Dentability and finite-dimensional decompositions

by

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Abstract. It is shown that a Banach space possesses the Radon-Nikodým property if and only if every subspace with a finite-dimensional Schauder decomposition has the Radon-Nikodým property.

Introduction. Let $X, \| \cdot \|$ be a Banach space with dual $X^*$. If $x \in X$ and $\varepsilon > 0$, then $B(x, \varepsilon)$ denotes the open ball with midpoint $x$ and radius $\varepsilon$. For sets $A \subseteq X$, let $c(A)$ be the convex hull and $\overline{c}(A)$ the closed convex hull of $A$. We will say that $A$ is dentable if for all $\varepsilon > 0$ there exists $x \in A$ satisfying $x \notin \overline{c}(A \setminus B(x, \varepsilon))$. The Banach space $X$ is said to be dentable if every nonempty, bounded subset of $X$ is dentable. We say that $X$ has the Radon-Nikodým property (RNP) provided for every measure space $(\Omega, \Sigma, \mu)$ with $\mu(\Omega) < \infty$ and every $\mu$-continuous measure $F: \Sigma \to X$ of finite variation, there exists a Bochner integrable function $f: \Omega \to X$ such that $F(E) = \int_E f d\mu$ for every $E \in \Sigma$. The RNP of $X$ is equivalent to the fact that any uniformly bounded $X$-valued martingale on a finite measure space is convergent a.e. (cf. [5], [21]).

It is known that $X$ is a dentable Banach space if and only if $X$ has RNP. For the history of the equivalence between those two properties, I refer the reader to the J. Diez and J. J. Uhl survey paper [6].

Recall that $(P_n, M_n)$ is a finite-dimensional Schauder decomposition for the Banach space $X$ if each $P_n$ is a continuous linear projection of $X$ onto the finite-dimensional $M_n$, $P_n P_m = 0$ if $n \neq m$ and $x = \sum P_n(x)$ for each $x \in X$. The partial sum operators $S_n$ are defined by $S_n = \sum P_i$.

Since $(S_n)_n$ is pointwise convergent to the identity operator, it is uniformly bounded. We denote by $\theta(M_n, n)$ the number $\sup_n \|S_n\_n\|$, which is called the Grynbaum constant of the decomposition. Our main result is the following:

Theorem 1. Assume $X$ without RNP. Then for each $\lambda > 1$ there exist a subspace $X'$ of $X$, a uniformly bounded $X'$-valued martingale $(\xi_n)_n$ on $[0, 1]$.

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and a sequence \((S_n : X \to X)_n\) of finite rank projections, such that:

1. \(x = \lim S_n(x)\) for each \(x \in X\),
2. \(\|S_n\| \leq \lambda\),
3. \(S_n S_m = S_n S_m = S_m\) if \(m \leq n\),
4. \(S_n \xi_{k_n} = \xi_{k_n}\),
5. \((\xi_{k_n})_n\) is nowhere convergent.

Theorem 1 if of course a refinement of the existence of non-convergent martingales in Banach spaces without RNP. Various authors pointed out that the RNP is separably determined (cf. [20], [10], [15], [15]). Theorem 1 yields the following improvement.

**Theorem 2.** If \(X\) fails RNP, then for every \(\lambda > 1\) there exists a subspace \(Y\) of \(X\) without RNP and with a finite-dimensional Schauder decomposition with Orlicz constant at most \(\lambda\).

Indeed, \(X\) fails RNP. If we take \(P_i = S_i, P_{n+1} = S_{n+1} - S_n\), then \((P_n, P_{\leq n})\) is a finite-dimensional Schauder decomposition of \(X\) with \(\Theta_{P_n, P_{\leq n}} \leq \lambda\).

Theorem 2 is related to the following question:

**Problem 1.** If every subspace of \(X\) possessing a Schauder basis possesses the RNP, then \(X\) also possess it?

To the best of my knowledge, this problem is still open.

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**Preliminary geometric lemmas.** In this section, \(O\) will be a fixed nonempty, bounded, closed and convex subset of \(X\).

We first introduce some terminology.

**Definition 1.** If \(x^* \in X^*\), define \(M(x^*, O) = \sup_{y^* \in O} \langle x^*, y^* \rangle\). For each \(a > 0\), let \(S(x^*, \alpha, O) = \{x \in O; \langle x^*, y^* \rangle \geq M(x^*, O) - a\}\) and \(\hat{S}(x^*, \alpha, O) = \{x \in O; \langle x^*, y^* \rangle > M(x^*, O) - a\}\). We will call \(S(x^*, \alpha, O)\) a plane and \(\hat{S}(x^*, \alpha, O)\) an open plane. If \(S\) is a slice, \(\hat{S}\) will denote the corresponding open plane.

The reader will easily verify the following property.

**Lemma 1.** If \(S(x^*, \alpha, O)\) is a slice, then there exist \(\epsilon > 0\) and \(\beta > 0\) such that \(\|x^* - y^*\| < \epsilon\) implies \(S(y^*, \beta, O) \subset \hat{S}(x^*, \alpha, O)\).

**Definition 2.** Let \(S = S(x^*, \alpha, O)\) be a slice. Define

\[ V(S) = \{x \in O; \exists y^* \in x^* \text{ with } \langle y^*, x^* \rangle = M(y^*, O) > M(y^*, O \setminus \hat{S})\}. \]

If further \(n \in N, s^*_1, \ldots, s^*_n \in X^*\) and \(\epsilon > 0\), then

\[ V(S, s^*_1, \ldots, s^*_n, \epsilon) = \{x \in O; \exists y^* \in x^* \text{ with } \langle y^*, x^* \rangle > \frac{\epsilon}{2}, \text{ with } \langle y^*, x^* \rangle = M(y^*, O) \}, \]

\[ S(y^*, \beta, O) \subset \hat{S} \] and \(\sup_{x^*} M(x^*, O) \leq \epsilon (1 \leq k \leq n)\).

