T\subset\mathring{S}$,

(2) $o(x_k^*|T)<\varepsilon$ $(1\leqslant k\leqslant n)$.

Hence we have

LEMMA 4. *If S is a slice, $x_1^*,\dots,x_n^*\in X^*$ and $\varepsilon>0$, then $V(S, x_1^*,\dots,x_n^*,\varepsilon)\neq\emptyset$.*

LEMMA 5. *If S is a slice, $x\in V(S)$ and D a closed convex subset of C with $x\notin D$, then $x\notin\bar{c}\big((C\setminus\mathring{S})\cup D\big)$.*

Proof. Take $y^*\in X^*$ satisfying $y^*(x) = M(y^*,C)> M(y^*, C\setminus\mathring{S})$. Suppose $x\in\bar{c}\big((C\setminus\mathring{S})\cup D\big)$, then $x = \lim_n\big(\lambda_n y_n+(1-\lambda_n)z_n\big)$, where $(y_n)_n$ is a sequence in $C\setminus\mathring{S}$, $(z_n)_n$ a sequence in D and $(\lambda_n)_n$ a sequence in $[0,1]$.

Hence $M(y^*,C)\leqslant\varliminf_n\big(\lambda_n M(y^*, C\setminus\mathring{S})+(1-\lambda_n)M(y^*,C)\big)$ showing that $\lim_n\lambda_n = 0$. It follows that $x = \lim_n z_n$ and thus $x\in D$, which is a contradiction.

LEMMA 6. *Let S be a slice, $x\in V(S)$, $x_1^*,\dots,x_n^*\in X^*$ and $\varepsilon>0$. Then $x\in\bar{c}\big(V(S,x_1^*,\dots,x_n^*,\varepsilon)\big)$.*

Proof. If $x\notin\bar{c}\big(V(S,x_1^*,\dots,x_n^*,\varepsilon)\big)$, we also have that $x\notin\bar{c}\big((C\setminus\mathring{S})\cup\cup V(S,x_1^*,\dots,x_n^*,\varepsilon)\big)$, by Lemma 5. By the separation theorem, there exists a slice T satisfying $T\subset\mathring{S}$ and $T\cap V(S,x_1^*,\dots,x_n^*,\varepsilon) = \emptyset$. But by Lemma 4, $V(T,x_1^*,\dots,x_n^*,\varepsilon)$ is a nonempty subset of $V(S,x_1^*,\dots,x_n^*,\varepsilon)$, a contradiction.

We now pass to the key lemma of this paper.

LEMMA 7. *Let S be a slice and U a weak open set such that $U\cap c\big(V(S)\big)\neq\emptyset$. Then there exist $n\in\mathbf{N}$, slices $S_1,\dots,S_n$ and positive numbers $\lambda_1,\dots,\lambda_n$ satisfying*

(1) $S_k\subset\mathring{S}$,

(2) $\sum_k\lambda_k = 1$,

(3) $\sum_k\lambda_k V(S_k)\subset U\cap c\big(V(S)\big)$.

Proof. If $x \in U \cap c(V(S))$, then there is a w-neighborhood $N(x, x_1^*, \ldots, x_p^*, \delta)$ of x contained in U. Since, by Lemma 6, $V(S) \subset \bar{c}(V(S, x_1^*, \ldots, x_p^*, \delta/2))$, we also have that $x \in \bar{c}(V(S, x_1^*, \ldots, x_p^*, \delta/2))$. Of course we can take $\|x_q^*\| \leqslant 1$ $(1 \leqslant q \leqslant p)$. Let then $n \in N$, $x_1, \ldots, x_n \in V(S, x_1^*, \ldots, x_p^*, \delta/2)$ and $\lambda_1, \ldots, \lambda_n$ positive numbers, with $\sum_k \lambda_k = 1$ and $\|x - \sum_k \lambda_k x_k\| < \delta/2$. For each $k = 1, \ldots, n$ a slice S is obtained so that $x_k \in S_k$, $S_k \subset \mathring{S}$ and $o(x_q^* | S_k) < \delta/2$ $(1 \leqslant q \leqslant p)$. Obviously $\sum_k \lambda_k V(S_k) \subset c(V(S))$. For every $q = 1, \ldots, p$, we find that

$$x_q^*\Big(\sum_k \lambda_k S_k\Big) = \sum_k \lambda_k x_q^*(S_k) \subset \sum_k \lambda_k \big(x_q^*(x_k) +]-\delta/2, \delta/2[\big)$$
$$= x_q^*\Big(\sum_k \lambda_k x_k\Big) +]-\delta/2, \delta/2[\subset x_q^*(x) +]-\delta, \delta[,$$

implying

$$\sum_k \lambda_k S_k \subset N(x, x_1^*, \ldots, x_p^*, \delta).$$

Hence $\sum_k \lambda_k V(S_k) \subset U \cap c(V(S))$.

Lemma 7 has the following immediate corollary, which will be used later.

LEMMA 8. *Let S be a slice, $\varepsilon > 0$ and U a w-open set such that $U \cap c(V(S)) \neq \emptyset$ and* $\operatorname{diam}(U \cap c(V(S))) \leqslant \varepsilon$. *Then there exist $n \in N$, slices $S_1, \ldots, S_n$ and positive numbers $\lambda_1, \ldots, \lambda_n$ satisfying $S_k \subset S$ $(1 \leqslant k \leqslant n)$, $\sum_k \lambda_k = 1$ and* $\operatorname{diam} \sum_k \lambda_k V(S_k) \leqslant \varepsilon$.

Banach spaces with property (*).

PROPOSITION 1. *For a Banach space X, the following properties are equivalent.*

(1) *For each nonempty, bounded, closed and convex subset A of X, the identity map on A has a $w - \|\ \|$ point of continuity.*

(2) *For each nonempty, bounded and convex subset A of X and for each $\varepsilon > 0$, there exists a w-open set U satisfying $U \cap A \neq \emptyset$ and* $\operatorname{diam}(U \cap A) \leqslant \varepsilon$.

