tions $C(f)F = C(f)G$, ces homomorphismes coïncident sur les mesures de Dirac $(\delta_x): x \in X$, et donc sur le sous-semi-groupe convexe faiblement fermé engendré par cet ensemble, c'est-à-dire sur $P[V(X)]$. Il s'ensuit que $F'$ et $G'$ coïncident sur l'enveloppe convexe équiliblée faiblement fermée de $P[V(X)]$, et donc sur un voisinage de 0. On en déduit que $F' = G'$, d'où $F = G$.

Références


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On basic sequences in non-locally convex spaces

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Abstract. The results in this paper concern linear topological spaces that are not necessarily locally convex and basic sequences in such spaces. The main result is a characterization of a block basic sequence which is a subsequence of a basis. Other sequences generated by a basis are considered.

0. Introduction. This paper deals with basic sequences in linear topological spaces which are not necessarily locally convex. The purpose of this paper is to study several types of sequences generated by a basis. One of these types is the block perturbation introduced by Pelczyński and Singer [11]; a second type is the $S_0$-basic sequence; and the third type studied has the form $(y_n)_{n=1}^\infty$, where $y_n = \sum_{i=1}^n x_i$ and $(x_n)_{n=1}^\infty$ is a basis.

Section 1 contains the essential definitions and terminology for the remainder of the paper.

Two important properties of basic sequences in $F$-spaces are contained in Section 2. An elementary proof of the "selective principle" of Bonanska and Pelczyński [3], in the context of an $F$-space, is given following closely the work of N. J. Kalton [7].

Section 3 contains a characterization of the space $S_0$.

Block perturbations are used in Section 4 to characterize the locally bounded spaces among $F$-spaces with a bounded, regular basis.

Section 5 contains results concerning when a block basic sequence is a subsequence of a basis.

The paper is concluded with an elementary proof of a result of L. Drewnowski [4] concerning the weak basis theorem. The author wishes to thank the referee for his/her helpful comments.

1. Definitions and terminology. Let $X$ denote an arbitrary linear topological space in this section unless otherwise specified. Let $(y_n)$ be a sequence contained in $X$; then $(y_n)$ denotes the closed linear span of $(y_n)$. Two basic sequences $(y_n)$ and $(z_n)$ are said to be equivalent provided that $\sum_{n=1}^\infty a_n y_n$ converges if and only if $\sum_{n=1}^\infty a_n z_n$ converges. A basic sequence

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(yₙ) is called unconditional if whenever \( \sum_{n=1}^{m} aₙ yₙ \) converges and \( (xₙ) \) is a sequence with \( eₙ = \pm 1 \) \((n = 1, 2, 3, \ldots)\), then \( \sum_{n=1}^{m} eₙ aₙ yₙ \) also converges.

A sequence \((yₙ)_{n=1}^{\infty}\) is called regular provided that there is a neighborhood \( N \) of zero such that \( yₙ \notin N \) \((n = 1, 2, 3, \ldots)\).

Let \( \| \cdot \| \) be a non-negative, real-valued function defined on \( X \), let \( p \) be in \((0, 1] \) and let \( \mathcal{K} \) be any positive constant. Consider the following properties:

1. \( \| x + y \| \leq \| x \| + \| y \| \)
2. \( \| x + y \| \leq \mathcal{K} (\| x \| + \| y \|) \)
3. \( \| x \| = \| -x \| \)
4. \( \| x \| = \| \lambda x \| \)
5. \( \| x \| \leq \| y \| \) if \( \| x \| \leq 1 \)
6. \( \lim_{n \to \infty} \| x_n \| = 0 \)

If \( \| \cdot \| \) satisfies (1) and (3), it is called a semi-norm; if \( \| \cdot \| \) satisfies (1), (3) and (4), it is called an \( \mathcal{F} \)-semi-norm; if \( \| \cdot \| \) satisfies (1) and (4) it is called a \( p \)-semi-norm; and if \( \| \cdot \| \) satisfies (2) and (3) it is called a quasi-norm.

In the case that, in addition \( \| x \| = 0 \) implies that \( x = 0 \), a semi-norm is a norm, an \( \mathcal{F} \)-semi-norm is an \( \mathcal{F} \)-norm, and a \( p \)-semi-norm is a \( p \)-norm.

It is well known that the topology of a metrizable space \( X \) may always be defined by an \( \mathcal{F} \)-norm. A complete linear metric space is called an \( \mathcal{F} \)-space. The phrase "let \( (X, \| \cdot \|) \) be an \( \mathcal{F} \)-space" will mean that \( X \) is an \( \mathcal{F} \)-space whose topology is defined by an \( \mathcal{F} \)-norm, \( \| \cdot \| \).

Let \( N \) be a neighborhood base at zero for the space \( X \). Let \( N \) be the \( \{0\} \) neighborhood base at zero for a linear topology on \( X \); this topology is called the Mackey topology of \( X \). If the original topology on \( X \) is denoted by \( \mu \) then the associated Mackey topology of \( X \) is denoted by \( \mathcal{M}(\mu) \) and the two linear topological spaces are designated by \( (X, \mu) \) and \( (X, \mathcal{M}(\mu)) \). If \( X \) is metrizable, then \( X \) with the Mackey topology is semi-metrizable, and the Mackey topology is the strongest locally convex topology on \( X \) with the same topological dual (see [12]). Thus, for \( X \) metrizable, the use here of the term "Mackey topology" is consistent with the use of the term in locally convex space theory. (Cf. page 263 of [5]).

2. Some properties of basic sequences. Proposition 2.1 below is due essentially to N. J. Kalton [9]. (See also [13]).

Theorem 2.2. Let \( (X, \| \cdot \|) \) be an \( \mathcal{F} \)-space with a basis \( (xₙ, fₙ)_{n=1}^{\infty} \). Assume that \( (yₙ) \) is a sequence satisfying

(1) \( \lim f_j(xₙ) = 0 \) \((j = 1, 2, 3, \ldots)\) and

(2) \( (yₙ) \) is regular.

