tion $C(j)F = C(j)G$, ces homomorphismes coïncident sur les mesures de Dirac $\{\delta_{j(x)}: x \in X\}$, et donc sur le sous-semigroupe convexe faiblement fermé engendré par cet ensemble, c'est-à-dire sur $P(V(X))$. Il s'ensuit que F' et G' coïncident sur l'enveloppe convexe équilibrée faiblement fermée de $P(V(X))$, et donc sur un voisinage de 0. On en déduit que $F' = G'$, d'où $F = G$.

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On basic sequences in non-locally convex spaces

by

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Abstract. The results in this paper concern linear topological spaces that are not necessarily locally convex and basic sequences in such spaces. The main result is a characterization of a block basic sequence which is a subsequence of a basis. Other sequences generated by a basis are considered.

0. Introduction. This paper deals with basic sequences in linear topological spaces which are not necessarily locally convex. The purpose of this paper is to study several types of sequences generated by a basis. One of these types is the block perturbation introduced by Pełczyński and Singer [10]; a second type is the block basic sequence; and the third type studied has the form $(y_n)_{n=1}^{\infty}$, where $y_n = \sum_{i=1}^{n} x_i$ and $(x_n)_{n=1}^{\infty}$ is a basis.

Section 1 contains the essential definitions and terminology for the remainder of the paper.

Two important properties of basic sequences in F-spaces are contained in Section 2. An elementary proof of the "selection principle" of Bessaga and Pełczyński [2], in the context of an F-space, is given following closely the work of N. J. Kalton [7].

Section 3 contains a characterization of the space c_0.

Block perturbations are used in Section 4 to characterize the locally bounded spaces among F-spaces with a bounded, regular basis.

Section 5 contains results concerning when a block basic sequence is a subsequence of a basis.

The paper is concluded with an elementary proof of a result of L. Drewnowski [4] concerning the weak basis theorem.

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1. Definitions and terminology. Let X denote an arbitrary linear topological space in this section unless otherwise specified. Let (y_n) be a sequence contained in X; then $[y_n]$ denotes the closed linear span of (y_n). Two basic sequences (y_n) and (z_n) are said to be *equivalent* provided that $\sum_{n=1}^{\infty} a_n y_n$ converges if and only if $\sum_{n=1}^{\infty} a_n z_n$ converges. A basic sequence

(y_n) is called *unconditional* if whenever $\sum_{n=1}^{\infty} a_n y_n$ converges and (ε_n) is a sequence with $\varepsilon_n = \pm 1$ $(n = 1, 2, 3, \ldots)$, then $\sum_{n=1}^{\infty} \varepsilon_n a_n y_n$ also converges. A sequence $(y_n)_{n=1}^{\infty}$ is called *regular* provided that there is a neighborhood N of zero such that $y_n \notin N$ $(n = 1, 2, 3, \ldots)$.

Let $\|\cdot\|$ be a non-negative, real-valued function defined on X, let p be in $(0, 1]$ and let K be any positive constant. Consider the following properties:

(1) $\|x+y\| \leqslant \|x\| + \|y\|$,
(2) $\|x+y\| \leqslant K(\|x\| + \|y\|)$,
(3) $\|tx\| = |t|\,\|x\|$,
(4) $\|tx\| = |t|^p \|x\|$,
(5) $\|tx\| \leqslant \|x\|$ if $|t| \leqslant 1$,
(6) $\lim_{t\to 0} \|tx\| = 0$.

If $\|\cdot\|$ satisfies (1) and (3), it is called a *semi-norm*; if $\|\cdot\|$ satisfies (1), (5) and (6), it is called an *F-semi-norm*; if $\|\cdot\|$ satisfies (1) and (4) it is called a *p-semi-norm*; and if $\|\cdot\|$ satisfies (2) and (3) it is called a *quasi-norm*. In the case that, in addition $\|x\| = 0$ implies that $x = 0$, a semi-norm is a *norm*, an F-semi-norm is an *F-norm*, and a p-semi-norm is a *p-norm*.

It is well known that the topology of a metrizable space X may always be defined by an F-norm. A complete linear metric space is called an *F-space*. The phrase "let $(X, \|\cdot\|)$ be an F-space" will mean that X is an F-space whose topology is defined by an F-norm, $\|\cdot\|$.

Let $\overline{N}$ be a neighborhood base at zero for the space X. Let $\overline{N} = \{\mathrm{co}(N) : N \in \overline{N}\}$. Then $\overline{N}$ is a neighborhood base at zero for a linear topology on X; this topology is called the *Mackey topology* of X. If the original topology on X is denoted by μ then the associated Mackey topology of X is denoted by $m(\mu)$ and the two linear topological spaces are designated by (X, μ) and $(X, m(\mu))$. If X is metrizable, then X with the Mackey topology is semi-metrizable, and the Mackey topology is the strongest locally convex topology on X with the same topological dual (see [12]). Thus, for X metrizable, the use here of the term "Mackey topology" is consistent with the use of the term in locally convex space theory. (Cf. page 203 of [5].)

2. Some properties of basic sequences. Proposition 2.1 below is due essentially to N. J. Kalton [6]. (See also [13].)

PROPOSITION 2.1. *Let $(X, \|\cdot\|)$ be an F-space with a basis $(x_n, f_n)_{n=1}^{\infty}$. For any subsequence $(x_{n_j})_{j=1}^{\infty}$, $(x_{n_j})_{n=1}^{\infty}$ is regular if and only if $(f_{n_j})_{j=1}^{\infty}$ is equicontinuous.*

Proof. Assume that (f_{n_j}) is equicontinuous. Then the function g, defined by $g(x) = \sup_j |f_{n_j}(x)|$, is continuous. It follows that $N = \{x \in X : g(x) < 1\}$ is a neighborhood of zero and $x_{n_j} \notin N$ $(j = 1, 2, 3, \ldots)$. Thus (x_{n_j}) is regular.