(3) *For each nonempty, bounded and convex subset A of X and for each $\varepsilon > 0$, there exists a w-open set U such that $U \cap A \neq \emptyset$ and $U \cap A$ has an ε-net.*

Proof. The implications (1)⇒(2) and (2)⇒(3) are clear. Assume (3), let A be nonempty, bounded, convex and $\varepsilon > 0$. We obtain a w-open set U and a finite number of balls $(B_i)_{1 \leqslant i \leqslant d}$ with radius $\varepsilon/2$ so that $U \cap A \neq \emptyset$ and $U \cap A \subset \bigcup_{i=1}^{d} B_i$. If $x \in U \cap A$, then by the separation theorem we obtain a w-open set V with $x \in V$, $V \subset U$ and $V \cap B_i = \emptyset$ whenever $x \notin \bar{B}_i$ $(1 \leqslant i \leqslant d)$. Therefore $V \cap A \neq \emptyset$ and $\operatorname{diam}(V \cap A) \leqslant \varepsilon$. Hence (2) holds. If we have (2) and A is nonempty, bounded, closed and convex, then by repeating application of (2) a sequence $(U_n)_n$ of convex w-open sets is obtained verifying $\bar{U}_{n+1} \subset U_n$, $U_n \cap A \neq \emptyset$ and $\operatorname{diam}(U_n \cap A) \leqslant 1/n$. It follows that $\bigcap_n (U_n \cap A)$ consists of a unique point of A which is clearly a $w - \|\ \|$ continuity point.

DEFINITION 4. If X satisfies (1), (2), (3) of Proposition 1, we will say that *X has property* (*).

Clearly the following implication is true:

PROPOSITION 2. *If X has RNP, then X has property* (*).

The converse is open. Thus

PROBLEM 2. *Does property (*) imply dentability?*

Property (*) plays an important role in the proof of Theorem 1. We need the following lemma:

LEMMA 9. *If X fails property (*), then there exist a nonempty, bounded subset $\mathscr{A}$ of X and $\varepsilon > 0$ so that for each $x \in \mathscr{A}$ and each subspace E of X with* $\operatorname{codim} E < \infty$ *the condition* $\operatorname{diam}(\mathscr{A} \cap (x + E)) \geqslant \varepsilon$ *is verified.*

Proof. Let A be a nonempty, bounded and convex subset of X and $\varepsilon > 0$ failing (2) of Proposition 1. We will show that the open convex set $\mathscr{A} = A + B(0, 1)$ satisfies the conclusion of the lemma.

More precisely, we prove by induction on n that if $x \in \mathscr{A}$, E is a subspace of X with $\operatorname{codim} E \leqslant n$ and N a w-neighborhood of x, then $\operatorname{diam}(\mathscr{A} \cap (x + E) \cap N) \geqslant \varepsilon$.

In the case $n = 0$, this is almost obvious. Assume now the statement correct for n and let $\operatorname{codim} E = n+1$. Take $x \in \mathscr{A}$ and $N = N(x, x_1^*, \ldots, x_p^*, \delta)$ with $\|x_q^*\| \leqslant 1$ $(1 \leqslant q \leqslant p)$ a w-neighborhood of x. It is clearly enough to obtain that $\operatorname{diam}(\mathscr{A} \cap (x + E) \cap N) \geqslant \varepsilon - \varkappa$, where $0 < \varkappa < \delta$ is arbitrarily chosen. Take $\varrho > 0$ with $B(x, \varrho) \subset \mathscr{A}$. There exists a subspace F of X and $x^* \in X^*$ with $\operatorname{codim} F = n$, $\|x^*\| = 1$ and $E = F \cap \operatorname{Ker} x^*$. Obviously x^* is not zero on F and we obtain $\gamma > 0$ such that $x^*(F \cap B(0, \varrho)) \supset [-\gamma, \gamma]$. Take $\iota = \min(\varkappa\gamma/2D, \delta/2)$, where $D = \operatorname{diam} \mathscr{A}$ and let $0 = N(x, x_1^*, \ldots, x_p^*, x^*, \iota)$. Since by induction hypothesis $\operatorname{diam}(\mathscr{A} \cap (x + F) \cap 0) \geqslant \varepsilon$, we only have to show that if $y \in \mathscr{A} \cap (x + F) \cap 0$, then $\operatorname{dist}(y, \mathscr{A} \cap (x + E) \cap N) \leqslant \varkappa/2$. Take $h \in F \cap B(0, \varrho)$ such that $x^*(h) = \gamma$ or $x^*(h) = -\gamma$ according to whether $x^*(x) \geqslant x^*(y)$ or $x^*(x) < x^*(y)$. Then $\lambda = \dfrac{x^*(x) - x^*(y)}{x^*(x) - x^*(y) + x^*(h)}$ belongs to $[0, \iota/\gamma]$. If $z = (1-\lambda)y + \lambda(x + h)$, then $z \in \mathscr{A}$ and $z - x = (1-\lambda)(y - x) + \lambda h \in F$. We verify that $x^*(z) = x^*(x)$ and thus $z \in x + E$. Since $\|y - z\| \leqslant \iota/\gamma D \leqslant \varkappa/2$, $z \in N(x, x_1^*, \ldots, x_p^*, \delta)$ and thus $z \in \mathscr{A} \cap (x + E) \cap N$. Therefore $\operatorname{dist}(y, \mathscr{A} \cap (x + E) \cap N) \leqslant \varkappa/2$, which completes the proof.

LEMMA 10. *Take* $\mathscr{A}$ *and* ε *as in Lemma 9. Then for every* $x \in \mathscr{A}$ *and every subspace* E *of* X *with* $\operatorname{codim} E < \infty$ *we obtain* $x \in \bar{c}\big((\mathscr{A} \setminus B(x, \delta)) \cap (x+E)\big)$, *whenever* $\delta < \varepsilon/2$.