Then there is a subsequence \( (yₙ_j) \) of \( (yₙ) \), with \( yₙ_j \to y_1 \), such that \( (yₙ_j) \) is a basic sequence equivalent to a top basic sequence of \( (xₙ) \).

Proof. Let \( \mu \) denote the original topology of \( X \) and let \( \mathcal{F} \) be the topology generated by the sequence of semi-norms \( (\| \cdot \|_n) \), where \( \| \cdot \|_n = \| fₙ \| \).

If \( \mathcal{F} \) is a Hausdorff vector topology on \( X \) that is weaker than \( \mu \), and \( yₙ \to 0 \) in the \( \mathcal{F} \) topology. Therefore, by Proposition 2.2 (i) of [7], there exist vector topologies \( \mathcal{G} \) and \( \beta \), with \( \| x \| \leq \beta \leq \mu \) such that \( \beta \) is metrizable and \( \mathcal{G} \)-polar (i.e., \( \beta \) has a neighborhood base at zero consisting of \( \mathcal{G} \)-closed sets), \( yₙ \to 0 \) in the \( \mathcal{G} \) topology, but \( yₙ \to 0 \) in the \( \beta \) topology. Then by Theorem 3.2 of [7] there exists a subsequence \( (yₙ_j) \) of \( (yₙ) \) such that \( (yₙ_j) \) is basic for the \( \beta \) topology and \( yₙ_j \to y_1 \). Now, using (1) and (2), it is easy
to choose a subsequence \((y_{n_j})\) of \((y_n)\), with \(y_{n_j} = y_{i_j}\), and a block basic sequence \((\epsilon_j)\) of \((\epsilon_n)\) so that \(|y_{n_j} - \epsilon_j| < \beta\) \((j = 1, 2, 3, \ldots)\). If \(\sum_{j=1}^\infty \lambda_j y_{n_j} = 0\) in the \(\mu\) topology, then \(\sum_{j=1}^\infty \lambda_j y_{i_j} = 0\) in the \(\beta\) topology, and hence \(\epsilon_j = 0\) \((j = 1, 2, 3, \ldots)\). Thus \(y_{n_j} = y_{i_j} + (y_{n_j} - y_{i_j})\), \((\epsilon_j)\) is a regular basic sequence \(\sum_{j=1}^\infty |y_{n_j} - y_{i_j}| < \infty\) and \(\sum_{j=1}^\infty \lambda_j y_{n_j} = 0\) implies that \(\epsilon_j = 0\) \((j = 1, 2, 3, \ldots)\). Therefore Lemma 4.3 of [7] implies that \((y_{n_j})\) is a basis sequence (and it is easy to see that \((y_{n_j})\) is equivalent to \((\epsilon_j)\).

3. A characterization of the space \(e_0^\infty\). Let \((x_n)_{n=1}^\infty\) be a basis of an \(F\)-space \(X\). Let \(y_n = \sum_{k=1}^n x_k\) \((n = 1, 2, 3, \ldots)\). In the space \(e_0^\infty\), where \((x_n)_{n=1}^\infty\) denotes the unit-vectors basis, the sequence \((y_n)_{n=1}^\infty\) is a basis. Theorem 3.2 shows that such a condition provides a characterization of \(e_0^\infty\).

A space \(X\) is called a locally bounded space if it contains a bounded neighborhood of zero. Aoki [1] and Rolewicz [11] have shown that an \(F\)-space \(X\) is a locally bounded space if and only if there is a number \(p, 0 < p < 1\), such that the topology on \(X\) may be defined by a \(p\)-norm.

The following lemma is contained, essentially, in Proposition 3.1 of [6].

**Lemma 3.1.** Let \((X, \|\cdot\|)\) be an \(F\)-space with a basis \((x_n)_{n=1}^\infty\). Then \((x_n)_{n=1}^\infty\) is bounded if and only if there is a bounded set \(A\) contained in \(X\) and \(\epsilon > 0\) such that \(\sup_{a \in A} |f_n(a)\| > \epsilon\) \((n = 1, 2, 3, \ldots)\).

**Proof.** Assume that \((x_n)_{n=1}^\infty\) is bounded. The \(\sup_{n \geq 1} |f_n(x_n)| = 1\) \((n = 1, 2, 3, \ldots)\).

Now assume that \(A\) is a bounded subset of \(X\) such that

\[
\sup_{a \in A} |f_n(a)| > \epsilon \quad (n = 1, 2, 3, \ldots).
\]

It may be assumed without loss of generality that

\[
\left\| \sum_{n \geq 1} f_n(a) x_n \right\| \leq \|a\|, \quad a \in X, \|a\| \leq \epsilon.
\]

Choose a sequence \((a_n)_{n=1}^\infty\) contained in \(A\) such that \(|f_n(a_n)| > \epsilon\). Suppose that \((x_n)_{n=1}^\infty\) is a sequence of scalars and \(\epsilon > 0\). Then

\[
\left\| \sum_{n \geq 1} f_n(a_n) a_n \right\| = \left\| f_n(t_n(a_n), a_n) \right\| \leq \|a_n\| \epsilon.
\]

Since \(A\) is bounded, \(t_n(a_n) a_n \to 0\); and since \(|f_n(a_n)| > \epsilon\), this implies that \(t_n a_n \to 0\). Therefore \((x_n)_{n=1}^\infty\) is bounded.}

**Theorem 3.2.** Let \(X\) be a locally bounded \(F\)-space with a bounded unconditional basis \((x_n, f_n)_{n=1}^\infty\). For \(n = 1, 2, 3, \ldots\), let \(y_n = -\sum_{i=1}^n x_i\). Then \((y_n)_{n=1}^\infty\) is a basis of \(X\) if and only if \((x_n)_{n=1}^\infty\) is equivalent to the unit-vectors \((e_i)\).