Suppose now that (f_{n_j}) is not equicontinuous. It may be assumed without loss of generality that $\|f_n(x)x_n\| \leqslant \|x\|$ $(n = 1, 2, 3, \ldots)$. Since (f_{n_j}) is not equicontinuous, there exist $\varepsilon > 0$, a subsequence (m_j) of (n_j), and a sequence (y_j) contained in X such that $\|y_j\| \leqslant 1/j$, but $|f_{m_j}(y_j)| \geqslant \varepsilon$. Choose an integer M such that $\varepsilon \geqslant 1/M$. Then, for $j = 1, 2, 3, \ldots$,

$$\frac{1}{j} \geqslant \|y_j\| \geqslant \|f_{m_j}(y_j)x_{m_j}\| \geqslant \|\varepsilon x_{m_j}\| \geqslant \left\| \frac{1}{M} x_{m_j} \right\| \geqslant \frac{1}{M} \|x_{m_j}\|.$$

Thus (x_{n_j}) is not regular. ∎

The next result is the "selection principle" of Bessaga and Pełczyński, given here in the context of an F-space. The proof, for Banach spaces, was given in [2]; the principle was also stated without proof in [2] for each of the cases of locally bounded F-spaces and locally convex F-spaces. An extension of the principle to the case of locally pseudo-convex F-spaces was made in [13]. The proof below follows rather easily from some results of N. J. Kalton [7]. Note that a sequence (z_n) is called a *block basic sequence* with respect to a basis (x_n) provided that there exists a strictly increasing sequence of positive integers, (m_n), and a sequence of scalars (a_n) such that

$$(*) \qquad z_n = \sum_{k=m_{n-1}+1}^{m_n} a_i x_i \qquad (n = 1, 2, 3, \ldots).$$

PROPOSITION 2.2. *Let X be an F-space with a basis $(x_n, f_n)_{n=1}^{\infty}$. Assume that (y_n) is a sequence satisfying*

(1) $\lim_n f_j(y_n) = 0$ $(j = 1, 2, 3, \ldots)$ *and*

(2) (y_n) *is regular.*

Then there is a subsequence (y_{n_j}) of (y_n), with $y_{n_1} = y_1$, that is a basic sequence equivalent to a block basic sequence of (x_n).

Proof. Let μ denote the original topology of X and let ϱ be the topology generated by the sequence of semi-norms $(\|\cdot\|_n)$, where $\|x\|_n = |f_n(x)|$. Then ϱ is a Hausdorff vector topology on X that is weaker than μ, and $y_n \to 0$ in the ϱ topology. Therefore, by Proposition 2.2 (i) of [7], there exist vector topologies α and β, with $\varrho \leqslant \alpha < \beta \leqslant \mu$ such that β is metrizable and α-polar (i.e., β has a neighborhood base at zero consisting of α-closed sets), $y_n \to 0$ in the α topology, but $y_n \not\to 0$ in the β topology. Then by Theorem 3.2 of [7] there exists a subsequence (y'_{n_j}) of (y_n) such that (y'_{n_j}) is basic for the β topology and $y'_{n_1} = y_1$. Now, using (1) and (2), it is easy to choose a subsequence (y_{n_j}) of (y'_{n_j}), with $y_{n_1} = y_1$, and a block basic sequence (z_j) of (x_n) so that $\|y_{n_j} - z_j\| < 1/2^j$ $(j = 1, 2, 3, \ldots)$. If $\sum_{n=1}^{\infty} t_j y_{n_j} = 0$ in the μ topology, then $\sum_{j=1}^{\infty} t_j y_{n_j} = 0$ in the β topology, and hence $t_j = 0$ $(j = 1, 2, 3, \ldots)$. Thus $y_{n_j} = z_j + (y_{n_j} - z_j)$, (z_j) is a regular basis sequence $\sum_{j=1}^{\infty} \|y_{n_j} - z_j\| < \infty$ and $\sum_{j=1}^{\infty} t_j y_{n_j} = 0$ implies that $t_j = 0$ $(j = 1, 2, 3, \ldots)$. Therefore Lemma 4.3 of [7] implies that (y_{n_j}) is a basis sequence (and it is easy to see that (y_{n_j}) is equivalent to (z_j)). ∎

3. A characterization of the space c_0. Let $(x_n)_{n=1}^{\infty}$ be a basis of an F-space X. Let $y_n = \sum_{i=1}^{n} x_i$ $(n = 1, 2, 3, \ldots)$. In the space c_0, where $(x_n)_{n=1}^{\infty}$ denotes the unit-vectors basis, the sequence $(y_n)_{n=1}^{\infty}$ is a basis. Theorem 3.2 shows that such a condition provides a characterization of c_0.

A space X is called a *locally bounded space* if it contains a bounded neighborhood of zero. Aoki [1] and Rolewicz [11] have shown that an F-space X is a locally bounded space if and only if there is a number p, $0 < p < 1$, such that the topology on X may be defined by a p-norm.

The following lemma is contained, essentially, in Proposition 3.1 of [6].

LEMMA 3.1. *Let $(X, \|\cdot\|)$ be an F-space with a basis $(x_n, f_n)_{n=1}^{\infty}$. Then $(x_n)_{n=1}^{\infty}$ is bounded if and only if there is a bounded set A contained in X and $\varepsilon > 0$ such that $\sup_{a \in A} |f_n(a)| > \varepsilon$ $(n = 1, 2, 3, \ldots)$.*

Proof. Assume that $(x_n)_{n=1}^{\infty}$ is bounded. The $\sup_n |f_n(x_n)| = 1$ $(n = 1, 2, 3, \ldots)$.

Now assume that A is a bounded subset of X such that

$$\sup_{a \in A} |f_n(a)| > \varepsilon \qquad (n = 1, 2, 3, \ldots).$$

It may be assumed without loss of generality that

$$\Big\| \sum_{i=m}^{n} f_i(x) x_i \Big\| \leqslant \|x\|, \qquad x \in X,\ m \leqslant n.$$

Choose a sequence $(a_n)_{n=1}^{\infty}$ contained in A such that $|f_n(a_n)| > \varepsilon$. Suppose that $(t_n)_{n=1}^{\infty}$ is a sequence of scalars and $t_n \to 0$. Then

$$\|t_n f_n(a_n) x_n\| = \|f_n(t_n a_n) x_n\| \leqslant \|t_n a_n\|.$$

Since A is bounded, $t_n f_n(a_n) x_n \to 0$; and since $|f_n(a_n)| > \varepsilon$, this implies that $t_n x_n \to 0$. Therefore $(x_n)_{n=1}^{\infty}$ is bounded. ∎

THEOREM 3.2. *Let X be a locally bounded F-space with a bounded unconditional basis $(x_n, f_n)_{n=1}^{\infty}$. For $n = 1, 2, 3, \ldots$, let $y_n = \sum_{i=1}^{n} x_i$. Then $(y_n)_{n=1}^{\infty}$ is a basis of X if and only if $(x_n)_{n=1}^{\infty}$ is equivalent to the unit-vectors of c_0.*

Proof. (i) The "if" part of the theorem is well-known.