Proof. If $x \notin \bar{c}\big((\mathscr{A} \setminus B(x, \delta)) \cap (x+E)\big) = D$, there is $x^* \in X^*$ such that $x^*(x) > M(x^*, D)$. Let $F = E \cap \operatorname{Ker} x^*$. Since $\operatorname{diam}(\mathscr{A} \cap (x+F)) \geqslant \varepsilon$, there is a point $y \in \mathscr{A} \cap (x+F)$ with $\|x-y\| > \delta$. Hence $y \in D$, contradicting $D \cap (x+F) = \emptyset$.

Proof of the main theorem. We start with the following lemma.

LEMMA 11. *Let* $\lambda > 1$ *and suppose there exist* $\alpha > 0$, *an increasing sequence* $(\mathscr{X}_p)_p$ *of finite-dimensional subspaces of* X *and for each* p *a projection* $\pi_p\colon \mathscr{X}_{p+1} \to \mathscr{X}_p$, *a finite subset* A_p *of* $\mathscr{X}_p$ *and* $\beta_p > 0$, *satisfying*

(1) $\prod_p \|\pi_p\| \leqslant \lambda$,

(2) $\bigcup_p A_p$ *is bounded in* X,

(3) *If* $z \in A_p$, *then there are vectors* $z_1, \ldots, z_r \in A_{p+1}$ *so that* $\|z - z_s\| \geqslant \alpha$, $\pi_p(z_s) = z$ *for each* $s = 1, \ldots, r$ *and* $\operatorname{dist}(z, c(z_1, \ldots, z_r)) < \beta_p$,

(4) $\beta = \sum_p \beta_p < \alpha/2$.

Then there exist a subspace $\mathscr{X}$ *of* X, *a uniformly bounded* $\mathscr{X}$*-valued martingale* $(\xi_n)_n$ *on* $[0,1]$ *and a sequence* $(S_n\colon \mathscr{X} \to \mathscr{X})_n$ *of finite rank projections satisfying* (1), (2), (3), (4), (5) *of Theorem* 1.

Proof. Take $\mathscr{X} = \overline{\bigcup_p \mathscr{X}_p}$. It is routine to obtain for each p a projection S_p from $\mathscr{X}$ onto $\mathscr{X}_p$ so that $\|S_p\| \leqslant \lambda$ and $S_p = \pi_p S_{p+1}$. Thus (1), (2), (3) hold.

(3) allows us to construct for each p a finite field $\mathscr{B}_p$ generated by subintervals of $[0,1]$ and a $\mathscr{B}_p$-measurable map $\eta_p\colon [0,1] \to A_p$, so that

(1) $\|\eta_p(t) - \eta_{p+1}(t)\| \geqslant \alpha$ whenever $t \in [0,1]$,

(2) $\pi_p \eta_{p+1} = \eta_p$,

(3) $\|\eta_p - E[\eta_{p-1} | \mathscr{B}_p]\|_\infty < \beta_p$.

This construction is less or more standard and we omit the details. The reader can find them in [9] or [5].

If we introduce inductively maps ξ_p by taking $\xi_1 = \eta_1$ and $\xi_{p+1} = \eta_{p+1} + \xi_p - E[\eta_{p+1} | \mathscr{B}_p]$, then $(\xi_p, \mathscr{B}_p)_p$ is clearly a martingale. By induction, it is easily seen that ξ_p ranges in $\mathscr{X}_p$ and $\|\xi_p - \eta_p\|_\infty < \beta_1 + \ldots + \beta_{p-1} < \beta$. It follows that $(\xi_p)_p$ is uniformly bounded. Furthermore $S_p \xi_{p+1} = \pi_p \xi_{p+1} = \eta_p + \xi_p - E[\eta_p | \mathscr{B}_p] = \xi_p$ and $\|\xi_p(t) - \xi_{p+1}(t)\| \geqslant \|\eta_p(t) - \eta_{p+1}(t)\| - \|\xi_p - \eta_p\|_\infty - \|\xi_{p+1} - \eta_{p+1}\|_\infty > \alpha - 2\beta > 0$.

Thus $(\xi_p)_p$ is nowhere convergent, which completes the proof.

In the proof of the main theorem, two cases will be distinguished:

I. X fails property $(*)$,

II. X has property $(*)$ and fails RNP.

We start with the first one, which is also the easiest.

LEMMA 12. *Let* $\mathscr{A}$ *and* δ *satisfy the condition of Lemma* 10. *Let* $(\varepsilon_p)_p$ *be a sequence of positive numbers. Then for each* $p \in \mathbf{N}$, *we can define a finite subset* A_p *of* $\mathscr{A}$ *and a subspace* E_p *of* X, *satisfying the following properties*:

(1) $\operatorname{codim} E_p < \infty$ $(p \in \mathbf{N})$,

(2) *If* $x \in \operatorname{span}(A_1, \ldots, A_p)$, *then there exists* $x^* \in X^*$ *with*

$$\|x^*\| = 1, \quad x^* | E_p = 0 \quad \textit{and} \quad \|x\| \leqslant (1+\varepsilon_p) x^*(x) \quad (p \in \mathbf{N}),$$

(3) $A_{p+1} = \bigcup_{x \in A_p} A_{p+1}^x$, where $A_{p+1}^x \cap B(x, \delta) = \emptyset$, $A_{p+1}^x \subset x + E_p$ *and* $\operatorname{dist}(x, c(A_{p+1}^x)) < \varepsilon_p$ $(p \in \mathbf{N})$.

Proof. We proceed by induction on $p \in \mathbf{N}$.