**Proof.** (i) The “if” part of the theorem is well-known.

(ii) Assume that \((x_n)_{n=1}^\infty\) is a basis of \(X\). It may be assumed without loss of generality that the topology of \(X\) is defined by a \(p\)-norm, \(\|\cdot\|\), for some \(p\) \((0 < p < 1)\). Let \(y_n = f_n - f_{n+1} + f_{n+2} - \ldots\) \((n = 1, 2, 3, \ldots)\). Then \((y_n)_{n=1}^\infty\) is the sequence of coefficients functions associated with \((x_n)_{n=1}^\infty\). Since \((x_n)_{n=1}^\infty\) is unconditional and \(X\) is locally bounded, it may be assumed without loss of generality that

\[
\left\| \sum_{n} \epsilon_n f_n(x) x_n \right\| \leq \|x\|
\]

whenever \(m \leq n, \|x\| \leq 1\) \((i = 1, 2, \ldots, n\) and \(x \in X\).

Suppose that \((x_n)_{n=1}^\infty\) is not regular. Then there is a subsequence \((y_{n_j})_{j=1}^\infty\) such that \(\|y_{n_j}\| \leq 1/2^j\) \((j = 1, 2, 3, \ldots)\). It follows that \(\sum y_{n_j}\) converges to some \(x\) in \(X\). But

\[
\sum_{j=1}^\infty f_j(x) x_{n_j} = \sum_{j=1}^\infty f_j(x) x_{n_j - f_{n_j+1}(x) y_{n_j - 1}}
\]

and

\[
\|f_j(x) y_{n_j - 1}\| \leq \|y_{n_j}\| \leq \|x\| \quad (j = 1, 2, 3, \ldots).
\]

Therefore \(\sum f_j(x) y_{n_j}\) does not converge to \(x\). It thus may be assumed that there exists \(\delta > 0\) such that \(\inf \|x\| > \delta\).

Let \(A = \{x_n; n = 1, 2, 3, \ldots\} + \{x_n; n = 1, 2, 3, \ldots\}\). Since \((x_n)_{n=1}^\infty\) is bounded, \(A\) is bounded. Then

\[
\sup_{a \in A} \|f_n(a)\| = \sup_{a \in A} (|f_n(a) - f_{n+1}(a)| + |f_{n+1}(a) - f_{n+2}(a)| + \ldots) \leq 1 - \delta = \frac{1}{2}.
\]

Therefore Lemma 3.1 implies that \((y_n)_{n=1}^\infty\) is bounded. Choose a number \(M\) such that \(\sup \|y_n\| \leq M\).

Suppose that \(a_1, a_2, \ldots, a_n\) are scalars. Then

\[
\delta \max \|a_i\|^p \leq \max \|x_i\| |a_i|^p = \max \|x_i\| |a_i|^p \leq \sum_{i=1}^n |a_i|^p
\]

\[
\leq \sum_{i=1}^n \max \|x_i\| |a_i|^p = (\max \|a_i\|^p) \sum_{i=1}^n |a_i| \leq M \max \|a_i\|^p.
\]

Therefore \(\sum a_i x_n\) converges if and only if \((a_n) \in e_0\).
4. Block perturbations. A block perturbation is obtained in the following way: Let \((X, \| \cdot \|)\) be an \(F\)-space with a basis \((e_n)_{n=1}^{\infty}\), let \((s_n)\) and \((y_n)\) be strictly increasing sequences of positive integers with \(m_{j+1} < s_j < m_j\) \((j = 1, 2, 3, \ldots)\) and let \((a_i)\) be a sequence of scalars. For each \(j\), let

\[
   a_j = \sum_{j=1}^{m_j-1} a_i e_i + \sum_{j=1}^{m_j} a_i e_i.
\]

Then let

\[
   y_n = \begin{cases} 
   s_n, & \text{if } n \neq m_j, \\
   s_n + y_j, & \text{if } n = m_j 
   \end{cases} \quad (n = 1, 2, 3, \ldots).
\]

If there is a constant \(K \geq 0\) such that \(\sup \|a_i\| \leq K\), then \((y_n)_{n=1}^{\infty}\) is called a block perturbation of \((a_n)_{n=1}^{\infty}\), with \(\| \cdot \|\)-bounded blocks. (Cf. [10].) In the case that \(X\) is locally bounded it will be assumed that the norm, \(\| \cdot \|\), above denotes a \(p\)-norm defining the topology of \(X\); hence \((a_n)_{n=1}^{\infty}\) is topologically bounded in this case. Pełczyński and Singer [10] have shown that each block perturbation of a normalized basis of a Banach space \(X\) is also a basis of \(X\). Part (a) of the next theorem is an extension of the result of Pełczyński and Singer. The proof of part (a) is omitted since it is the same as that given in [10].

**Theorem 4.1.** Let \(X\) be an \(F\)-space with a regular, bounded basis \((e_n, f_n)_{n=1}^{\infty}\).

(a) If \(X\) is a locally bounded space, then each block perturbation of \((a_n)_{n=1}^{\infty}\) is a basis.

(b) If \(X\) is not a locally bounded space, then for each \(F\)-norm, \(\| \cdot \|\), defining the topology of \(X\), there is a block perturbation of \((a_n)_{n=1}^{\infty}\), with \(\| \cdot \|\)-bounded blocks, which is not a basis of \(X\).

**Proof.** (b) Assume that \(X\) is not locally bounded and \(\| \cdot \|\) is an \(F\)-norm defining the topology of \(X\). An equivalent \(F\)-norm, \(\| \cdot \|\), may be defined on \(X\) satisfying each of the following properties:

1. \(\|x\| \geq \|x\|\)
2. \(\|x\| \geq \sum_{x \in X} f_x(x) a_i\) if \(s = a_{i+1}\) and \(a_{i+1}\) are positive integers with \(m \leq n\).