(ii) Assume that $(y_n)_{n=1}^{\infty}$ is a basis of X. It may be assumed without loss of generality that the topology of X is defined by a p-norm, $\|\cdot\|$, for some p $(0 < p \leqslant 1)$. Let $g_n = f_n - f_{n+1}$ $(n = 1, 2, 3, \ldots)$. Then $(g_n)_{n=1}^{\infty}$ is the sequence of coefficient functionals associated with $(y_n)_{n=1}^{\infty}$. Since $(x_n)_{n=1}^{\infty}$ is unconditional and X is locally bounded, it may be assumed without loss of generality that

$$\Big\| \sum_{i=m}^{n} a_i f_i(x) x_i \Big\| \leqslant \|x\|$$

whenever $m \leqslant n$, $|a_i| \leqslant 1$ $(i = 1, 2, \ldots, n)$ and $x \in X$.

Suppose that $(x_n)_{n=1}^{\infty}$ is not regular. Then there is a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that $\|x_{n_j}\| \leqslant 1/2^j$ $(j = 1, 2, 3, \ldots)$. It follows that $\sum_{j=1}^{\infty} x_{n_j}$ converges to some x in X. But

$$\sum_{i=1}^{n_j-1} g_i(x) y_i = \sum_{i=1}^{n_j-1} f_i(x) x_i - f_{n_j}(x) y_{n_j-1};$$

and

$$\|f_{n_j}(x) y_{n_j-1}\| = \|y_{n_j-1}\| \leqslant \|x_1\| \qquad (j = 1, 2, 3, \ldots).$$

Therefore $\sum_{n=1}^{\infty} g_n(x) y_n$ does not converge to x. It thus may be assumed that there exists $\delta > 0$ such that $\inf \|x_n\| \geqslant \delta$.

Let $A = \{x_n\colon n = 1, 2, 3, \ldots\} + \frac{1}{2}\{x_n\colon n = 1, 2, 3, \ldots\}$. Since $(x_n)_{n=1}^{\infty}$ is bounded, A is bounded. Then

$$\sup_{a \in A} |g_n(a)| = \sup_{a \in A} |(f_n - f_{n+1})(a)| \geqslant |(f_n - f_{n+1})(x_n + \tfrac{1}{2} x_{n+1})| = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Therefore Lemma 3.1 implies that $(y_n)_{n=1}^{\infty}$ is bounded. Choose a number M such that $\sup_n \|y_n\| \leqslant M$.

Suppose that $a_1, a_2, \ldots, a_n$ are scalars. Then

$$\begin{aligned} \delta \max_{1 \leqslant i \leqslant n} |a_i|^p &\leqslant \max_{1 \leqslant i \leqslant n} \|x_i\| |a_i|^p = \max_{1 \leqslant i \leqslant n} \|a_i x_i\| \leqslant \Big\| \sum_{i=1}^{n} a_i x_i \Big\| \\ &\leqslant \Big\| \sum_{i=1}^{n} (\max_{1 \leqslant j \leqslant n} |a_j|) x_i \Big\| = (\max_{1 \leqslant j \leqslant n} |a_j|)^p \Big\| \sum_{i=1}^{n} x_i \Big\| \\ &= (\max_{1 \leqslant j \leqslant n} |a_j|^p) \|y_n\| \leqslant M \max_{1 \leqslant j \leqslant n} |a_j|^p. \end{aligned}$$

Therefore $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $(a_n) \in c_0$. ∎

4. Block perturbations. A block perturbation is obtained in the following way: Let $(X, \|\cdot\|)$ be an F-space with a basis $(x_n)_{n=1}^{\infty}$, let (m_j) and (n_j) be strictly increasing sequences of positive integers with $m_{j-1} < n_j < m_j$ $(j = 1, 2, 3, \ldots)$ and let (a_i) be a sequence of scalars. For each j, let

$$u_j = \sum_{i=m_{j-1}+1}^{n_j-1} a_i x_i + \sum_{i=n_j+1}^{m_j} a_i x_i.$$

Then let

$$y_n = \begin{cases} x_n, & \text{if } n \neq n_j \\ x_{n_j} + u_j, & \text{if } n = n_j \end{cases} \quad (n = 1, 2, 3, \ldots).$$

If there is a constant $K > 0$ such that $\sup_j \|u_j\| \leqslant K$, then $(y_n)_{n=1}^{\infty}$ is called a *block perturbation* of $(x_n)_{n=1}^{\infty}$, with $\|\cdot\|$-bounded blocks. (Cf. [10].) In the case that X is locally bounded it will be assumed that the norm, $\|\cdot\|$, above denotes a p-norm defining the topology of X; and hence $(u_n)_{n=1}^{\infty}$ is topologically bounded in this case. Pełczyński and Singer [10] have shown that each block perturbation of a normalized basis of a Banach space X is also a basis of X. Part (a) of the next theorem is an extension of the result of Pełczyński and Singer. The proof of part (a) is omitted since it is the same as that given in [10].

THEOREM 4.1. *Let X be an F-space with a regular, bounded basis $(x_n, f_n)_{n=1}^{\infty}$.*

(a) *If X is a locally bounded space, then each block perturbation of $(x_n)_{n=1}^{\infty}$ is a basis.*

(b) *If X is not a locally bounded space, then for each F-norm, $|\cdot|$, defining the topology of X, there is a block perturbation of $(x_n)_{n=1}^{\infty}$, with $|\cdot|$-bounded blocks, which is not a basis of X.*

Proof. (b) Assume that X is not locally bounded and $|\cdot|$ is an F-norm defining the topology of X. An equivalent F-norm, $\|\cdot\|$, may be defined on X satisfying each of the following properties:

(1) $\|x\| \geqslant |x|$ and

(2) $\|x\| \geqslant \|\sum_{i=m}^{n} f_i(x) x_i\|$, if m and n are positive integers with $m \leqslant n$ $(x \in X)$.