(a) Let $A_1 = \{x_1\}$, where x_1 is an arbitrary point in $\mathscr{A}$. Consider a finite subset $\mathscr{E}_1$ of the unit sphere of X^* such that if $x \in \operatorname{span}(A_1)$, then there is $x^* \in \mathscr{E}_1$ with $\|x\| \leqslant (1+\varepsilon_1) x^*(x)$. Take $E_1 = \bigcap_{x^* \in \mathscr{E}_1} \operatorname{Ker} x^*$.

(b) Assume now A_p and E_p obtained. Let $x \in A_p$ be fixed. Since $x \in \bar{c}\big((\mathscr{A} \setminus B(x, \delta)) \cap (x+E_p)\big)$, there is a finite subset A_{p+1}^x of $\mathscr{A}$ so that $A_{p+1}^x \cap B(x, \delta) = \emptyset$, $A_{p+1}^x \subset x + E_p$ and $\operatorname{dist}(x, c(A_{p+1}^x)) < \varepsilon_p$. Define $A_{p+1} = \bigcup_{x \in A_p} A_{p+1}^x$. Again a finite subset $\mathscr{E}_{p+1}$ of the unit sphere of X^* can be obtained such that if $x \in \operatorname{span}(A_1, \ldots, A_{p+1})$, then $\|x\| \leqslant (1+\varepsilon_{p+1}) x^*(x)$ for some $x^* \in \mathscr{E}_{p+1}$. Take $E_{p+1} = \bigcap_{x^* \in \mathscr{E}_{p+1}} \operatorname{Ker} x^*$.

Clearly this completes the construction.

Proof of the theorem in case I. Take $\lambda > 1$ and let $(\varepsilon_p)_p$ be a sequence of positive numbers satisfying $\sum_p \varepsilon_p \leqslant \min(\ln \lambda, \delta/2)$ and hence $\prod_p (1+\varepsilon_p) \leqslant \lambda$. Let A_p and E_p be as in Lemma 12. For each $p \in \mathbf{N}$, take $\mathscr{X}_p = \operatorname{span}(A_1, \ldots, A_p)$, which is finite-dimensional. Clearly $A_{p+1} \subset A_p + E_p$ and thus $\mathscr{X}_{p+1} = \mathscr{X}_p + (E_p \cap \mathscr{X}_{p+1})$. Using (2), we see that $\|x\| \leqslant (1+\varepsilon_p) \|x+y\|$ whenever $x \in \mathscr{X}_p$ and $y \in E_p \cap \mathscr{X}_{p+1}$. Therefore there exists a projection π_p of $\mathscr{X}_{p+1}$ onto $\mathscr{X}_p$ with $\|\pi_p\| \leqslant 1 + \varepsilon_p$. If we take $\alpha = \delta$ and $\beta_p = \varepsilon_p$, the conditions of Lemma 11 are satisfied, finishing the proof.

We now pass to the case of a Banach space X with property $(*)$, failing RNP. Let C be a fixed, nonempty, bounded, closed and convex subset of X, which is not dentable.

LEMMA 13. *If* S *is a slice and* $\varepsilon > 0$, *then there exists a slice* T *with* $T \subset S$ *and* $\operatorname{dist}\big(x, \tilde{c}(E(S))\big) < \varepsilon$ *if* $x \in T$.

Proof. Let $S = S(x^*, \alpha, C)$ and take

$$D = \{x^{**} \in \tilde{C};\ x^{**}(x^*) \leqslant M(x^*, C) - \alpha\}.$$

We remark that $C = c\big(c(E(S)) \cup D\big)$. Let $d = \operatorname{diam} \tilde{C}$ and take $T = S(x^*, \beta, C)$, where $0 < \beta < \varepsilon\alpha/d$. If $x \in T$, there is $x_1^{**} \in \tilde{c}(E\ S))$, x_2^{**}

$\in D$ and $\lambda \in [0,1]$ with $x = (1-\lambda)x_1^{**} + \lambda x_2^{**}$. But then $M(x^*, C) - \beta \leqslant x^*(x) = (1-\lambda)x_1^{**}(x^*) + \lambda x_2^{**}(x^*) \leqslant M(x^*, C) - \lambda\alpha$, implying $\lambda < \varepsilon/d$. Hence $\|x - x_1^{**}\| = \lambda\|x_1^{**} - x_2^{**}\| < \varepsilon$, proving the lemma.

LEMMA 14. *There is $\iota > 0$ such that for every slice S, the set $E(S)$ has no ι-net.*

Proof. This follows immediately from the lemma of Huff and Morris [10] and Lemma 13.

LEMMA 15. *Let $n \in \boldsymbol{N}$, $S_1, \ldots, S_n$ slices, $x \in X$, $a_1, \ldots, a_n \in \boldsymbol{R}$ with $\sum_k |a_k| \leqslant 1$ and $\varepsilon > 0$. Then there are slices $T_1, \ldots, T_n$ and $x^* \in X^*$ with $\|x^*\| = 1$, so that*

(1) *$T_k \subset S_k$ $(1 \leqslant k \leqslant n)$,*

(2) *If $x_k \in c(V(T_k))$ $(1 \leqslant k \leqslant n)$, then*

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant x^*(x) + \sum_k a_k x^*(x_k) + \varepsilon.$$

Proof. Let $s = \sup\{\|x + \sum_k a_k x_k\|;\ x_k \in V(S_k)\ (1 \leqslant k \leqslant n)\}$. Take $x_k \in V(S_k)$ $(1 \leqslant k \leqslant n)$, such that $\|x + \sum_k a_k x_k\| \geqslant s - \varepsilon/2$ and let $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x) + \sum_k a_k x^*(x_k) = \|x + \sum_k a_k x_k\|$. For each $k = 1, \ldots, n$ we define D_k by taking

$$D_k = \begin{cases} \{x \in C;\ x^*(x) \leqslant x^*(x_k) - \varepsilon/2\} & \text{if} \quad a_k \geqslant 0, \\ \{x \in C;\ x^*(x) \geqslant x^*(x_k) + \varepsilon/2\} & \text{if} \quad a_k < 0. \end{cases}$$