(\(\cap n\)) means oscillation.

Using Lemma 1 and the Bishop-Phelps density result on the supporting functionals [1], we obtain immediately

**Lemma 2.** If \(S\) is a slice, then \(V(S) \neq \emptyset\).

**Definition 3.** Suppose \(S = S(x^*, \alpha, O)\) a slice. Let \(\hat{O}\) be the \(w^*\)-closure of \(O\) in \(X^*\) and \(\text{ex}(\hat{O})\) the extreme points of \(\hat{O}\). Define

\[ E(S) = \{x^* \in \text{ex}(\hat{O}); \langle x^*, y^* \rangle > M(y^*, O) - a\}. \]

It follows from the Krein–Milman theorem that \(E(S)\) is nonempty.

**Lemma 3.** If \(S\) is a slice, \(s^*_1, \ldots, s^*_n \in X^*\) and \(\epsilon > 0\), then there is a slice \(T\) such that:

1. \(T \subset \hat{S}\),
2. \(\sup_{x^*} M(x^*, T) < \epsilon (1 \leq k \leq n)\).

Hence we have

**Lemma 4.** If \(S\) is a slice, \(s^*_1, \ldots, s^*_n \in X^*\) and \(\epsilon > 0\), then \(V(S, s^*_1, \ldots, s^*_n, \epsilon) \neq \emptyset\).

**Lemma 5.** If \(S\) is a slice, \(x \in V(S)\) and \(D\) a closed convex subset of \(O\) with \(x \notin D\), then \(x \notin \text{ex}(O \setminus \hat{S})\).

Proof. Take \(x^* \in X^*\) satisfying \(\langle x^*, x \rangle = M(y^*, O) > M(y^*, O \setminus \hat{S})\).

Suppose \(x \in \text{ex}(O \setminus \hat{S})\), then \(x = \lim_{n \to \infty} (y_n + (1 - \lambda_n) s^*_n)\) where \(y_n\) is a sequence in \(O \setminus \hat{S}\), \(s^*_n\) a sequence in \(D\) and \(\lambda_n\) a sequence in \([0, 1]\).

Hence \(M(y^*, O) < \lim_{n \to \infty} \{M(y^*, O \setminus \hat{S}) + (1 - \lambda_n) M(y^*, O)\}\) showing that \(\lim_{n \to \infty} \lambda_n = 0\). It follows that \(x = \lim_{n \to \infty} x \) and thus \(x \in D\), which is a contradiction.

**Lemma 6.** Let \(S\) be a slice, \(x \in V(S)\), \(s^*_1, \ldots, s^*_n \in X^*\) and \(\epsilon > 0\). Then \(x \notin \text{ex}(V(S, s^*_1, \ldots, s^*_n, \epsilon))\).

Proof. If \(x \notin \text{ex}(V(S, s^*_1, \ldots, s^*_n, \epsilon))\), then we also have that \(x \notin \text{ex}(O \setminus \hat{S}) \cup V(S, s^*_1, \ldots, s^*_n, \epsilon)\), by Lemma 5. By the separation theorem there exists a slice \(T\) satisfying \(T \subset \hat{S}\) and \(x \notin \cap V(S, s^*_1, \ldots, s^*_n, \epsilon) = \emptyset\). But by Lemma 4, \(V(T, s^*_1, \ldots, s^*_n, \epsilon)\) is a nonempty subset of \(V(S, s^*_1, \ldots, s^*_n, \epsilon)\), a contradiction.

We now pass to the key lemma of this paper.

**Lemma 7.** Let \(S\) be a slice and \(U\) a weak open set such that \(U \cap V(S) \neq \emptyset\). Then there exist \(n \in N, s_1, \ldots, s_n\) and positive numbers \(\lambda_1, \ldots, \lambda_n\) satisfying

1. \(S_1 = \hat{S}\),
2. \(\sum_{i=1}^{n} \lambda_i = 1\),
3. \(\sum_{i=1}^{n} \lambda_i V(S_i) \subset U \cap V(S)\).
Proof. If \( x \in U \cap \sigma(V(S)) \), then there is a \( w \)-neighborhood \( N(x, a_1, \ldots, a_n) \) of \( x \) contained in \( U \). Since, by Lemma 6, \( V(S) = \delta(V(S), a_1, \ldots, a_n, \delta(2)) \), we also have that \( x \in \sigma(V(S), a_1, \ldots, a_n, \delta(2)) \). Of course we can take \( |a_i| \leq 1 \) \((1 \leq i \leq p)\). Let \( n \in N(x, a_1, \ldots, a_n) \in V(S, a_1, \ldots, a_n, \delta(2)) \) and \( \lambda_1, \ldots, \lambda_n \) positive numbers, with \( \sum \lambda_i = 1 \) and \( |\| \sum \lambda_i a_i \| | < \delta(2) \). For each \( \epsilon = 1, \ldots, n \) a slice \( S_\epsilon \) is obtained so that \( a_\epsilon \in S_\epsilon, S_\epsilon \subset B \) and \( o(S_\epsilon) < \delta(2) \) \((1 \leq \epsilon \leq p)\). Obviously \( \sum \lambda_i V(S_\epsilon) = c(V(S)) \). For every \( q = 1, \ldots, p \), we find that

\[
\sum_{\epsilon=1}^{p} \lambda_i \epsilon(S_\epsilon) = \sum_{\epsilon=1}^{p} \lambda_i \epsilon(S_\epsilon) < \sum_{\epsilon=1}^{p} \lambda_i \epsilon(S_\epsilon) + \delta(2) = \sum_{\epsilon=1}^{p} \lambda_i \epsilon(S_\epsilon) + \delta(2) \leq \delta(2) \leq \delta(2),
\]

implying

\[
\sum_{\epsilon=1}^{p} \lambda_i \epsilon(S_\epsilon) = N(x, a_1, \ldots, a_n, \delta).
\]

Hence \( \sum \lambda_i V(S_\epsilon) = U \cap \sigma(V(S)) \).