Since the basis $(x_n)_{n=1}^{\infty}$ is regular, it is easy to see by property (2) that there is a constant $\varepsilon > 0$ so that for each $x \in X$ there is a scalar $a = a(x)$ such that

(3) $\|ax\| \geqslant 2\varepsilon$.

Since X is not locally bounded, $\{x\colon \|x\| \leqslant \varepsilon\}$ is not bounded. Thus, using condition (3) and the boundedness of $(x_n)_{n=1}^{\infty}$, there exists a sequence $(z_n)_{n=1}^{\infty}$ contained in X and a sequence $(t_n)_{n=1}^{\infty}$ of scalars such that $\|z_n\| = \varepsilon$, $\inf_j \|t_j z_j\| > 0$, $(t_i)_{i=1}^{\infty}$ is a decreasing sequence, $\lim_i t_i = 0$ and $\|t_n x_K\| < 1/2^n$ $(n = 1, 2, 3, \ldots; K = 1, 2, 3, \ldots)$. Let $\delta = \inf_j \|t_j z_j\|$, let $m_0 = 0$, and let $n_1 = 1$. Choose m_1 such that $\sum_{i=1}^{n-1} \|t_1 f_i(z_1) x_i\| \geqslant \delta/2$. The collection $\{|f_i(z_n)| = i = 1, 2, \ldots, m;\ n = 1, 2, 3, \ldots\}$ is bounded for each m since $\varepsilon = \|z_n\| \geqslant \|f_i(z_n) x_i\|$ for all n and all i and since condition (3) holds. Therefore there is an integer $n_2 > n_1$ satisfying:

$$\Big\|\sum_{i=1}^{n_1} t_{n_2} f_i(z_{n_2}) x_i\Big\| < \delta/4.$$

Also there is an integer $m_2 < m_1$ such that

$$\Big\|\sum_{i=m_1+1}^{m_2-1} t_{n_2} f_i(z_{n_2}) x_i\Big\| \geqslant \delta/2.$$

Using induction, it is easy to see that there exist strictly increasing sequences $(n_j)_{j=1}^{\infty}$ and $(m_j)_{j=1}^{\infty}$ such that

$$\Big\|\sum_{i=m_{j-1}+1}^{m_j-1} t_{n_j} f_i(z_{n_j}) x_i)\Big\| \geqslant \delta/2, \quad j = 1, 2, 3, \ldots$$

Let

$$y_n = \begin{cases} x_n, & \text{if } n \neq m_j, \\ x_{n_j} + u_j, & \text{if } n = m_j, \end{cases}$$

where $u_j = \sum_{i=m_{j-1}+1}^{m_j-1} f_i(z_{n_j}) x_i$. Then

$$|u_j| \leqslant \|u_j\| = \Big\|\sum_{i=m_{j-1}+1}^{m_j-1} f_i(z_{n_j}) x_i\Big\| \leqslant \|z_{n_j}\| = \varepsilon;$$

hence $(y_n)_{n=1}^{\infty}$ is a block perturbation of $(x_n)_{n=1}^{\infty}$ with $|\cdot|$-bounded blocks.

Since $\|t_n x_K\| \leqslant 1/2^n$ $(n, K = 1, 2, 3, \ldots)$, $\sum_{j=1}^{\infty} t_{n_j} x_{m_j}$ converges to some element $x \in X$.

Suppose that $(y_n)_{n=1}^{\infty}$ is a basis of X. Then $x = \sum_{n=1}^{\infty} a_n y_n$ for some sequence $(a_n)_{n=1}^{\infty}$ of scalars. It follows that

$$t_{n_j} = f_{m_j}\Big(\sum_{i=1}^{\infty} t_{n_i} x_{m_i}\Big) = f_{m_j}(x) = f_{m_j}\Big(\sum_{n=1}^{\infty} a_n y_n\Big)$$
$$= \sum_{n=1}^{\infty} a_n f_{m_j}(y_n) = a_{m_j} \quad (j = 1, 2, 3, \ldots).$$

Also, for $m \neq m_j$,

$$0 = f_m\Big(\sum_{i=1}^{\infty} t_{n_i} x_{m_i}\Big) = f_m\Big(\sum_{n=1}^{\infty} a_n y_n\Big) = \sum_{n=1}^{\infty} a_n f_m(y_n)$$
$$= a_m + a_{m_j} f_m(z_{n_j}) = a_m + t_{n_j} f_m(z_{n_j}).$$

Therefore

$$a_{m_j} = t_{n_j} \ (j = 1, 2, 3, \ldots) \quad \text{and} \quad a_m = -t_{n_j} f_m(z_{n_j}) \ (m_{j-1} < m < m_j;\ j = 1, 2, 3, \ldots).$$

Choose N such that for $j \geqslant N$, $\big\|\sum_{i=j}^{\infty} t_{n_i} x_{m_i}\big\| \leqslant \delta/4$. If $j \geqslant N$, then

$$\Big\| x - \sum_{i=1}^{m_j-1} a_i y_i \Big\| \geqslant \Big\| \sum_{i=1}^{j-1} t_{n_i} x_{m_i} - \sum_{i=1}^{m_j-1} a_i y_i \Big\| - \Big\| \sum_{i=j}^{\infty} t_{n_i} x_{m_i} \Big\|$$
$$\geqslant \Big\| - \sum_{i=m_{j-1}+1}^{m_j-1} t_{n_i} f_i(z_{n_i}) x_i \Big\| - \delta/4 \geqslant \delta/2 - \delta/4 = \delta/4.$$

This shows that $\sum_{n=1}^{\infty} a_n y_n$ cannot converge to x. It must then be concluded that $(y_n)_{n=1}^{\infty}$ is not a basis of X. ■

Remark. The space (s) is a non-locally bounded F-space with a bounded basis. It is easy to see, following the proof given by Pełczyński and Singer for Theorem 4.1 part (a), that each block perturbation of a basis of (s) is also a basis of (s). Hence the condition of regularity may not be removed in Theorem 4.1.