Since the basis \((a_n)_{n=1}^{\infty}\) is regular, it is easy to see by property (2) that there is a constant \(n > 0\) so that for each \(x \in X\) there is a scalar \(a = a(x)\) such that

3. \(\|ax\| \geq \|x\|\).

Since \(X\) is not locally bounded, \((x: \|x\| \leq \varepsilon)\) is not bounded. Thus, using condition (3) and the boundedness of \((a_n)_{n=1}^{\infty}\), there exists a sequence \((e_n, f_n)_{n=1}^{\infty}\) contained in \(X\) and a sequence \((t_n)_{n=1}^{\infty}\) of scalars such that \(\|e_n\| = e_n\), \(\inf \|f_n\| > 0\), \((t_n)_{n=1}^{\infty}\) is a decreasing sequence, \(\lim t_n = 0\) and \(\|a_n e_i f_n(x)\| < 1/2^n (n = 1, 2, 3, \ldots; i = 1, 2, 3, \ldots)\). Let \(\delta = \inf \|f_n\|\), let \(m_0 = 0\), and let \(n_1 = 1\). Choose \(m_1\) such that \(\|f_n(x)\| > \delta/2\). The collection \(\{(f_n(x))_n = 1, 2, \ldots, m_0; n = 1, 2, 3, \ldots\}\) is bounded for each \(n\) since \(e = \|e_n\| \geq \|f_n(x)\|\) for all \(x\) and \(n\) and since condition (3) holds. Therefore there is an integer \(n_1 > n_1\) satisfying:

\[
\left\| \sum_{j=1}^{n_1} t_n f_n(x_n) e_n \right\| < \delta/2.
\]

Also there is an integer \(m_1 < n_1\) such that

\[
\left\| \sum_{j=m_0}^{m_1-1} t_n f_n(x_n) e_n \right\| > \delta/2.
\]

Using induction, it is easy to see that there exist strictly increasing sequences \((a_n)_{n=1}^{\infty}\) and \((m_n)_{n=1}^{\infty}\) such that

\[
\left\| \sum_{n=m_0}^{n_1-1} t_n f_n(x_n) e_n \right\| > \delta/2, \quad n = 1, 2, 3, \ldots
\]

Let

\[
   y_n = \begin{cases} 
   s_n, & \text{if } n \neq m_j, \\
   s_n + y_j, & \text{if } n = m_j 
   \end{cases}
\]

where

\[
   y_j = \sum_{n=m}^{m_1-1} f_n e_n a_x e_n.
\]

Then

\[
   \|y_j\| = \sum_{n=m_0}^{n_1-1} f_n e_n a_x e_n \leq \|e_n\| = \varepsilon,
\]

hence \((y_n)_{n=1}^{\infty}\) is a block perturbation of \((a_n)_{n=1}^{\infty}\) with \(\| \cdot \|\)-bounded blocks.

Since \(\|a_n e_i f_n(x)\| \leq 1/2^n (n = 1, 2, 3, \ldots; i = 1, 2, 3, \ldots)\), \(\sum_{i=1}^{\infty} a_n f_n(x) a_i e_i \) converges to some element \(x \in X\).

Suppose that \((a_n)_{n=1}^{\infty}\) is a basis of \(X\). Then \(x = \sum_{n=1}^{\infty} a_n y_n\) for some sequence \((a_n)_{n=1}^{\infty}\) of scalars. It follows that

\[
   t_n = f_n \sum_{i=1}^{m_n} t_{i_n} e_i a_{i_n} = f_n(x) = f_n \sum_{i=1}^{m_n} a_{i_n} y_n = a_{i_n} (j = 1, 2, 3, \ldots).
\]
Also, for $m \neq m_j$,

\[
0 = f_m \left( \sum_{n=1}^{m_j} a_n y_n \right) - f_m \left( \sum_{n=1}^{m} a_n y_n \right) = \sum_{n=1}^{m} a_n f_m(y_n).
\]

Therefore

\[
a_{n_j} = a_{m_j} (j = 1, 2, 3, \ldots) \quad \text{and} \quad a_m = -n m_j (m_{j-1} < m < m_j; j = 1, 2, 3, \ldots).
\]

Choose $N$ such that for $j \geq N \delta$, \(|\sum_{n=m_j}^{m} a_{n_j} y_{n_j}| \leq \delta/4$. If $j \geq N$, then

\[
\left\| x - \sum_{n=1}^{m} a_n y_n \right\| \geq \sum_{j=1}^{j-1} \left| t_{n_j} y_{n_j} - \sum_{i=1}^{m_j} a_{m_i} y_{m_i} \right| - \left\| \sum_{i=m_j+1}^{m} a_{n_i} y_{n_i} \right\| \geq - \sum_{j=m_{j-1}+1}^{m} t_{n_j} y_{n_j} - \delta/4 \geq \delta/2 - \delta/4 = \delta/4.
\]

This shows that \(\sum_{n=1}^{m} a_n y_n\) cannot converge to $x$. It must then be concluded that \(y_{n_j}\) is not a basis of $X$. \(\blacksquare\)

Remark. The space $(s)$ is a non-locally bounded $F$-space with a bounded basis. It is easy to see, following the proof given by Pelczynski and Singer for Theorem 4.1 part (a), that each block perturbation of a basis of $(s)$ is also a basis of $(s)$. Hence the condition of regularity may not be removed in Theorem 4.1.

5. Block extensions. Let \((y_{n_j})_{n_j=1}^{m_j}\) be a block basic sequence of the form $(*)$ given in Section 2. If \((y_{n_j})_{n_j=1}^{m_j}\) is a sequence in $X$ with $y_{n_j} = x_j$ and $y_{n_j} \in \{x_{n_j}\}_{n_j=1}^{m_j+1}$, then the sequence \((y_{n_j})_{n_j=1}^{m_j+1}\) is called a block extension of \((x_{n_j})_{n_j=1}^{m_j+1}\). Zippin [16] has proved that each block basic sequence of a basis of a Banach space $(s)$ is a block extension that is a basis of $X$. Locally convex spaces are characterized by means of block extensions in this section (see Theorem 5.8).