Lemma 7 has the following immediate corollary, which will be used later.

**Lemma 8.** Let \( S \) be a slice, \( \epsilon > 0 \) and \( U \) a \( w \)-open set such that \( U \cap \sigma(V(S)) \neq \emptyset \) and \( \text{diam}(U \cap \sigma(V(S))) \leq \epsilon \). Then there exist \( n \in N(x, \ldots, a_n, \delta, \sum \lambda_i = 1 \) and \( \text{diam}(\sum \lambda_i V(S_\epsilon)) \leq \epsilon \).

**Banach spaces with property \((*)\).**

**Proposition 1.** For a Banach space \( X \), the following properties are equivalent.

1. For each nonempty, bounded, closed and convex subset \( A \) of \( X \), the identity map on \( A \) has a \( w \)-neighborhood of \( A \) with \( \text{diam}(A) \leq \epsilon \).
2. For each nonempty, bounded and convex subset \( A \) of \( X \) and for each \( \epsilon > 0 \), there exists a \( w \)-open set \( U \) satisfying \( U \cap A \neq \emptyset \) and \( \text{diam}(U \cap A) \leq \epsilon \).
3. For each nonempty, bounded and convex subset \( A \) of \( X \) and for each \( \epsilon > 0 \), there exists a \( w \)-open set \( U \) such that \( U \cap A \neq \emptyset \) and \( U \cap A \) has an \( \epsilon \)-neighborhood.

Proof. The implications \((1) \Rightarrow (2) \Rightarrow (3) \) are clear. Assume \((3) \), let \( A \) be nonempty, bounded, convex and \( \epsilon > 0 \). We find a \( w \)-open set \( U \) and a finite number of balls \( \{B_i \} \) of radius \( \epsilon/2 \) so that \( U \cap A \neq \emptyset \) and \( U \cap A \subset \bigcup_{i=1}^{n} B_i \). If \( x \in U \cap A \), then by the separation theorem we obtain a \( w \)-open set \( V \) with \( x \in V, V \subset U \) and \( V \cap B_i = \emptyset \) whenever \( x \notin B_i \) \((1 \leq i \leq n) \). Therefore \( V \cap A \neq \emptyset \) and \( \text{diam}(V \cap A) \leq \epsilon \). Hence \((2) \) holds. If we have \((2) \) and \( A \) is nonempty, bounded, closed and convex, then by repeating application of \((3) \) a sequence \( \{U_n \} \) of convex \( w \)-open sets is obtained verifying \( U_n \subset U_{n+1} \) and \( U_n \cap A \neq \emptyset \) and \( \text{diam}(U_n \cap A) \leq \epsilon/n \). It follows that \( \bigcap_{\epsilon=n}^{\infty} (U_n \cap A) \) consists of a unique point of \( A \) which is clearly a \( w \)-neighborhood of \( A \).
LEMMA 10. Take $\alpha$ and $\delta$ as in Lemma 9. Then for every $x \in \alpha$ and every subspace $E$ of $X$ with $\text{codim} E < \infty$ we obtain $x \in \partial(\alpha \cap (x + E))$ or $\text{dim}(\alpha \cap (x + E)) = \infty$. Proof. If $x \in \partial(\alpha \cap (x + E))$ then there is $x^* \in X^*$ such that $\text{lin}(x^*) > M(x, \alpha)$. Let $E = E \cap \text{ker} x^*$. Since $\text{dim}(E \cap (x + F)) \geq \varepsilon$, there is a point $y \in \alpha \cap (x + F)$ with $\|x - y\| > \delta$. Hence $y \in D$, contradicting $D \cap (x + F) = \emptyset$.

Proof of the main theorem. We start with the following lemma.

LEMMA 11. Let $\lambda > 1$ and suppose there exist $\alpha > 0$, an increasing sequence $(E_n)_{n \in N}$ of finite-dimensional subspaces of $X$ and for each $p$ a projection $\pi_p : \ell_2 \to \ell_2$, a finite subset $A_p$ of $X_p^*$ and $\beta_p > 0$, satisfying:

1. $\|\pi_p\| \leq \lambda$
2. $\bigcup_{n \in N} A_{n}$ is bounded in $X_{p}$
3. If $x \in A_{n}$, then there are vectors $z_1, \ldots, z_r \in \alpha_{n}$ such that $\|x - z_i\| \geq \delta$, $\pi_n(z_i) = z_i$ for each $i = 1, \ldots, r$ and $\text{dist}(z_i, \alpha_{n+1}) < \beta_n$.
4. $\beta_n \leq 2/\lambda$.

Then there exists a subspace $E$ of $X$, a uniformly bounded $x$-valued martingale $(\xi_n)$ on $[0, 1]$ and a sequence $(\delta_n : E \to \ell_2)$ of finite rank projections satisfying (1), (2), (3), (4), (5) of Theorem 1.