5. Block extensions. Let $(z_n)_{n=1}^{\infty}$ be a block basic sequence of the form $(*)$ given in Section 2. If $(y_n)_{n=1}^{\infty}$ is a sequence in X with $y_{n_j} = z_j$ and $y_n \in [x_i]_{i=m_{j-1}+1}^{m_j}$. Whenever $m_{j-1} < n \leqslant m_j$ $(j = 1, 2, 3, \ldots;\ n = 1, 2, 3, \ldots)$, then the sequence $(y_n)_{n=1}^{\infty}$ is called a *block extension* of $(z_n)_{n=1}^{\infty}$. Zippin [16] has proved that each block basic sequence of a basis of a Banach space X has a block extension that is a basis of X. Locally convex spaces are characterized by means of block extensions in this section (see Theorem 5.8).

The proof of the next lemma is the same as that of Zippin in [16].

LEMMA 5.1. *Let F be an n-dimensional Hausdorff linear topological space and assume that the topology on F is defined by a p-norm, $\|\cdot\|$, for some $p \in (0, 1]$. If H and G are any two hyperplanes in F, then there exists an isomorphism $T\colon H \to G$ satisfying*

(i) $\frac{1}{3}\|x\| \leqslant \|Tx\| \leqslant 3\|x\|$ $(x \in H)$ *and*
(ii) $Tx = x$ $(x \in H \cap G)$.

THEOREM 5.2. *Let (X, μ) be a locally bounded F-space whose topology is defined by a p-norm, $\|\cdot\|$. Let $(x_n)_{n=1}^{\infty}$ be a basis of X and let $(z_n)_{n=1}^{\infty}$ be a normalized block basic sequence of the form*

$$(*) \qquad z_j = \sum_{i=n_{j-1}+1}^{n_j} a_i x_i \quad (j = 1, 2, 3, \ldots).$$

Then the following are equivalent:

(1) *The block basis $(z_n)_{n=1}^{\infty}$ has a block extension that is a basis of X.*

(2) *There is a constant K such that for each j there is a continuous linear functional g_j defined on $[x_i]_{i=n_{j-1}+1}^{n_j}$ satisfying*

$$\text{(a)} \quad g_j(z_j) = 1 \quad \textit{and} \quad \text{(b)} \quad \sup_{\|x\| \leqslant 1} |g_j(x)| \leqslant K.$$

(3) *The block basis $(z_n)_{n=1}^{\infty}$ is regular for the topology $m(\mu)$.*

Proof. For $x \in X$, define $\|x\|_1 = \inf\{t > 0\colon x \in tA\}$, where $A = \mathrm{co}(\{x\colon \|x\| \leqslant 1\})$. Then $\|\cdot\|_1$ is a norm and $m(\mu)$ is defined by $\|\cdot\|_1$.

(1) implies (2): Assume that $(z_n)_{n=1}^{\infty}$ has a block extension $(y_n)_{n=1}^{\infty}$ that is a basis of X. Then there is a constant K' such that for positive integers l, m and n, with $l \leqslant n$, and scalars $t_l, t_{l+1}, \ldots, t_{m+n}$

$$\Big\| \sum_{i=l}^{n} t_i y_i \Big\| \leqslant K' \Big\| \sum_{i=l}^{m+n} t_i y_i \Big\|.$$

It follows that $T_j\colon X_j \to X_j$ defined by

$$T_j\Big(\sum_{i=n_{j-1}+1}^{n_j} t_i y_i\Big) = \sum_{i=n_{j-1}+1}^{n_j-1} t_i y_i \quad (\text{where } X_j = [x_i]_{i=n_{j-1}+1}^{n_j})$$

is a continuous projection of norm less than or equal to K' and $I - T_j$ is a continuous projection of X_j onto $[z_j]$ of norm less than or equal to $K'+1$ $(j = 1, 2, 3, \ldots)$. Let $K = K'+1$ and define a functional g_j on X_j by $g_j\Big(\sum_{i=n_{j-1}+1}^{n_j} t_i y_i\Big) = t_{n_j}$. Then $g_j(z_j) = 1$ and

$$(I - T_j)\Big(\sum_{i=n_{j-1}+1}^{n_j} t_i y_i\Big) = t_{n_j} z_j = g_j\Big(\sum_{i=n_{j-1}+1}^{n_j} t_i y_i\Big) z_j.$$

If $\Big\|\sum_{i=n_{j-1}+1}^{n_j} t_i y_i\Big\| \leqslant 1$, then $\Big\| g_j\Big(\sum_{i=n_{j-1}+1}^{n_j} t_i y_i\Big) z_j \Big\| \leqslant K$.

Since $\|z_j\| = 1$ $(j = 1, 2, \ldots)$,

$$\sup\{|g_j(x)|\colon x \in X_j,\ \|x\| \leqslant 1\} \leqslant K^{1/p}.$$

(2) implies (1): The proof that conditions (a) and (b) imply (1) has been given essentially in [16]. It is therefore omitted here.

(2) implies (3): Assume that condition (2) holds, and suppose that $(z_n)_{n=1}^{\infty}$ is not regular for the topology $m(\mu)$. Then $\|z_{n_j}\|_1 \to 0$ for some subsequence (n_j). Since $\sup_{\|x\|\leqslant 1} |g_j(x)| \leqslant K$ implies that $\sup_{\|x\|_1\leqslant 1} |g_j(x)| \leqslant K$ $(j = 1, 2, 3, \ldots)$, $(g_n)_{n=1}^{\infty}$ is an equicontinuous family for the topology $m(\mu)$. Hence $\lim_j g_{n_j}(z_{n_j}) = 0$. But $g_{n_j}(z_{n_j}) = 1$ for each j, so $(z_n)_{n=1}^{\infty}$ must be regular for $m(\mu)$.