The proof of the next lemma is the same as that of Zippin in [16].

Lemma 5.1. Let $F$ be an $n$-dimensional Hausdorff linear topological space and assume that the topology on $F$ is defined by a $p$-norm, \(\|\cdot\|\), for some $p \in (0, 1]$. If $H$ and $G$ are any two hyperplanes in $F$, then there exists an isomorphism $T : H \to G$ satisfying

(i) \(\frac{1}{2} \|x\| \leq \|Tx\| \leq 3 \|x\| (x \in H)\) and

(ii) $Te = x$ ($x \in H \cap G$).

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\[
\text{Theorem 5.2. Let } (X, \mu) \text{ be a locally bounded } F\text{-space whose topology is defined by a } p\text{-norm, } \|\cdot\|. \text{ Let } (x_{n_j})_{n_j=1}^{m_j} \text{ be a basis of } X \text{ and let } (y_{n_j})_{n_j=1}^{m_j} \text{ be a normalized block basic sequence of the form}
\]

\[
(*) \quad y_{n_j} = \sum_{i=n_j+1}^{m_j} a_i x_i \quad (j = 1, 2, 3, \ldots).
\]

Then the following are equivalent:

(1) The block basis \((x_{n_j})_{n_j=1}^{m_j}\) has a block extension that is a basis of $X$.

(2) There is a constant $K$ such that for each $j$ there is a continuous linear functional $g_j$ defined on \([x_{n_j}]_{n_j=1}^{m_j+1}\), satisfying

\[
g_j(x_j) = 1 \quad \text{and} \quad \sup \{g_j(x) \leq K \}.
\]

(3) The block basis \((x_{n_j})_{n_j=1}^{m_j}\) is regular for the topology $\mu(\mu)$.

Proof. For $x \in X$, define $\|x\| = \inf \{t > 0 : x \in tA\}$, where $A = \text{co}\{x : \|x\| < 1\}$. Then $\|x\|_1$ is a norm and $\mu(\mu)$ is defined by $\|x\|_1$. (1) implies (2). Assume that \((x_{n_j})_{n_j=1}^{m_j}\) has a block extension \((y_{n_j})_{n_j=1}^{m_j+1}\) that is a basis of $X$. Then there is a constant $K'$ such that for positive integers $l, m$ and $n$, with $l < n$ and scalar $t_1, t_2, \ldots, t_{m+n}$

\[
\left\| \sum_{i=1}^{m+n} t_i y_i \right\| \leq K' \left\| \sum_{i=1}^{m+n} t_i y_i \right\|.
\]

It follows that $T_j : x_{n_j} \to x_j$ defined by

\[
T_j \left( \sum_{i=n_j+1}^{m_j} t_i y_i \right) = \sum_{i=n_j+1}^{m_j} t_i y_i \quad (x_{n_j} = \{y_{n_j} : y_{n_j} \in x_{n_j}\})
\]

is a continuous projection of norm less than or equal to $K'$ and $I - T_j$ is a continuous projection of $x_{n_j}$ onto \(x_j\) of norm less than or equal to $K' + 1$ ($j = 1, 2, 3, \ldots$). Let $K = K' + 1$ and define a functional $g_j$ on $x_j$ by $g_j \left( \sum_{i=n_j+1}^{m_j} t_i y_i \right) = t_j$. Then $g_j(x_j) = 1$ and

\[
(I - T_j) \left( \sum_{i=n_j+1}^{m_j} t_i y_i \right) = t_j x_j = g_j \left( \sum_{i=n_j+1}^{m_j} t_i y_i \right). x_j.
\]

If $\sup \{g_j(x) \leq K \}$, then $\|g_j(x) \leq K \leq K' + 1$.

Since $\|x_j\|_1 = 1$ ($j = 1, 2, \ldots$),

\[
\sup \{g_j(x) : x \in x_j, \|x\|_1 \leq 1 \leq K' + 1\}
\]

(2) implies (1). The proof that conditions (a) and (b) imply (1) has been given essentially in [16]. It is therefore omitted here.
(3) implies (2): Assume that condition (2) holds, and suppose that $(v_{n,k})_{n-k}$ is not regular for the topology $\mathcal{m}(\mu)$. Then $\liminf_{n \to \infty} |v_{n,k}| = 0$ for some subsequence $(v_{n,k})$. Since $\sup_{s \in \mathbb{C}} |g_j(s)| < 1$ implies that $\liminf_{n \to \infty} |v_{n,k}| = 0$, it follows that $(v_{n,k})_{n-k}$ is an equicontinuous family for the topology $\mathcal{m}(\mu)$. Hence $\limsup_{n \to \infty} |v_{n,k}| = 0$. But $g_j(x_j) = 1$ for each $j$, so $(v_{n,k})_{n-k}$ must be regular for $\mathcal{m}(\mu)$.

(3) implies (2): Now assume that there is a constant $\delta > 0$ such that $|v_{n,k}| > \delta$ for $f = 0, 1, 2, 3, \ldots$. Then for each $j$ there is a continuous linear functional $g_j$ defined on $X$ such that $g_j(e_j) = 1$ and $\sup_{s \in \mathbb{C}} |g_j(s)| = \frac{1}{|v_{n,k}|}$.

This implies that $\sup_{s \in \mathbb{C}} |g_j(s)| < 1/\delta$ for $f = 0, 1, 2, 3, \ldots$. Therefore condition (2) holds.

**Corollary 5.3.** Let $X$ be a locally bounded $F$-space with a basis $(v_{n,k})_{n,k}$. Then each block basic sequence of $(v_{n,k})_{n,k}$ has a block extension that is a basis of $X$ if and only if $X$ is a Banach space.