Proof. Take $E = \bigcup_{n \in N} E_n$. It is routine to obtain for each $p$ a projection $S_p$ from $X_p$ onto $E_p$ so that $\|S_p\| \leq \lambda$ and $S_p \pi_{p+1} = \pi_p S_p+1$. Thus (1), (3), (5) hold.

(3) allows us to construct for each $p$ a finite field $\mathcal{F}_p$ generated by subintervals of $[0, 1]$ and a $\mathcal{F}_p$-measurable map $\eta_p : [0, 1] \to A_p$, so that:

1. $|\eta_p(t) - \eta_p(0)| \geq \delta$ whenever $t \in [0, 1]$
2. $\eta_p(t) \in \alpha_p$
3. $\|\eta_p(t) - E[\eta_p(0) \mid \mathcal{F}_p]\| \leq \beta_p$

This construction is less or more standard and we omit the details. The reader can find them in [9] or [5].

If we introduce inductively maps $\xi_p : \eta_p \to \eta_{p+1}$ by setting $\xi_p = \eta_p$ and $\xi_{p+1} = \eta_{p+1} + E[\eta_{p+1} \mid \mathcal{F}_p]$, then $(\xi_p, \mathcal{F}_p)$ is clearly a martingale. By induction, it is easily seen that $\xi_p$ ranges in $X$ and $\|\xi_p - \eta_p\| < \beta_1 + \cdots + \beta_p < \delta$. It follows that $(\xi_p)$ is uniformly bounded. Furthermore $\xi_{p+1} - \eta_{p+1} = E[\eta_{p+1} \mid \mathcal{F}_p]$ and $\|\xi_p(t) - \xi_p(0)\| \geq \|\eta_p(t) - \eta_p(0)\| - \|\xi_p(t) - \xi_p(0)\| \geq \|\eta_p(t) - \eta_p(0)\| - \|\eta_p(0) - \eta_p(0)\| > 0$.

Thus $(\xi_p)$ is nowhere convergent, which completes the proof.

In the proof of the main theorem, two cases will be distinguished:

I. $X$ fails property (*).
II. $X$ has property (*) and fails RNP.

We start with the first one, which is also the easiest.

LEMMA 12. Let $\alpha$ and $\delta$ satisfy the condition of Lemma 10. Let $(\varepsilon_p)$ be a sequence of positive numbers. Then for each $p \in N$, we can define a finite subset $A_p$ of $\alpha$ and a subspace $E_p$ of $X$, satisfying the following properties:

1. $\text{codim}(E_p) < \infty$ ($p \in N$)
2. If $x \notin \text{span}(A_{p+1}, \ldots, A_p)$, then there exists $x^* \in X^*$ with $\|x^* - x\| > \delta$.
3. $\|x^*\| = 1$, $x^* \mid E_p = 0$, and $\|x^*\| \leq (1 + \varepsilon_p) x^*(x)$ ($p \in N$).
4. $A_{p+1} = \bigcup_{q=0}^{p} A_q$, where $A_q \cap B(x, \alpha_{q+1}) = \emptyset$, $A_q \cap B(x, \alpha_q) = x + E_q$, and $\text{dist}(x, \alpha_{q+1}) < \varepsilon_q$ ($p \in N$).

Proof. We proceed by induction on $p \in N$.

(a) Let $A_0 = \{x_0\}$, where $x_0$ is an arbitrary point in $\alpha$. Consider a finite subset $A_1$ of the unit sphere of $X^*$ such that $x_0 \notin \text{span}(A_1)$, then there exists $x^* \in A_1$ with $\|x^*\| < (1 + \delta) x^*(x_0)$. Take $E_1 = \text{ker} x^*$.

(b) Assume now $A_p$ and $E_p$ obtained. Let $x \in A_p$. Then $x \notin \text{span}(A_{p+1})$. Since $x \notin \text{span}(B(x, \delta) \cap (x + E_p))$, there is a finite subset $A^{p+1}_{p+1}$ of $\alpha$ so that $A_{p+1} \cap B(x, \delta) = \emptyset$, $A_{p+1} \cap \text{span}(A_{p+1}) = x + E_p$, and $\text{dist}(x, \alpha_{p+1}) < \varepsilon_p$. Define $A_{p+1} = \bigcup_{q=0}^{p+1} A_q$. Again a finite subset $A_{p+1}$ of the unit sphere of $X^*$ can be obtained such that $x \notin \text{span}(A_{p+1})$, then $\|x^*\| \leq (1 + \varepsilon_p) x^*(x)$ for some $x^* \in A_{p+1}$. Take $E_{p+1} = \bigcup_{q=0}^{p+1} \text{ker} x^*$.

Clearly this completes the construction.

Proof of the theorem in case 1. Take $\lambda > 1$ and let $(\varepsilon_p)$ be a sequence of positive numbers satisfying $\sum_p \varepsilon_p \leq \text{min}(1, \lambda, \delta/\beta)$. Hence $\sum_p (1 + \varepsilon_p) \leq \lambda$. Let $A_p$ and $E_p$ be as in Lemma 12. For each $p \in N$, take $x^* = \text{span}(A_{p+1}, \ldots, A_p)$, which is finite-dimensional. Clearly $A_{p+1} \subset A_p + E_p$ and thus $x^* = \bigcap_{q=0}^{p+1} (E_q \cap \text{span}(A_{q+1}))$. Using (2), we see that $\|x^*\| \leq (1 + \varepsilon_p) x^*(x)$ whenever $x \notin \text{span}(A_{p+1})$. Therefore there exists a projection $\pi_p$ of $X$ onto $E_p$ with $\|\pi_p\| \leq 1 + \varepsilon_p$. If we take $\alpha = \delta$ and $\beta_p = \varepsilon_p$, the conditions of Lemma 11 are satisfied, finishing the proof.