By Lemma 5, we obtain a slice T_k so that $T_k \subset S_k$ and $T_k \cap D_k = \emptyset$. To verify condition (2), we can clearly replace $c(V(T_k))$ by $V(T_k)$. If now $y_k \in V(T_k)$ $(1 \leqslant k \leqslant n)$, we get

$$\begin{aligned}\Big\|x + \sum_k a_k y_k\Big\| &\leqslant s \leqslant \Big\|x + \sum_k a_k x_k\Big\| + \varepsilon/2 = x^*(x) + \sum_{k, a_k \geqslant 0} a_k x^*(x_k) + \\ &+ \sum_{k, a_k < 0} a_k x^*(x_k) + \varepsilon/2 \leqslant x^*(x) + \sum_{k, a_k \geqslant 0} a_k\big(x^*(y_k) + \varepsilon/2\big) \\ &+ \sum_{k, a_k < 0} a_k\big(x^*(y_k) - \varepsilon/2\big) + \varepsilon/2 \leqslant x^*(x) + \sum_k a_k x^*(y_k) + \varepsilon,\end{aligned}$$

what must be obtained.

LEMMA 16. *Let $n \in \boldsymbol{N}$, $S_1, \ldots, S_n$ slices and E a finite-dimensional subspace of X. Then there are slices $T_1, \ldots, T_n$ and $M > 0$ such that*

(1) *$T_k \subset S_k$ $(1 \leqslant k \leqslant n)$,*

(2) *If $x \in E$, $x_k \in T_k$ $(1 \leqslant k \leqslant n)$ and $a_k \in \boldsymbol{R}$ $(1 \leqslant k \leqslant n)$, then*

$$\|x\| + \sum_k |a_k| \leqslant M\Big\|x + \sum_k a_k x_k\Big\|.$$

Proof. Assume C in the unit ball of X. Using Lemma 14, we obtain for each $k = 1, \ldots, n$ a point $x_k^{**} \in E(S_k)$ such that E and $x_1^{**}, \ldots, x_n^{**}$ are linearly independent. Thus there are elements $(y_k^*)_{1 \leqslant k \leqslant n}$ in X^* satisfying $y_k^*|E = 0$ $(1 \leqslant k \leqslant n)$ and $x_k^{**}(y_l^*) = \delta_{k,l}$ $(1 \leqslant k, l \leqslant n)$. Take $M > 0$ such that $\|y_k^*\| \leqslant (M-1)/4n$ $(1 \leqslant k \leqslant n)$. Clearly there are slices $T_1, \ldots, T_n$ so that $T_k \subset S_k$ $(1 \leqslant k \leqslant n)$ and $|y_l^*(x) - \delta_{k,l}| < 1/2n$ if $x \in T_k$ $(1 \leqslant k, l \leqslant n)$.

If now $x \in E$, $x_k \in T_k$ $(1 \leqslant k \leqslant n)$ and $a_k \in \boldsymbol{R}$ $(1 \leqslant k \leqslant n)$, it follows for each $l = 1, \ldots, n$:

$$\frac{M-1}{4n}\Big\|x + \sum_k a_k x_k\Big\| \geqslant \Big|\sum_k a_k y_l^*(x_k)\Big| \geqslant \Big(1 - \frac{1}{2n}\Big)|a_l| - \frac{1}{2n}\sum_{k \neq l}|a_k|$$

and by addition

$$\frac{M-1}{4}\Big\|x + \sum_k a_k x_k\Big\| \geqslant \Big(1 - \frac{1}{2n}\Big)\sum_k |a_k| - \frac{n-1}{2n}\sum_k |a_k| = \frac{1}{2}\sum_k |a_k|.$$

Thus

$$\sum_k |a_k| \leqslant \frac{M-1}{2}\Big\|x + \sum_k a_k x_k\Big\|$$

and therefore

$$\|x\| \leqslant \Big\|x + \sum_k a_k x_k\Big\| + \Big\|\sum_k a_k x_k\Big\| \leqslant \frac{M+1}{2}\Big\|x + \sum_k a_k x_k\Big\|,$$

completing the proof.

LEMMA 17. *Let $n \in \boldsymbol{N}$, $S_1, \ldots, S_n$ slices, E a finite-dimensional subspace of X and $\varepsilon > 0$. Then there are slices $T_1, \ldots, T_n$ and a finite subset $\mathscr{E}$ of X^*, satisfying:*

(1) *$T_k \subset S_k$ $(1 \leqslant k \leqslant n)$,*

(2) *$\|x^*\| = 1$ for each $x^* \in \mathscr{E}$,*

(3) *If $x \in E$ and $a_1, \ldots, a_n \in \boldsymbol{R}$, then there is some $x^* \in \mathscr{E}$ such that*

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant (1+\varepsilon)\Big(x^*(x) + \sum_k a_k x^*(x_k)\Big)$$

whenever $x_k \in c(V(T_k))$ $(1 \leqslant k \leqslant n)$.

Proof. Assume C in the unit ball of X. Let the slices $T_1', \ldots, T_n'$ and $M > 0$ satisfy the conditions of Lemma 16 applied to $S_1, \ldots, S_n$ and E. Take $\delta > 0$ with $(1 - 5\delta M)^{-1} \leqslant 1 + \varepsilon$. Let $(y_i)_{i \in I}$ be a δ-net in the unit ball of E and $\big((a_k^j)_{1 \leqslant k \leqslant n}\big)_{j \in J}$ a δ-net in the unit ball of $\boldsymbol{R}^n$ with the l^1-norm. By successive applications of Lemma 15, we obtain slices $T_1, \ldots, T_n$ and a finite family $\mathscr{E} = (x_{i,j}^*)_{i \in I, j \in J}$ in the unit sphere of X^*, satisfying $T_k \subset T_k'$ $(1 \leqslant k \leqslant n)$ and

$$\Big\|y_i + \sum_k a_k^j x_k\Big\| \leqslant x_{i,j}^*(y_i) + \sum_k a_k^j x_{i,j}^*(x_k) + \delta,$$

if $x_k \in c(V(T_k))$ $(1 \leqslant k \leqslant n)$ and $i \in I, j \in J$.