**Proof.** If $X$ is a Banach space, then the previously mentioned result of Zippin in [16] states that each block basic sequence of $(v_{n,k})_{n,k}$ has a block extension that is a basis of $X$.

Conversely, if $X$ is not a Banach space, then $X$ is not locally convex. Let $\mu$ denote the topology of $X$. Then there is a sequence $(v_{n,k})_{n,k}$ contained in $X$ such that $v_{n,k} \to 0$ in the topology $\mathcal{m}(\mu)$, but $v_{n,k} \neq 0$ in the topology $\mu$.

Let $||| \cdot |||$ be a $\mu$-norm defining $\mu$, and let $||| \cdot |||$ be defined by $||| s ||| = \inf \{ t > 0 : s \in tA \}$, where $A = \{ x : |x| < 1 \}$.

It may be assumed without loss of generality that there is a constant $\delta > 0$ such that $||| v_{n,k} ||| > \delta$ for some $n \geq 2$. Since $v_{n,k} \to 0$ in the weak topology also, it is easy to see that there is a subsequence $(v_{n_k,k})_{n_k,k}$ and a block basic sequence $(v_{n_k,k})_{n_k,k}$ such that $\lim\sup_{n_k} |v_{n_k,k}| < 1/\delta$ for $f = 0, 1, 2, 3, \ldots$. It follows that $|v_{n,k}| > \delta$ for some constants $\delta' > 0$ and that $|v_{n,k}| \leq |v_{n_k,k}| + |v_{n_k,k} - v_{n_k,k}|| - |v_{n_k,k} - v_{n_k,k}|| = 0$.

So $\lim\sup_{n_k} |v_{n,k}| = 0$. Since for each $n_k |v_{n,k}| > \delta'$, it follows that $\lim\sup_{n_k} |v_{n,k}| = 0$.

Therefore by Theorem 5.2 $(v_{n,k})_{n,k}$ is a block basic sequence of the unit-vectors basis, $(v_{n,k})_{n,k}$, in $l_1(p < 1)$. Assume that $v_{n,k} = \sum_{i=1}^n a_i e_i (j = 1, 2, 3, \ldots)$ and that $(v_{n,k})_{n,k}$ is normalized with respect to $||| \cdot |||$, the usual $\mu$-norm of $l_1$.

(a) The sequence $(v_{n,k})_{n,k}$ has a block extension that is a basis of $l_1$, if and only if $(v_{n,k})_{n,k}$ is regular in the $l_1$-topology restricted to $l_1$.

(b) If $\lim\sup_{n_k} |v_{n,k}| > 0$, then $(v_{n,k})_{n,k}$ has a block extension that is a basis of $l_1$.

**Proof.** (a) This is immediate from Theorem 5.2 since the associated Mackey topology of $l_1$ is the $l_1$-topology restricted to $l_1$.

(b) Choose $m_k$ such that $|v_{n,k}| = \max_{|v_{n,k}| < m_k}$.

Let $F_j = \{ n_1 + 1, n_2 + 1, \ldots, n_j \}$ and $G_j = F_j \cup (v_{n,j})$ (j = 1, 2, 3, \ldots).

For each $j$ also define a linear functional $g_j$ on $[v_{n,j}]_{n,k}$ by $g_j = \sum_{i=1}^n t_i e_i + f_{n,j}$, where $\sum_{i=1}^n t_i e_i + f_{n,j}$ is the unique representation of a vector $v \in [v_{n,j}]_{n,k}$ in terms of the basis $(e_i : i \in G_j) \cup (v_{n,j})$.

If $\sum_{i=1}^n t_i e_i + f_{n,j} \leq 1$, then $|v_{n,j}| \leq 1$. Hence if $\sum_{i=1}^n t_i e_i + f_{n,j} \leq 1$, then $|v_{n,j}| \leq 1$. Therefore, by part (2) of Theorem 5.2, $(v_{n,k})_{n,k}$ has a block extension that is a basis of $l_1$.

**Corollary 5.4.** Let $X$ be a locally bounded $F$-space with a basis $(v_{n,k})_{n,k}$. Assume that $(v_{n,k})_{n,k}$ is a block basis of the form (c) given in Theorem 5.3. If $\sum_{i=1}^n |a_i e_i| < \infty$, then $(v_{n,k})_{n,k}$ has a block extension that is a basis of $X$.

**Proof.** It may be assumed without loss of generality that $\sum_{i=1}^n |a_i e_i| < \infty$, and that $|v_{n,k}| = 1$ for all $n \geq 1, 2, 3, \ldots$. Choose $m_k$ so that $v_{n,k} < m_k$ and $|v_{n,k}| = \max_{|v_{n,k}| < m_k}$ for $j = 1, 2, 3, \ldots$.

Let $F_j = \{ n_1 + 1, n_2 + 1, \ldots, n_j \}$ and $G_j = F_j \cup (v_{n,j})$.

Suppose that $\sum_{i=1}^n t_i e_i + f_{n,j} < 1$. Let $g_j = \frac{1}{|v_{n,j}|} f_{n,j}$. Then $g_j$ defines a vector $v \in [v_{n,j}]_{n,k}$.

If $\sum_{i=1}^n t_i e_i + f_{n,j} < 1$, then $|v_{n,j}| < 1$. But $|v_{n,j}| < 1$. Hence $g_j = 1$.

Thus $|g_j| = \left| \sum_{i=1}^n t_i e_i + f_{n,j} \right| < \sum_{i=1}^n |t_i e_i + f_{n,j}| < 1$.

Therefore by Theorem 5.2 $(v_{n,k})_{n,k}$ is a block basic sequence of the unit-vectors basis, $(e_i)_{i=1}^n$, in $l_1(0 < p < 1)$. Assume that $v_{n,k} = \sum_{i=1}^n a_i e_i (j = 1, 2, 3, \ldots)$ and that $(v_{n,k})_{n,k}$ is normalized with respect to $||| \cdot |||$, the usual $\mu$-norm of $l_1$.