We now pass to the case of a Bausch space $X$ with property (*), failing RNP. Let $\mathcal{B}$ be a fixed, nonempty, bounded, closed, and convex subset of $X$, which is not dentable.

LEMMA 13. If $S$ is a slice and $\alpha > 0$, then there exists a slice $T$ with $T \subset S$ and $\text{dist}(x, E) > \alpha$ for $x \in T$.

Proof. Let $S = \text{span}(\alpha^*, \delta, \alpha)$ and take $D = \text{lin}(\alpha; \delta + \alpha^* x^*(x) + M(x^*, \alpha) - \alpha)$. We remark that $C \ni \text{lin}(\alpha^* x^*(x) + \delta)$. Let $\delta = \text{dist} S$, and take $T = \text{lin}(\alpha^*, \delta, \alpha)$, where $\delta < \alpha$. If $x \in T$, then $\alpha^* \in \text{lin}(\partial S)$, and $\alpha^* \in \text{lin}(\partial S)$.

We can find, for example, $\alpha^* \in \text{lin}(\partial S)$, and $\alpha^* \in \text{lin}(\partial S)$.
e D and \( k \in [0, 1] \) with \( w = (1 - \lambda) x^* + \lambda x^* \). But then \( M(x^*, c) - \beta \leq a^*(x) = (1 - \lambda) x^*(a^*) + \lambda x^*(a^*) \leq M(x^*, c) - \beta, \) implying \( \lambda < e^d. \) Hence \( \| x - x^* \| = \| x^* - x \| < \varepsilon, \) proving the lemma.

**Lemma 14.** There is \( \varepsilon > 0 \) such that for every slice \( S \), the set \( E(S) \) has no \( \varepsilon \)-net.

**Proof.** This follows immediately from the lemma of Huff and Morris [10] and Lemma 13.

**Lemma 15.** Let \( n \in N, S_1, \ldots, S_n \) slices, \( x \in X, a_1, \ldots, a_n \in R \) with \( \sum \| a_k \| = 1 \) and \( \varepsilon > 0. \) Then there are slices \( T_1, \ldots, T_n \) and \( x^* \in X^* \) with \( \| x^* \| = 1, \) so that

1. \( T_k \leq S_k \) (1 \( k \leq n), \)
2. If \( a_k \in \mathcal{V}(T_k) \) (1 \( k \leq n), \) then

\[
\left\| x + \sum_{k=1}^{n} a_k y_k \right\| < \| x^* \| \left( x + \sum_{k=1}^{n} a_k y_k \right) + \varepsilon.
\]

**Proof.** Let \( s = \sup \{ \left\| x + \sum_{k=1}^{n} a_k y_k \right\| ; \, a_k \in \mathcal{V}(S_k) \} (1 \leq k \leq n) \). Take \( x \in X \) such that \( \| x + \sum_{k=1}^{n} a_k y_k \| = s - e^2 \) and let \( a^* \in X^* \) with \( \| a^* \| = 1 \) and \( \| x^* \| = \sum_{k=1}^{n} a_k a^*(y_k) = \left\| x + \sum_{k=1}^{n} a_k y_k \right\| \). For each \( k = 1, \ldots, n \), we define \( D_k \) by taking

\[
D_k = \left\{ x \in X ; \, a^*(x) < a^*(x_k - e^2) \right\} \quad \text{if} \quad a_k > 0,
\]

\[
\left\{ x \in X ; \, a^*(x) > a^*(x_k - e^2) \right\} \quad \text{if} \quad a_k < 0.
\]

By Lemma 5, we obtain a slice \( T_k \) so that \( T_k \leq S_k \) and \( T_k \cap D_k = \emptyset. \) To verify condition (2), we can clearly replace \( \mathcal{V}(T_k) \) by \( \mathcal{V}(T_k). \) If now \( y_k \in \mathcal{V}(T_k) \) (1 \( k \leq n), \) we get

\[
\left\| x + \sum_{k=1}^{n} a_k y_k \right\| < s \leq \left\| x + \sum_{k=1}^{n} a_k y_k \right\| + \varepsilon/2 = a^*(x) + \sum_{k=1}^{n} a_k a^*(y_k) + \left( \sum_{k=1}^{n} a_k a^*(y_k) \right) + \left( \sum_{k=1}^{n} a_k a^*(y_k) \right) + \left( \sum_{k=1}^{n} a_k a^*(y_k) \right) + \varepsilon/2 + \varepsilon = a^*(x) + \sum_{k=1}^{n} a_k a^*(y_k) + \varepsilon/2 + \varepsilon + \varepsilon/2 = a^*(x) + \sum_{k=1}^{n} a_k a^*(y_k) + \varepsilon/2,
\]

what must be obtained.