(3) implies (2): Now assume that there is a constant $\delta > 0$ such that $\|z_j\|_1 \geqslant \delta$ $(j = 1, 2, 3, \ldots)$. For each j there is a continuous linear functional g_j defined on X_j such that

$$g_j(z_j) = 1 \quad \text{and} \quad \sup_{\|x\|\leqslant 1} |g_j(x)| = \frac{1}{\|z_j\|_1}.$$

This implies that $\sup_{\|x\|\leqslant 1} |g_j(x)| \leqslant 1/\delta$ $(j = 1, 2, 3, \ldots)$. Therefore condition (2) holds. ∎

COROLLARY 5.3. *Let X be a locally bounded F-space with a basis $(x_n)_{n=1}^{\infty}$. Then each block basic sequence of $(x_n)_{n=1}^{\infty}$ has a block extension that is a basis of X if and only if X is a Banach space.*

Proof. If X is a Banach space then the previously mentioned result of Zippin in [16] states that each block basic sequence of $(x_n)_{n=1}^{\infty}$ has a block extension that is a basis of X.

Conversely, if X is not a Banach space, then X is not locally convex. Let μ denote the topology of X. Then there is a sequence $(y_n)_{n=1}^{\infty}$ contained in X such that $y_n \to 0$ in the topology $m(\mu)$, but $y_n \not\to 0$ in the topology μ.

Let $\|\cdot\|$ be a p-norm defining μ, and let $\|\cdot\|_1$ be defined by

$$\|x\|_1 = \inf\{t > 0 \colon x \in tA\}, \quad \text{where} \quad A = \mathrm{co}(\{x \colon \|x\| \leqslant 1\}).$$

It may be assumed without loss of generality that there is a constant $\delta > 0$ such that $\|y_n\| \geqslant \delta$ $(n = 1, 2, 3, \ldots)$. Since $y_n \to 0$ in the weak topology also, it is easy to see that there is a subsequence $(y_{n_j})_{j=1}^{\infty}$ and a block basic sequence $(z_n)_{n=1}^{\infty}$ such that $\|y_{n_j} - z_j\| \leqslant 1/j$ $(j = 1, 2, 3, \ldots)$. It follows that $\|z_j\| \geqslant \delta'$ for some constant $\delta' > 0$ and that $\|z_j\|_1 \leqslant \|y_{n_j}\|_1 + \|z_j - y_{n_j}\|_1 \leqslant \|y_{n_j}\|_1 + \|z_j - y_{n_j}\|$ $(j = 1, 2, 3, \ldots)$. But $\lim_j (\|y_{n_j}\|_1 + \|z_j - y_{n_j}\|) = 0$, so $\lim_j \|z_j\|_1 = 0$. Since for each j $\|z_j\| \geqslant \delta'$, it follows that $\lim_j \left\| \frac{z_j}{\|z_j\|} \right\|_1 = 0$. Therefore by Theorem 5.2 $(z_n/\|z_n\|)_{n=1}^{\infty}$ (and consequently $(z_n)_{n=1}^{\infty}$) has no block extension that is a basis of X. ∎

COROLLARY 5.4. *Let $(z_n)_{n=1}^{\infty}$ be a block basic sequence of the unit-vectors basis, $(e_n)_{n=1}^{\infty}$, in l_p $(0 < p < 1)$. Assume that $z_j = \sum_{i=n_{j-1}+1}^{n_j} = a_i e_i$ $(j = 1, 2, 3, \ldots)$ and that $(z_n)_{n=1}^{\infty}$ is normalized with respect to $\|\cdot\|$, the usual p-norm of l_p.*

(a) *The sequence $(z_n)_{n=1}^{\infty}$ has a block extension that is a basis of l_p if and only if $(z_n)_{n=1}^{\infty}$ is regular in the l_1-topology restricted to l_p.*

(b) *If $\inf_j \max_{n_{j-1}<i\leqslant n_j} |a_i| > 0$, then $(z_n)_{n=1}^{\infty}$ has a block extension that is a basis of l_p.*

Proof. (a) This is immediate from Theorem 5.2 since the associated Mackey topology of l_p is the l_1-topology restricted to l_p.

(b) Choose m_j such that $|a_{m_j}| = \max_{n_{j-1}<i\leqslant n_j} |a_i|$; let

$$F_j = \{n_{j-1}+1, n_{j-1}+2, \ldots, n_j\} \quad \text{and} \quad G_j = F_j/\{m_j\} \quad (j = 1, 2, 3, \ldots).$$

For each j also define a linear functional g_j on $[e_j]_{i\in F_j}$ by $g_j \sum_{i\in G_j} (t_i e_i + t z_j) = t$, where $\sum_{i\in G_j} t_i e_i + t z_j$ is the unique representation of a vector $x \in [e_i]_{i\in F_j}$ in terms of the basis $\{e_i \colon i \in G_j\} \cup \{z_j\}$. If $\left\| \sum_{i\in G_j} t_i e_i + t z_j \right\| \leqslant 1$, then $|t a_{m_j}|^p \leqslant 1$. Hence if $\|x\| \leqslant 1$ for $x \in [e_i]_{i\in F_j}$, $|g_j(x)| \leqslant 1/|a_{m_j}| \leqslant 1/\delta$. Therefore, by part (2) of Theorem 5.2, $(z_n)_{n=1}^{\infty}$ has a block extension that is a basis of l_p. ∎

COROLLARY 5.5. *Let X be a locally bounded F-space with a basis $(x_n, f_n)_{n=1}^{\infty}$. Assume that $(z_n)_{n=1}^{\infty}$ is a block basis of the form $(*)$ given in Theorem 3.5. If $\sup_j (n_j - n_{j-1}) < \infty$, then $(z_n)_{n=1}^{\infty}$ has a block extension that is a basis.*

Proof. It may be assumed without loss of generality that $\left\| \sum_{i=1}^{m} t_i x_i \right\| \leqslant \left\| \sum_{i=1}^{n} t_i x_i \right\|$ for $m \leqslant n$, and that $\|z_n\| = 1$ $(n = 1, 2, 3, \ldots)$. Choose m_j so that $n_{j-1} < m_j \leqslant n_j$ and $\|a_{m_j} x_{m_j}\| = \max\{\|a_i x_i\| \colon n_{j-1} < i \leqslant n_j\}$. Let $F_j = \{n_{j-1}+1, n_{j-1}+2, \ldots, n_j\}$ and $G_j = F_j/\{m_j\}$.