Lindenstrauss and Tzafriri have considered projections in Banach spaces onto subspaces spanned by block basic sequences. In [8] they...
proved that a Banach space $X$ with an unconditional basis $(a_n)_{n=1}^\infty$ is isomorphic to either $c_0$ or $l_p$, $1 \leq p < \infty$, if and only if for each permutation $\pi$ of the integers and each block basic sequence $(a_{n_k})_{k=1}^\infty$ of $(a_n)_{n=1}^\infty$ there exists a projection in $X$ onto the block subspace $[a_{n_k}]_{k=1}^\infty$. Corollary 5.7 is an extension of this result to the case of a locally bounded $F$-space.

**Theorem 5.6.** Let $X$ be a locally bounded $F$-space with a basis $(a_{n_k})_{k=1}^\infty$. Assume that for each block basic sequence $(a_{n_k})_{k=1}^\infty$, $(a_{n_k})_{k=1}^\infty$ is complemented in $X$. Then $X$ must be locally convex, and hence normed.

**Proof.** Let $|| \cdot ||$ be a $p$-norm defining the topology of $X$. By Corollary 5.3 it suffices to show that each normalized block basic sequence of $(a_{n_k})_{k=1}^\infty$ has a block extension that is a basis of $X$.

Let $(a_{n_k})_{k=1}^\infty$ be a normalized block basic sequence of $(a_{n_k})_{k=1}^\infty$, say $a_j = \sum_{k=1}^{\infty} c_k a_k$ for $1 \leq j \leq \infty$. By Theorem 5.2 it suffices to show that there exists a constant $K > 0$ and a sequence of continuous linear functionals $(g_n)_{n=1}^\infty$, such that

(a) $g_j$ is defined on $[a_{n_k}]_{k=1}^\infty$, (b) $g_j(a_k) = c_k$, and (c) $||g_j|| = K$ for each $j$.

By renorming the subspace $[a_{n_k}]_{k=1}^\infty$, it may be assumed without loss of generality that $\sum_{j=1}^{\infty} b_j a_j = ||a||$ for each $a_j$, whenever $\sum_{j=1}^{\infty} b_j a_j$ converges in $[a_{n_k}]_{k=1}^\infty$.

Since $(a_{n_k})_{k=1}^\infty$ is complemented in $X$, there is a projection $P$ of $X$ onto $[a_{n_k}]_{k=1}^\infty$. Thus, by renorming $X$, it may be assumed without loss of generality that $||a|| = ||P a|| + ||I - P a||$ for $x \in X$. For each $j$, define a linear function $g_j$ by the following:

$$g_j(a) = b_j, \text{ if } P a = \sum_{j=1}^{\infty} b_j a_j.$$  

Then $g_j(a_j) = 1$ for each $j$. Also if $x = \sum_{j=1}^{\infty} b_j a_j$ and $||x|| \leq 1$, let $P x = \sum_{j=1}^{\infty} b_j a_j$ then

$$||g_j(x)||^p = ||b_j a_j||^p = ||b_j a_j||^p \leq \sum_{j=1}^{\infty} ||b_j a_j||^p = ||P x|| \leq ||P x||.$$

Therefore conditions (a), (b), and (c) are satisfied.

The next corollary is an immediate consequence of Theorem 3.11 and the result of Lindenstrauss and Tafriri mentioned in the paragraph following Corollary 5.5.

**Corollary 5.7.** A locally bounded $F$-space $X$ with an unconditional basis $(a_{n_k})_{k=1}^\infty$ is isomorphic to either $c_0$ or $l_p$, $1 \leq p < \infty$, if and only if for each permutation $\pi$ of the integers and each block basic sequence $(a_{n_k})_{k=1}^\infty$ of $(a_n)_{n=1}^\infty$ there exists a projection in $X$ whose range is $[a_{n_k}]_{k=1}^\infty$.

The next theorem is a refinement of some of the ideas of Theorem 5.2 and Corollary 5.3.

**Theorem 5.8.** Let $X$ be an $F$-space with a basis $(a_{n_k})_{k=1}^\infty$. If each block basic sequence of $(a_{n_k})_{k=1}^\infty$ has a block extension that is a basis of $X$, then $X$ is locally convex.

**Proof.** Assume that each block basic sequence has a block extension that is a basis of $X$. Let $\mu$ denote the topology of $X$.

Suppose that $X$ is not locally convex. Then there exists a sequence $(a_{n_k})_{k=1}^\infty$, contained in $X$ such that $a_{n_k} \to 0$ in the topology $\mu$, but $(a_{n_k})_{k=1}^\infty$ is regular for the topology $\mu$. It follows that there exists a block basic sequence $(a_{n_k})_{k=1}^\infty$ of $(a_{n_k})_{k=1}^\infty$ such that $a_{n_k} \to 0$ in the topology $\mu$, but $(a_{n_k})_{k=1}^\infty$ is regular for the topology $\mu$.

Let $(g_{n_k})_{k=1}^\infty$ denote a block extension of $(a_{n_k})_{k=1}^\infty$, that is a basis of $X$, with $g_{n_k} = a_k$ for $1 \leq k \leq \infty$. Let $(g_{n_k})_{k=1}^\infty$, denote the sequence of coefficients associated with $(g_{n_k})_{k=1}^\infty$. By an observation of J. H. Shapiro (see p. 129 of [13]), $(g_{n_k})_{k=1}^\infty$ may be considered a basis of the completion of $(X, m(\mu))$. By Proposition 2.1, $(g_{n_k})_{k=1}^\infty$ is equiconvergent for the topology $\mu$; hence $(g_{n_k})_{k=1}^\infty$ is also equiconvergent for the topology $m(\mu)$. But then, according to Proposition 3.1, $(g_{n_k})_{k=1}^\infty$ must be regular for the topology $m(\mu)$. However, this is impossible since $a_{n_k} \to 0$ in the topology $m(\mu)$.