**Lemma 16.** Let \( n \in N, S_1, \ldots, S_n \) slices and \( E \) a finite-dimensional subspace of \( X. \) Then there are slices \( T_1, \ldots, T_n \) and \( M > 0 \) such that

1. \( T_k \leq S_k \) (1 \( k \leq n), \)
2. If \( a \in E, a \in T_k \) (1 \( k \leq n), \) and \( a_k \in R \) (1 \( k \leq n), \) then

\[
\| a \| + \sum_{k=1}^{n} a_k a_k \leq M \| x + \sum_{k=1}^{n} a_k y_k \|.
\]

**Proof.** Assume \( C \) in the unit ball of \( X. \) Using Lemma 14, we obtain for each \( k = 1, \ldots, n, \) a point \( x_k^* \in E(S_k) \) such that \( E \) and \( a_k^*, \ldots, a_k^* \) are linearly independent. Thus there are elements \((y_k^*)_{k \in \mathbb{N}} \) in \( X^* \) satisfying \( y_k^*(E) = 0 \) (1 \( k \leq n), \) \( y_k^*(y^*) = \delta_k \) (1 \( k \leq n), \) and \( M > 0 \) such that \( \| y_k^* \| \leq M \) (1 \( k \leq n), \) i.e., \( T_k \leq S_k \) (1 \( k \leq n), \) and \( y_k^*(x - x_k^*) \leq 2/n \) if \( x \in T_k \) (1 \( k \leq n). \)

If now \( x \in E, a_k \in T_k \) (1 \( k \leq n), \) then \( a_k \in R \) (1 \( k \leq n), \) it follows for each \( \lambda = 1, \ldots, n, \)

\[
\frac{M - 1}{2n} \| x + \sum_{k=1}^{n} a_k y_k \| \leq \left\| \sum_{k=1}^{n} a_k y_k \right\| \leq \| x + \sum_{k=1}^{n} a_k y_k \|.
\]

and by addition

\[
\frac{M - 1}{4n} \| x + \sum_{k=1}^{n} a_k y_k \| \leq \left\| \sum_{k=1}^{n} a_k y_k \right\| \leq \left( 1 - \frac{1}{2n} \right) \sum_{k=1}^{n} \| a_k \| - \frac{M - 1}{4n} \sum_{k=1}^{n} \| a_k \|.
\]

Thus

\[
\sum_{k=1}^{n} \| a_k \| \leq \frac{M - 1}{2} \| x + \sum_{k=1}^{n} a_k y_k \|
\]

and therefore

\[
\| a \| \leq \left\| x + \sum_{k=1}^{n} a_k y_k \right\| + \sum_{k=1}^{n} \| a_k \| \leq \frac{M - 1}{2} \| x + \sum_{k=1}^{n} a_k y_k \|
\]

completing the proof.

**Lemma 17.** Let \( n \in N, S_1, \ldots, S_n \) slices, \( E \) a finite-dimensional subspace of \( X \) and \( \varepsilon > 0. \) Then there are slices \( T_1, \ldots, T_n \) and a finite subset \( \mathcal{S} \) of \( X^*, \) satisfying:

1. \( T_k \leq S_k \) (1 \( k \leq n), \)
2. \( \| a^* \| = 1 \) for each \( a^* \in \mathcal{S}, \)
3. If \( a \in E \) and \( a_1, \ldots, a_n \in R, \) then there is some \( a^* \in \mathcal{S} \) such that

\[
\left\| x + \sum_{k=1}^{n} a_k y_k \right\| \leq (1 + e) \left\| x^* + \sum_{k=1}^{n} a_k a^*(y_k) \right\|
\]

whenever \( y_k \in \mathcal{V}(T_k) \) (1 \( k \leq n). \)

**Proof.** Assume \( C \) in the unit ball of \( X. \) Let the slices \( T_1, \ldots, T_n \) and \( M > 0 \) satisfy the conditions of Lemma 16 applied to \( S_1, \ldots, S_n \) and \( E. \) Take \( \delta > 0 \) with \( (1 - 5M)^{-1} \leq 1 - \varepsilon. \) Let \( (y_k^*)_k \) be a \( \delta \)-net in the unit ball of \( E \) and \( \left\| a_k \right\| \leq \left\| a_k \right\| \leq \delta \)-net in the unit ball of \( R^* \) with the \( \mu \)-norm. By successive applications of Lemma 15, we obtain slices \( T_1, \ldots, T_n \) and a finite family \( \mathcal{S} = (s_k^*)_{k \in \mathbb{N}} \) in the unit sphere of \( X^*, \) satisfying \( T_k \leq T_k^* \) (1 \( k \leq n) \) and

\[
\left\| y_k^* + \sum_{k=1}^{n} a_k y_k \right\| \leq \left\| x^* + \sum_{k=1}^{n} a_k a^*(y_k) \right\| + \delta,
\]

etc.
if \( a_k \in c(V(T_k)) \) \((1 \leq k \leq n)\) and \( i \in I, j \in J \).

We verify (3). Let \( \varepsilon \in E, a_1, \ldots, a_n \in R \) and take
\[
\varepsilon = \left( |\varepsilon| + \sum_{k} |a_k| \right)^{-1}.
\]
By construction there is \( i \in I \) and \( j \in J \) so that
\[
|\varepsilon - y_i| < \delta \quad \text{and} \quad \sum_{k} |a_k a_i - a_j^*| < \delta.
\]
If \( a_k \in c(V(T_k)) \) \((1 \leq k \leq n)\), it follows:
\[
|\varepsilon + \sum a_k a_i| \leq \varepsilon^{-1} \left( |\varepsilon| + \sum_{k} |a_k a_i| \right) + 2\delta
\]
\[
\leq \varepsilon^{-1} \left( s^*_{i,j}(y_i) + \sum_{k} a_k s^*_{i,j}(a_k) + 3\delta \right) = s^*_{i,j}(x) + \sum_{k} a_k s^*_{i,j}(a_k) + 3\delta \varepsilon^{-1}.
\]
Since \( \varepsilon^{-1} = |\varepsilon| + \sum_{k} |a_k| \leq M \left( |\varepsilon| + \sum_{k} |a_k a_i| \right) \), we obtain
\[
|\varepsilon + \sum a_k a_i| \leq s^*_{i,j}(x) + \sum_{k} a_k s^*_{i,j}(a_k) + 5\delta M \left( |\varepsilon| + \sum_{k} a_k a_i \right).
\]
Therefore
\[
\left( 1 - 5\delta M \right) |\varepsilon + \sum a_k a_i| \leq (1 + \varepsilon) \left( s^*_{i,j}(x) + \sum_{k} a_k s^*_{i,j}(a_k) \right)
\]
and hence
\[
|\varepsilon + \sum a_k a_i| \leq (1 + \varepsilon) \left( s^*_{i,j}(x) + \sum_{k} a_k s^*_{i,j}(a_k) \right).
\]
This completes the proof.