We verify (3). Let thus $x \in E$, $a_1, \ldots, a_n \in \boldsymbol{R}$ and take

$$\varrho = \Big(\|x\| + \sum_k |a_k|\Big)^{-1}.$$

By construction there is $i \in I$ and $j \in J$ so that

$$\|\varrho x - y_i\| < \delta \quad \text{and} \quad \sum_k |\varrho a_k - a_k^j| < \delta.$$

If $x_k \in c(V(T_k))$ $(1 \leqslant k \leqslant n)$, it follows:

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant \varrho^{-1}\Big(\Big\|y_i + \sum_k a_k^j x_k\Big\| + 2\delta\Big)$$
$$\leqslant \varrho^{-1}\Big(x_{i,j}^*(y_i) + \sum_k a_k^j x_{i,j}^*(x_k) + 3\delta\Big) \leqslant x_{i,j}^*(x) + \sum_k a_k x_{i,j}^*(x_k) + 5\delta\varrho^{-1}.$$

Since $\varrho^{-1} = \|x\| + \sum_k |a_k| \leqslant M\|x + \sum_k a_k x_k\|$, we obtain

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant x_{i,j}^*(x) + \sum_k a_k x_{i,j}^*(x_k) + 5\delta M\Big\|x + \sum_k a_k x_k\Big\|.$$

Therefore

$$(1 - 5\delta M)\Big\|x + \sum_k a_k x_k\Big\| \leqslant x_{i,j}^*(x) + \sum_k a_k x_{i,j}^*(x_k)$$

and hence

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant (1+\varepsilon)\Big(x_{i,j}^*(x) + \sum_k a_k x_{i,j}^*(x_k)\Big).$$

This completes the proof.

LEMMA 18. *If $\mathscr{E}$ is a finite subset of X^* and $\varepsilon > 0$, then there exists a finite-dimensional subspace E of X and $\delta > 0$, such that:*

If $x \in X$ and $|x^*(x)| < \delta$ for each $x^* \in \mathscr{E}$, then there is some $y \in E$ with $\|y\| < \varepsilon$ and $x^*(x) = x^*(y)$ for all $x^* \in \mathscr{E}$.

Proof. Assume $\mathscr{E} = \{x_1^*, \ldots, x_n^*\}$. Obviously $f\colon X \to \boldsymbol{R}^n$, $\| \|_\infty$ given by $f(x) = (x_1^*(x), \ldots, x_n^*(x))$ is an operator mapping X on some subspace $\mathscr{S}$ of $\boldsymbol{R}^n$. Let E be a finite-dimensional subspace of X satisfying $f(E) = \mathscr{S}$. By the open map principle we obtain $\delta > 0$ such that $f(E \cap B(0, \varepsilon)) \supset \mathscr{S} \cap B(0, \delta)$.

Let now $x \in X$ with $|x^*(x)| < \delta$ for each $x^* \in \mathscr{E}$. Then $f(x) \in \mathscr{S} \cap B(0, \delta)$ and therefore there is $y \in E$ with $\|y\| < \varepsilon$ and $f(y) = f(x)$.

LEMMA 19. *Let $n \in \boldsymbol{N}$, $S_1, \ldots, S_n$ slices, E a finite-dimensional subspace of X and $\varepsilon > 0$. Then there are slices $T_1, \ldots, T_n$ and a finite-dimensional subspace F of X, verifying the following properties:*

(1) *$T_k \subset S_k$ $(1 \leqslant k \leqslant n)$,*

(2) *For each $k = 1, \ldots, n$, let $x_k \in c(V(T_k))$ and $(x_{k,i})_i$ a finite number of points in T_k. Then for each $k = 1, \ldots, n$, there are points $(y_{k,i})_i$ in X, satisfying:*

(1) *$y_{k,i} - x_{k,i} \in F$ $(1 \leqslant k \leqslant n, i)$,*

(2) *$\|y_{k,i} - x_{k,i}\| < \varepsilon$ $(1 \leqslant k \leqslant n, i)$,*

(3) *If $x \in E$, $a_1, \ldots, a_n \in \boldsymbol{R}$ and $(b_{1,i})_i, \ldots, (b_{n,i})_i \in \boldsymbol{R}$, then*

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant (1+\varepsilon)\Big\|x + \sum_k a_k x_k + \sum_k \sum_i b_{k,i}(y_{k,i} - x_k)\Big\|.$$

Proof. Consider first slices $T_1', \ldots, T_n'$ and a finite subset $\mathscr{E}$ of X^* verifying (1), (2), (3) of Lemma 17. Next, take a finite-dimensional subspace F of X and $\delta > 0$ satisfying the condition of Lemma 18. By Lemma 3, there are slices $T_1, \ldots, T_n$ such that $T_k \subset T_k'$ $(1 \leqslant k \leqslant n)$ and $o(x^*|T_k) < \delta$ $(x^* \in \mathscr{E},\ 1 \leqslant k \leqslant n)$. For each $k = 1, \ldots, n$, let $x_k \in c(V(T_k))$ and $(x_{k,i})_i$ a finite number of points in T_k.

For each $k = 1, \ldots, n$ and i, we have that $|x^*(x_k - x_{k,i})| < \delta$ whenever $x^* \in \mathscr{E}$.

Hence there is $w_{k,i} \in F$ with $\|w_{k,i}\| < \varepsilon$ and $x^*(x_k - x_{k,i}) = x^*(w_{k,i})$ for every $x^* \in \mathscr{E}$. Take $y_{k,i} = x_{k,i} + w_{k,i}$. It remains to verify (3) of (2).