Therefore $X$ must be locally convex.

**6. The weak basis theorem.** Section 6 concludes with an elementary proof of a result concerning weak bases which has been proved by L. Drewnowski [4]. It is well known that each weak basis of a Fréchet space is a basis; this fact was first stated for Banach spaces by Banach and improvements were made by Bessaga and Pełczyński [3] and by McArthur [9]. W. J. Stiles [15] showed that the theorem fails in $l_p$, $0 < p < 1$ and J. H. Shapiro [14] proved that the theorem fails in any $F$-space which has a weak basis and which admits a continuous norm. Finally, L. Drewnowski [4] has shown that the weak basis theorem fails in non-locally convex $F$-space.

**Theorem 6.1.** Let $(X, \mu)$ be an $F$-space with a basis $(a_{n_k})_{k=1}^\infty$. Let $X$ denote the completion of $(X, m(\mu))$, and assume that $X$ is isomorphic to $(e)$. Then the weak basis theorem holds in $X$ if and only if $X$ is isomorphic to $(e)$.

**Proof.** Assume that $X$ is not isomorphic to $(e)$. It may be assumed, without loss of generality, that $\lim x_n = 0$. Since $X$ is isomorphic to $(e)$, $X$ is not locally convex. Therefore there exists a sequence $(y_{n_k})_{k=1}^\infty$ contained in $X$ such that $y_{n_k} \to 0$ in the topology $m(\mu)$, but $(y_{n_k})_{k=1}^\infty$ is regular in the topology $\mu$. 

Now \((x_i, y_i)_{i=1}^m\) may be considered a basis of \(X\). Since each basis of \((e)\) is equivalent to the unit-vectors basis of \((e)\), it may be assumed that the topology of \(X\) is defined by an \(F\)-norm, \(\|\cdot\|\), with the property that

\[
\left\| \sum_{n=1}^m a_n x_n \right\| = \sum_{n=1}^m \frac{|a_n|}{\beta_n + |a_n|}
\]

for each sequence of scalars \((a_n)_{n=1}^\infty\). Since \(\lim_{n\to\infty} f(y_n) = 0\) (i.e., \(1, 2, 3, \ldots\)) and \((y_n)_{n=1}^\infty\) is regular in the topology \(\mu\), it is easy to see that there exists a regular block basis sequence \((x_n)_{n=1}^\infty\), with

\[
x_j = \sum_{n=1}^m a_{nj} x_n \quad m_j = 0, j = 1, 2, 3, \ldots \quad \text{having the property:}
\]

\[
\sup \|x_j\| < 1/2^j \quad (j = 1, 2, 3, \ldots)
\]

It may be assumed, without loss of generality, that \(a_1 = 0\).

Let \(u_n = x_n + e_n \quad (n = 1, 2, 3, \ldots)\); then \((u_n)_{n=1}^\infty\) is a regular sequence in the topology \(\mu\). Suppose that \(\sum_{n=1}^m t_n u_n = 0\) (convergence in \(X\)). Then

\[
t_0 = f_1 \left( \sum_{n=1}^\infty t_n u_n \right) = t_0 + \sum_{n=1}^m a_n t_n = 1. \quad \text{Given any integer } n \text{ with } n > 1, \text{ there is a unique integer } j \text{ with } j < n \text{ such that } a_j \text{ occurs in the basis expansion of } s_j; \text{ thus } 0 = f_1 \left( \sum_{n=1}^\infty t_n u_n \right) = t_0 + \sum_{n=1}^m a_n t_n. \quad \text{Hence, by induction, it is easy to see that } t_n = 0 \quad (n = 1, 2, 3, \ldots). \quad \text{Since } (u_n)_{n=1}^\infty \text{ is a basic sequence in } X \quad \text{and } \sum_{n=1}^\infty |u_n| < \infty. \quad \text{Lemma 4.3 of [9] implies that } (u_n)_{n=1}^\infty \text{ is a basic sequence in } X. \quad \text{Define an infinite matrix } A \text{ in the following way: the } (i, j) \text{ entry is zero unless } i = j \text{ or } m_{i-1} < j < m_i; \text{ the } (i, i) \text{ entry is one; } \text{the } (i, m_{i-1} + k) \text{ entry is } a_{m_{i-1} + k} \text{ for } k = 1, 2, \ldots, m_i - m_{i-1} \quad \text{and } i > 1; \text{ and the } (1, k) \text{ entry is } a_{k} \text{ for } k = 1, 2, \ldots, m. \quad \text{Since } A \text{ has a left inverse and } \sum_{n=1}^\infty b_n u_n \quad \text{converges in } X \text{ for each sequence of scalars } (b_n)_{n=1}^\infty, \text{ each element } x_n \text{ can be represented in } X \text{ in terms of the basic sequence } (u_n)_{n=1}^\infty. \quad \text{Therefore } (u_n)_{n=1}^\infty \text{ is a basis of } X. \quad \text{This implies that } (u_n)_{n=1}^\infty \text{ is a weak basis of } X \text{ [because the coefficient functionals associated with } (u_n)_{n=1}^\infty \text{ are continuous].}

To finish the proof it suffices to show that \((u_n)_{n=1}^\infty\) is not a basis of \(X\). To that end, assume that \((u_n)_{n=1}^\infty\) is a basis of \(X\). Let \((x_n)_{n=1}^\infty\) denote the associated sequence of coefficient functionals. Since \((x_n)_{n=1}^\infty\) is a regular basis, \((y_n)_{n=1}^\infty\) is equicoherent by Proposition 2.3. Then again by Proposition 2.3, \((u_n)_{n=1}^\infty\) is regular in the topology \(m(\mu)\). Since \(u_n \to 0\) in \(m(\mu)\), it must be the case that \((u_n)_{n=1}^\infty\) is not a basis of \(X\).  

References


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