**Lemma 18.** If \( \delta \) is a finite subset of \( X^* \) and \( \varepsilon > 0 \), then there exists a finite-dimensional subspace \( E \) of \( X \) and \( \delta > 0 \), such that:
If \( \varepsilon \in E \) and \( |\varepsilon^*(x)| < \delta \) for each \( \varepsilon \in \delta \), then there is some \( y \in E \) with \(|y| < \varepsilon \) and \( \varepsilon^*(y) = \varepsilon^*(y) \) for all \( \varepsilon \in \delta \).

**Proof.** Assume \( \delta = \{a_1^*, \ldots, a_n^*\} \). Obviously \( f: X \to R^n, \| \| \) given by \( f(x) = (a_1^*(x), \ldots, a_n^*(x)) \) is an operator mapping \( X \) on some subspace \( \mathcal{S} \) of \( R^n \). Let \( E \) be a finite-dimensional subspace of \( X \) satisfying \( f(E) = \mathcal{S} \). By the open map principle we obtain \( \delta > 0 \) such that \( f(E \cap R(0, \delta)) = \mathcal{S} \cap R(0, \delta) \).

Let now \( \varepsilon \in E \) with \( |\varepsilon^*(x)| < \delta \) for each \( \varepsilon \in \delta \). Then \( f(x) \in \mathcal{S} \cap R(0, \delta) \) and therefore there is \( y \in E \) with \(|y| < \varepsilon \) and \( f(y) = f(x) \).

**Lemma 19.** Let \( n \in N, S_1, \ldots, S_n \) slices, \( E \) be a finite-dimensional subspace of \( X \) and \( \varepsilon > 0 \). Then there are slices \( T_1, \ldots, T_n \) and a finite-dimensional subspace \( F \) of \( X \), verifying the following properties:
(1) \( T_k \subseteq S_k \) \((1 \leq k \leq n)\).
(2) For each \( k = 1, \ldots, n \), let \( x_k \in c(V(T_k)) \) and \( \{a_k, b_k\} \) a finite number

of points in \( T_k \). Then for each \( k = 1, \ldots, n \), there are points \( (y_k, \epsilon)^{i_k}_{j_k} \) in \( X \), satisfying:
(1) \( y_k - a_k \in F \) \((1 \leq k \leq n, i)\),
(2) \( |y_k - a_k| < \delta \) \((1 \leq k \leq n, i)\),
(3) \( \sum_{k} a_k a_k \in F \) and \( (b_1, \ldots, b_n) \in R \), then
\[
\|x + \sum a_k a_k - b_k a_k \| < (1 + \varepsilon) \| x + \sum a_k a_k + \sum b_k a_k \|.
\]

**Proof.** Consider first slices \( T_1, \ldots, T_n \) and a finite subset \( \delta \) of \( X^* \) verifying (1), (2), (3) of Lemma 17. Next, take a finite-dimensional subspace \( F \) of \( X \) and \( \delta > 0 \) satisfying the condition of Lemma 18. By Lemma 18, there are slices \( T_1, \ldots, T_n \) such that \( T_k \subseteq T_k \) \((1 \leq k \leq n)\) and \( |\varepsilon^*(x_k)| < \delta \) \((1 \leq k \leq n)\). For each \( k = 1, \ldots, n \), let \( a_k \in c(V(T_k)) \) and \( \{a_k, b_k\} \) a finite number of points in \( T_k \).

For each \( k = 1, \ldots, n \), and \( i, j \), we have that \( |\varepsilon^*(a_k - a_j)| < \delta \) whenever \( \varepsilon \in \delta \).

Hence there is \( a_k \in F \) with \( |\varepsilon^*(a_k)| < \delta \) and \( \varepsilon^*(a_k - a_k) = \varepsilon^*(a_k) \) for every \( \varepsilon \in \delta \). Take \( y_k = a_k + \varepsilon^*(a_k) \). It remains to verify (3) of (2).

Let thus \( \varepsilon \in E, a_k \in R \) and \( (b_1, \ldots, b_n) \in R \). Let \( \varepsilon \in \delta \) be the functional in \( \delta \) associated to \( a_1, \ldots, a_n \) by Lemma 16.

We obtain
\[
\|x + \sum a_k a_k - b_k a_k\| < (1 + \varepsilon) \|x + \sum a_k a_k + \sum b_k a_k\|.
\]
This completes the proof.