Let thus $x \in E$, $a_1, \ldots, a_n \in \boldsymbol{R}$ and $(b_{1,i})_i, \ldots, (b_{n,i})_i \in \boldsymbol{R}$. Let $x^* \in \mathscr{E}$ be the functional in $\mathscr{E}$ associated to x and $a_1, \ldots, a_n$ by Lemma 16.

We obtain

$$\Big\|x + \sum_k a_k x_k\Big\| \leqslant (1+\varepsilon)\Big(x^*(x) + \sum_k a_k x^*(x_k)\Big)$$
$$= (1+\varepsilon)\Big(x^*(x) + \sum_k a_k x^*(x_k) + \sum_k \sum_i b_{k,i} x^*(y_{k,i} - x_k)\Big)$$
$$\leqslant (1+\varepsilon)\Big\|x + \sum_k a_k x_k + \sum_k \sum_i b_{k,i}(y_{k,i} - x_k)\Big\|.$$

This completes the proof.

LEMMA 20. *Let $(\varepsilon_p)_p$ be a sequence of positive numbers. Then, for each $p \in \boldsymbol{N}$, we can define a finite subset Ω_p of $\boldsymbol{N}^p$, a finite subset A_p of X, finite-dimensional subspaces E_p, F_p of X and for each $\omega \in \Omega_p$ a $\lambda_\omega > 0$ and a slice S_ω of C, such that the following conditions hold:*

(1) *Ω_p is the projection of Ω_{p+1} on the p first coordinates $(p \in \boldsymbol{N})$,*

(2) *$A_p \subset E_p$ $(p \in \boldsymbol{N})$,*

(3) *$E_p \subset E_{p+1}$, $F_p \subset E_{p+1}$ $(p \in \boldsymbol{N})$,*

(4) *$S_{\omega,i} \subset S_\omega$ $(p \in \boldsymbol{N}, (\omega, i) \in \Omega_{p+1})$,*

(5) *$\sum_i \lambda_{\omega,i} = 1$ $(p \in \boldsymbol{N}, \omega \in \Omega_p)$,*

(6) *There is $x_\omega \in A_{p+1} \cap c(V(S_\omega))$ so that*

$$\sum_i \lambda_{\omega,i} V(S_{\omega,i}) \subset B(x_\omega, \varepsilon_{p+1}) \qquad (p \in \boldsymbol{N}, \omega \in \Omega_p),$$

(7) E_p, *the slices* $(S_\omega)_{\omega\in\Omega_p}$ *and* F_p *satisfy condition* (2) *of Lemma* 19, *with* $\varepsilon = \varepsilon_p$ $(p \in N)$,

(8) $S_\omega \cap B(A_p, \iota) = \emptyset$ $(p \in N,\ \omega \in \Omega_p)$.

Proof. We proceed inductively on $p \in N$.

(a) Take $\Omega_1 = \{1\}$, $A_1 = \emptyset$, $E_1 = \{0\}$ and $\lambda_1 = 1$. Let S_1 be a slice and F_1 a finite-dimensional subspace of X satisfying Lemma 19 applied to C, $\{0\}$, ε_1.

(b) Assume now Ω_p, A_p, E_p, F_p and for each $\omega \in \Omega_p$, λ_ω, S_ω obtained. Let $\omega \in \Omega_p$ be fixed. Lemma 8 and the fact that X is $(*)$ yields us some $n_\omega \in N$, slices $(T'_{\omega,i})_{1\leqslant i\leqslant n_\omega}$ and positive numbers $(\lambda_{\omega,i})_{1\leqslant i\leqslant n_\omega}$ such that $T'_{\omega,i} \subset S_\omega$ $(1 \leqslant i \leqslant n_\omega)$, $\sum_i \lambda_{\omega,i} = 1$ and $\operatorname{diam} \sum_i \lambda_{\omega,i} V(T'_{\omega,i}) < \varepsilon_{p+1}$. Take $\Omega_{p+1} = \{(\omega, i);\ \omega \in \Omega_p,\ 1 \leqslant i \leqslant n_\omega\}$. For each $\omega \in \Omega_p$, choose a point x_ω in $\sum_i \lambda_{\omega,i} V(T'_{\omega,i})$ and define $A_{p+1} = \{x_\omega;\ \omega \in \Omega_p\}$. For each $(\omega, i) \in \Omega_{p+1}$, let $T''_{\omega,i}$ be a slice satisfying $T''_{\omega,i} \subset T'_{\omega,i}$ and $T''_{\omega,i} \cap B(A_{p+1}, \iota) = \emptyset$, which can be found by Lemma 14. Let $E_{p+1} = \operatorname{span}(E_p, F_p, A_{p+1})$. Let $(S_{\omega,i})_{(\omega,i)\in\Omega_{p+1}}$ be slices and F_{p+1} a finite-dimensional subspace of X satisfying Lemma 19 applied to $(T''_{\omega,i})_{(\omega,i)\in\Omega_{p+1}}$, E_{p+1} and ε_{p+1}. It is easily seen that all conditions are fulfilled.

Proof of the theorem in case II. Take $\lambda > 1$ and let $(\varepsilon_p)_p$ be a sequence of positive numbers such that $\sigma = \sum_p \varepsilon_p < \min(\ln\lambda, \iota/6)$. We use the construction of Lemma 20 and consider points x_ω satisfying (6). Let $p \in N$ be fixed. Remark that $x_{\omega,i} \in S_\omega$ for each $(\omega, i) \in \Omega_{p+1}$. It follows from (7) that there are points $(y_{\omega,i})_{(\omega,i)\in\Omega_{p+1}}$ so that $y_{\omega,i} - x_{\omega,i} \in F_p$, $\|y_{\omega,i} - x_{\omega,i}\| < \varepsilon_p$ for each $(\omega, i) \in \Omega_{p+1}$ and such that

$$\|x\| \leqslant (1+\varepsilon_p)\Big\|x + \sum_{\omega,i} b_{\omega,i}(y_{\omega,i} - x_\omega)\Big\|$$

whenever $x \in \operatorname{span}(E_p, x_\omega \text{ with } \omega \in \Omega_p)$ and $(b_{\omega,i})_{(\omega,i)\in\Omega_{p+1}} \subset \boldsymbol{R}$.