Suppose that $\sup_j (n_j - n_{j-1}) = K < \infty$. Let $g_j = \frac{1}{a_{m_j}} f_{m_j}$. Then if $\left\| \sum_{i\in G_j} t_i x_i + t z_j \right\| \leqslant 1$, $\|t a_{m_j} x_{m_j}\| \leqslant 1$. But

$$1 = \|z_j\| = \left\| \sum_{i=n_{j-1}+1}^{n_j} a_i x_i \right\| \leqslant (n_j - n_{j-1}) \|a_{m_j} x_{m_j}\|.$$

Thus

$$\left| g_j \sum_{i\in G_j} t_i x_i + t z_j \right| = |t| \leqslant \frac{1}{\|a_{m_j} x_{m_j}\|^{1/p}} - (n_j - n_{j-1})^{1/p} \leqslant K^{1/p}.$$

Theorem 5.2 then implies that $(z_n)_{n=1}^{\infty}$ has a block extension that is a basis of X. ∎

Lindenstrauss and Tzafriri have considered projections in Banach spaces onto subspaces spanned by block basic sequences. In [8] they proved that a Banach space X with an unconditional basis $(x_n)_{n=1}^{\infty}$ is isomorphic to either c_0 or l_p, $1 \leqslant p < \infty$, if and only if for each permutation π of the integers and each block basic sequence $(z_n)_{n=1}^{\infty}$ of $(x_{\pi(n)})_{n=1}^{\infty}$ there exists a projection in X onto the block subspace $[z_n]_{n=1}^{\infty}$. Corollary 5.7 is an extension of this result to the case of a locally bounded F-space.

THEOREM 5.6. *Let X be a locally bounded F-space with a basis $(x_n)_{n=1}^{\infty}$. Assume that for each block basic sequence $(z_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$, $[z_n]_{n=1}^{\infty}$ is complemented in X. Then X must be locally convex, and hence normed.*

Proof. Let $\|\cdot\|$ be a p-norm defining the topology of X. By Corollary 5.3 it suffices to show that each normalized block basic sequence of $(x_n)_{n=1}^{\infty}$ has a block extension that is a basis of X.

Let $(z_n)_{n=1}^{\infty}$ be a normalized block basic sequence of $(x_n)_{n=1}^{\infty}$, say $z_j = \sum_{i=n_{j-1}+1}^{n_j} a_i x_i$ $(j = 1, 2, 3, \ldots)$. By Theorem 5.2 it suffices to show that there exist a constant $K > 0$ and a sequence of continuous linear functionals $(g_n)_{n=1}^{\infty}$ such that

(a) g_j is defined on $[x_i]_{i=n_{j-1}+1}^{n_j}$,

(b) $g_j(z_j) = 1$, and

(c) $\sup_{\|x\|\leqslant 1} |g_j(x)| \leqslant K$ $(j = 1, 2, 3, \ldots)$.

By renorming the subspace $[z_n]_{n=1}^{\infty}$ it may be assumed without loss of generality that $\left\|\sum_{j=1}^{\infty} b_j z_j\right\| \geqslant \|b_i z_i\|$ for each i, whenever $\sum_{j=1}^{\infty} b_j z_j$ converges in $[z_n]_{n=1}^{\infty}$.

Since $[z_n]_{n=1}^{\infty}$ is complemented in X, there is a projection P of X onto $[z_n]_{n=1}^{\infty}$. Thus, by renorming X, it may be assumed without loss of generality that $\|x\| = \|Px\| + \|(I-P)x\|$ $(x \in X)$. For each j, define a linear function g_j by the following: $g_j(x) = t_j$ if $Px = \sum_{i=1}^{\infty} t_i z_i$. Then $g_j(z_j) = 1$ $(j = 1, 2, 3, \ldots)$. Also if $x = \sum_{i=n_{j-1}+1}^{n_j} b_i x_i$ and $\|x\| \leqslant 1$, let $Px = \sum_{i=1}^{\infty} t_i z_i$; then

$$|g_j(x)|^p = |t_j|^p = \|t_j z_j\| \leqslant \left\|\sum_{i=1}^{\infty} t_i z_i\right\| = \|Px\| \leqslant \|P\|.$$

Therefore conditions (a), (b) and (c) are satisfied. ∎

The next corollary is an immediate consequence of Theorem 3.11 and the result of Lindenstrauss and Tzafriri mentioned in the paragraph following Corollary 5.5.

COROLLARY 5.7. *A locally bounded F-space X with an unconditional basis $(x_n)_{n=1}^{\infty}$ is isomorphic to either c_0 or l_p $(1 \leqslant p < \infty)$ if and only if for each permutation π of the integers and each block basis $(z_n)_{n=1}^{\infty}$ of $(x_{\pi(n)})_{n=1}^{\infty}$ there exists a projection in X whose range is $[z_n]_{n=1}^{\infty}$.*

The next theorem is a refinement of some of the ideas of Theorem 5.2 and Corollary 5.3.

THEOREM 5.8. *Let X be an F-space with a basis $(x_n)_{n=1}^{\infty}$. If each block basic sequence of $(x_n)_{n=1}^{\infty}$ has a block extension that is a basis of X, then X is locally convex.*

Proof. Assume that each block basic sequence has a block extension that is a basis of X. Let μ denote the topology of X.

Suppose that X is not locally convex. Then there exists a sequence $(u_n)_{n=1}^{\infty}$ contained in X such that $u_n \to 0$ in the topology $m(\mu)$, but $(u_n)_{n=1}^{\infty}$ is regular for the topology μ. It follows that there exists a block basic sequence $(z_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that $z_n \to 0$ in the topology $m(\mu)$, but $(z_n)_{n=1}^{\infty}$ is regular for the topology μ.