**Lemma 20.** Let \( (a_k) \) be a sequence of points of \( X \). Then, for each \( p \in N \), we can define a finite subset \( Q_p \) of \( N^p \), a finite subset \( A_p \) of \( X \), finite-dimensional subspaces \( E_p, F_p \) of \( X \) and for each \( p \in N \), \( a_p > 0 \) and a slice \( S_{p+1} \) of \( G_p \) such that the following conditions hold:
(1) \( Q_p \) is the projection of \( Q_{p+1} \) on the \( p \) first coordinates \( (p \in N) \),
(2) \( A_p \subseteq E_p \) \((p \in N)\),
(3) \( E_p \subseteq E_{p+1}, F_p \subseteq F_{p+1} \) \((p \in N)\),
(4) \( S_{p+1} \subseteq S_p \) \((p \in N, p + 1 \in Q_{p+1})\),
(5) \( \sum a_p = 1 \) \((p \in N, o \in Q_p)\),
(6) There is \( \xi_p \in A_{p+1} \cap c(V(S)) \) so that
\[
\sum a_p S_{p+1} \subseteq B(\xi_p, r_{p+1}) \quad (p \in N, o \in O_p),
\]
(7) Let $\mathcal{S}_{\gamma}$ be the slices $(S_{\gamma})_{\gamma \in \mathbb{N}}$ and $F_\gamma$ satisfy condition (2) of Lemma 19, with $\epsilon = \epsilon_\gamma$ ($\gamma \in \mathbb{N}$).

Proof. We proceed inductively on $\gamma \in \mathbb{N}$.

(a) Take $\gamma = 1$, $A_\gamma = \emptyset$, $F_1 = \emptyset$, and $\lambda_1 = 1$. Let $S_{\gamma}$ be a slice and $F_{\gamma}$ a finite-dimensional subspace of $X$ satisfying Lemma 19 applied to $C_0$, $\epsilon_1$.

(b) Assume now $\gamma_\mu$, $A_{\gamma_\mu}$, $F_{\gamma_\mu}$, and for each $\omega \in \mathcal{O}_{\gamma_\mu}$, $\lambda_{\gamma_\mu}$, obtained. Let $\omega \in \mathcal{O}_{\gamma_\mu}$ be fixed. Lemma 5 and the fact that $X$ is $(\epsilon)$ yields an $n_{\omega} \in N$ such that $\mathcal{S}_{\gamma_\mu} \setminus \mathcal{S}_{\gamma_\mu + 1}$ is a finite-dimensional subspace of $X$ satisfying Lemma 19 applied to $C_0$, $\epsilon_{\gamma_\mu}$. Hence, $\mathcal{S}_{\gamma_\mu} \setminus \mathcal{S}_{\gamma_\mu + 1}$ is a finite-dimensional subspace of $X$ satisfying Lemma 19 applied to $C_0$, $\epsilon_{\gamma_\mu}$. It is easily seen that all conditions are fulfilled.

Proof of the theorem in case II. Take $\lambda > 1$ and let $(\epsilon_\gamma)$ be a sequence of positive numbers such that $\epsilon = \sum \epsilon_\gamma < \min \{1, \lambda \}$. We use the construction of Lemma 20 and consider points $\omega_n$ satisfying (6). Let $p \in N$ be fixed. Let $\lambda_{\gamma_\mu}$ be a sequence of positive numbers such that $\epsilon = \sum \lambda_{\gamma_\mu} < \min \{1, \lambda \}$.

We introduce inductively finite-dimensional subspaces $F_{\gamma}$ of $X$, by taking $F_1 = \mathbb{R}$.

Proof. We proceed inductively on $\gamma \in \mathbb{N}$. The reader will verify that $F_{\gamma + 1} = \mathbb{R}$.

References
A generalization of Khintchine's inequality and its application in the theory of operator ideals

by

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Abstract. We prove a generalization of Khintchine's inequality which can be used to estimate the absolutely r-summing norm and the r-factorable norm of the identity map from $l^w_1$ into $l^v_1$ for certain exponents $w$ and $v$. This result fills in the remaining gaps in the limit order diagrams of the operator ideals $I^w_v$ and $I^v_w$.

In the following $\mathcal{U}(E, F)$ denotes the set of all (bounded linear) operators from $E$ into $F$, where $E$ and $F$ are arbitrary Banach spaces.

An operator $S \in \mathcal{U}(E, F)$ is called absolutely r-summing ($1 \leq r < \infty$) if there exists a constant $\sigma$ such that

$$\left(\sum_{k=1}^{n} \left| \langle x_k, u_k \rangle \right|^r \right)^{1/r} \leq \sup \left( \sum_{k=1}^{n} \| x_k \| \right)^{1/r} : \| u_k \| \leq 1$$

for all finite families of elements $x_1, \ldots, x_n \in E$. The class $I^w_v$ of these operators is an ideal with the norm $\| S \|_{I^w_v} = \inf \| \sigma \|$, where the infimum is taken over all admissible factorizations.

Let us denote by $\mathcal{L}_P^w(E, F, \mu)$ the operator ideal of $\mathcal{L}_P(E, F, \mu)$, where $\mathcal{L}_P(E, F, \mu)$ is the space of all $\mathcal{L}_P$-valued measurable mappings $E \to F$, and $\mathcal{L}_P^w(E, F, \mu)$ is the space of all $\mathcal{L}_P$-valued measurable mappings $E \to F$ which are $w$-summing. The class $\mathcal{L}_P^w(E, F, \mu)$ is an ideal with the norm $\| S \|_{\mathcal{L}_P^w(E, F, \mu)} = \inf \| \sigma \|$, where the infimum is taken over all admissible factorizations.

Let us denote by $\mathcal{L}_P(E, F, \mu)$ the space of all $\mathcal{L}_P$-valued measurable mappings $E \to F$, and $\mathcal{L}_P^w(E, F, \mu)$ is the space of all $\mathcal{L}_P$-valued measurable mappings $E \to F$ which are $w$-summing. The class $\mathcal{L}_P^w(E, F, \mu)$ is an ideal with the norm $\| S \|_{\mathcal{L}_P^w(E, F, \mu)} = \inf \| \sigma \|$, where the infimum is taken over all admissible factorizations.