We introduce inductively finite-dimensional subspaces $\mathscr{X}_p$ of X, by taking $\mathscr{X}_1 = \operatorname{span}(x_\omega;\ \omega \in \Omega_1)$ and

$$\mathscr{X}_{p+1} = \operatorname{span}\big(\mathscr{X}_p,\ y_{\omega,i} - x_\omega \text{ with } (\omega, i) \in \Omega_{p+1}\big).$$

Using induction on p, the reader will verify that $\mathscr{X}_p \subset \operatorname{span}(E_p, x_\omega \text{ with } \omega \in \Omega_p)$. Hence $\|x\| \leqslant (1+\varepsilon_p)\|x+y\|$ if $x \in \mathscr{X}_p$ and $y \in \operatorname{span}\big(y_{\omega,i} - x_\omega \text{ with } (\omega, i) \in \Omega_{p+1}\big)$, showing that there exists a projection π_p of $\mathscr{X}_{p+1}$ onto $\mathscr{X}_p$ with $\|\pi_p\| \leqslant 1 + \varepsilon_p$.

By induction on $p \in N$, we define vectors $(z_\omega)_{\omega\in\Omega_p}$ taking $z_\omega = x_\omega$ if $\omega \in \Omega_1$ and $z_{\omega,i} = z_\omega + (y_{\omega,i} - x_\omega)$ if $(\omega, i) \in \Omega_{p+1}$. Clearly $z_\omega \in \mathscr{X}_p$ if $\omega \in \Omega_p$ and $\pi_p(z_{\omega,i}) = z_\omega$ if $(\omega, i) \in \Omega_{p+1}$. Moreover $\|x_\omega - z_\omega\| < \varepsilon_1 + \ldots + \varepsilon_{p-1} < \sigma$ whenever $p > 1$ and $\omega \in \Omega_p$. Finally

$$\Big\|z_\omega - \sum_i \lambda_{\omega,i} z_{\omega,i}\Big\| \leqslant \Big\|x_\omega - \sum_i \lambda_{\omega,i} x_{\omega,i}\Big\| + \Big\|\sum_i \lambda_{\omega,i} x_{\omega,i} - \sum_i \lambda_{\omega,i} y_{\omega,i}\Big\| < \varepsilon_{p+1} + \varepsilon_p$$

and

$$\|z_\omega - z_{\omega,i}\| \geqslant \|x_\omega - x_{\omega,i}\| - \|x_\omega - z_\omega\| - \|x_{\omega,i} - z_{\omega,i}\| > \iota - 2\sigma > 0.$$

We only have to take $\alpha = \iota - 2\sigma$, $A_p = \{z_\omega;\ \omega \in \Omega_p\}$ and $\beta_p = \varepsilon_p + \varepsilon_{p+1}$ to fulfil the conditions of Lemma 11.

Added in proof. It follows from recent work of H. Rosenthal and the author [22] that Problem 2 stated above has negative solution.

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A generalization of Khintchine's inequality and its application in the theory of operator ideals

by

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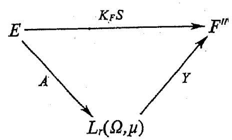
Abstract. We prove a generalization of Khintchine's inequality which can be used to estimate the absolutely r-summing norm and the r-factorable norm of the identity map from l_u^n into l_v^n for certain exponents u and v. This results fill in the remaining gaps in the limit order diagrams of the operator ideals $\mathfrak{P}_r$ and $\mathfrak{L}_r$.

In the following $\mathfrak{L}(E, F)$ denotes the set of all (bounded linear) operators from E into F, where E and F are arbitrary Banach spaces.

An operator $S \in \mathfrak{L}(E, F)$ is called *absolutely r-summing* $(1 \leqslant r < \infty)$ if there exists a constant σ such that

$$\Big\{\sum_1^n \|Sx_i\|^r\Big\}^{1/r} \leqslant \sigma \sup\Big[\Big\{\sum_1^n |\langle x_i, a\rangle|^r\Big\}^{1/r}: \|a\| \leqslant 1\Big]$$

for all finite families of elements $x_1, \ldots, x_n \in E$. The class $\mathfrak{P}_r$ of these operators is an ideal with the norm $\boldsymbol{P}_r(S) := \inf\sigma$. An operator $S \in \mathfrak{L}(E, F)$ is called *r-factorable* $(1 \leqslant r \leqslant \infty)$ if there exists a commutative diagram



with $A \in \mathfrak{L}(E, L_r(\Omega, \mu))$ and $Y \in \mathfrak{L}(L_r(\Omega, \mu), F'')$. Here (Ω, μ) is a measure space and K_F denotes the evaluation map from F into F''. The class $\mathfrak{L}_r$ of these operators is an ideal with the norm $\boldsymbol{L}_r(S) := \inf\|Y\|\,\|A\|$, where the infimum is taken over all admissible factorizations.

Let us denote by I the identity map from l_u^n into l_v^n, where l_u^n and l_v^n are the Minkowski spaces with $1 \leqslant u, v \leqslant \infty$. It is well known that the asymptotic properties of $\boldsymbol{A}(I\colon l_u^n \to l_v^n)$ give important information about the operator ideal $\mathfrak{A}$ with the norm $\boldsymbol{A}$. In particular, we are interested to know the so-called *limit order* $\lambda(\boldsymbol{A}, u, v)$ which is defined to be the infimum