Let $(y_n)_{n=1}^{\infty}$ denote a block extension of $(z_n)_{n=1}^{\infty}$ that is a basis of X, with $y_{n_j} = z_j$ $(j = 1, 2, 3, \ldots)$. Let $(g_n)_{n=1}^{\infty}$ denote the sequence of coefficient functionals associated with $(y_n)_{n=1}^{\infty}$. By an observation of J. H. Shapiro (see p. 1296 of [14]), $(y_n, g_n)_{n=1}^{\infty}$ may be considered a basis of the completion of $(X, m(\mu))$. By Proposition 2.1, $(g_{n_j})_{n=1}^{\infty}$ is equicontinuous for the topology μ; hence $(g_{n_j})_{n=1}^{\infty}$ is also equicontinuous for the topology $m(\mu)$. But then, according to Proposition 2.1, $(y_{n_j})_{n=1}^{\infty}$ must be regular for the topology $m(\mu)$. However, this is impossible since $z_n \to 0$ in the topology $m(\mu)$.

Therefore X must be locally convex. ∎

6. The weak basis theorem. Section 6 concludes with an elementary proof of a result concerning weak bases which has been proved by L. Drewnowski [4]. It is well known that each weak basis of a Fréchet space is a basis; this fact was first stated for Banach spaces by Banach and improvements were made by Bessaga and Pełczyński [3] and by McArthur [9]. W. J. Stiles [15] showed that the theorem fails in l_p $(0 < p < 1)$ and J. H. Shapiro [14] proved that the theorem fails in any F-space which has a weak basis and which admits a continuous norm. Finally, L. Drewnowski [4] has shown that the weak basis theorem fails in each non-locally convex F-space.

THEOREM 6.1. *Let (X, μ) be an F-space with a basis $(x_n, f_n)_{n=1}^{\infty}$. Let $\hat{X}$ denote the completion of $(X, m(\mu))$, and assume that $\hat{X}$ is isomorphic to (s). Then the weak basis theorem holds in X if and only if X is isomorphic to (s).*

Proof. Assume that X is not isomorphic to (s). It may be assumed, without loss of generality, that $\lim_n x_n = 0$. Since $\hat{X}$ is isomorphic to (s), X is not locally convex. Therefore there exists a sequence $(y_n)_{n=1}^{\infty}$ contained in X such that $y_n \to 0$ in the topology $m(\mu)$, but $(y_n)_{n=1}^{\infty}$ is regular in the topology μ.

Now $(x_n, f_n)_{n=1}^{\infty}$ may be considered a basis of $\hat{X}$. Since each basis of (s) is equivalent to the unit-vectors basis of (s), it may be assumed that the topology of $\hat{X}$ is defined by an F-norm, $\|\cdot\|$, with the property that

$$\Big\|\sum_{n=1}^{\infty} a_n x_n\Big\| = \sum_{n=1}^{\infty} \frac{1}{2^n}\frac{|a_n|}{1+|a_n|}$$

for each sequence of scalars $(a_n)_{n=1}^{\infty}$.

Since $\lim_{n\to\infty} f_i(y_n) = 0$ $(i = 1, 2, 3, \ldots)$ and $(y_n)_{n=1}^{\infty}$ is regular in the topology μ, it is easy to see that there exists a regular block basic sequence $(z_n)_{n=1}^{\infty}$, with $z_j = \sum_{i=m_{j-1}+1}^{m_j} a_i x_i$; $m_0 = 0$, $j = 1, 2, 3, \ldots$, having the property: $\sup_t \|tz_j\| \leqslant 1/2^j$ $(j = 1, 2, 3, \ldots)$. It may be assumed, without loss of generality, that $a_1 = 0$.

Let $u_n = x_n + z_n$ $(n = 1, 2, 3, \ldots)$; then $(u_n)_{n=1}^{\infty}$ is a regular sequence in the topology μ. Suppose that $\sum_{n=1}^{\infty} t_n u_n = 0$ (convergence in $\hat{X}$). Then $0 = f_1\big(\sum_{n=1}^{\infty} t_n u_n\big) = t_1 + t_1 a_1 = t_1$. Given any integer n, with $n > 1$, there is a unique integer j, with $j < n$, such that x_n occurs in the basis expansion of z_j; thus $0 = f_n\big(\sum_{i=1}^{\infty} t_i u_i\big) = t_n + t_j a_n$. Hence, by induction, it is easy to see that $t_n = 0$ $(n = 1, 2, 3, \ldots)$. Since $(x_n)_{n=1}^{\infty}$ is a basic sequence in $\hat{X}$ and $\sum_{n=1}^{\infty} \|z_n\| < \infty$, Lemma 4.3 of [9] implies that $(u_n)_{n=1}^{\infty}$ is a basic sequence in $\hat{X}$.

Define an infinite matrix A in the following way: the (i, j) entry is zero unless $i = j$ or $m_{i-1} < j \leqslant m_i$; the (i, i) entry is one; the $(i, m_{i-1}+k)$ entry is $a_{m_{i-1}+k}$ for $k = 1, 2, \ldots, m_i - m_{i-1}$ and $i > 1$; and the $(1, k)$ entry is a_k for $k = 1, 2, \ldots, m$. Since A has a left inverse and $\sum_{n=1}^{\infty} b_n u_n$ converges in $\hat{X}$ for each sequence of scalars $(b_n)_{n=1}^{\infty}$, each element x_n can be represented in $\hat{X}$ in terms of the basic sequence $(u_n)_{n=1}^{\infty}$. Therefore $(u_n)_{n=1}^{\infty}$ is a basis of $\hat{X}$. This implies that $(u_n)_{n=1}^{\infty}$ is a weak basis of X (because the coefficient functionals associated with $(u_n)_{n=1}^{\infty}$ are continuous).

To finish the proof it suffices to show that $(u_n)_{n=1}^{\infty}$ is not a basis of X. To that end, assume that $(u_n)_{n=1}^{\infty}$ is a basis of X. Let $(g_n)_{n=1}^{\infty}$ denote the associated sequence of coefficient functionals. Since $(u_n)_{n=1}^{\infty}$ is a regular basis, $(g_n)_{n=1}^{\infty}$ is equicontinuous by Proposition 2.2. Therefore $(g_n)_{n=1}^{\infty}$ is also equicontinuous for the topology $m(\mu)$. But then, again by Proposition 2.2, $(u_n)_{n=1}^{\infty}$ is regular in the topology $m(\mu)$. Since $u_n \to 0$ in $m(\mu)$, it must be the case that $(u_n)_{n=1}^{\infty}$ is not a basis of X. ∎

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