which is positive if $\varepsilon > 0$ is sufficiently small. Thus $u^{-a-\varepsilon}\Phi(u)$ is non-decreasing for some small $\varepsilon > 0$. Similarly, form (4.1), (4.4) we have $\frac{d}{du}\big(u^{-b+\varepsilon}\Phi(u)\big) < 0$ if $\varepsilon > 0$ is small. Hence $\Phi \sim [a+\varepsilon, b-\varepsilon]$ for sufficiently small $\varepsilon > 0$. This completes the proof of Theorem 2 (i). We omit the proof of Theorem 2 (ii) as the argument is similar to that of part (i), except that we now put $h(u) = \int_1^u t^{-a-1}\varphi(t)\,dt$ for $u \geqslant u_0 \geqslant 1$.

**5. Proofs of Theorems 3, 4 and 5.** It is obvious that if $\Phi_i \in Y\langle a_i, b_i\rangle$ ($a_i \geqslant 0$, $i = 1, 2$), then $\Phi_1\Phi_2 \in Y\langle a_1+a_2, b_1+b_2\rangle$. Since

$$\frac{\Phi_2\circ\Phi_1(u)}{u^{p_1p_2}} = \frac{\Phi_2\big(\Phi_1(u)\big)}{\big(\Phi_1(u)\big)^{p_2}}\left(\frac{\Phi_1(u)}{u^{p_1}}\right)^{p_2} \qquad (p_1, p_2 \geqslant 0),$$

we have $\Phi_2\circ\Phi_1 \in Y\langle a_1a_2, b_1b_2\rangle$. Theorems 3 and 4 follow readily from Theorem 2 and Lemmas 1, 2, 3.

We come now to prove Theorem 5. Since $\varphi_i \in M(a_i, b_i)$ ($i = 1, 2$), by Theorem 2 there exist $\Phi_i \in Y\langle a_i, b_i\rangle$ ($i = 1, 2$) in $[u_0, \infty)$ such that

$$K_1\Phi_i(u) \leqslant \varphi_i(u) \leqslant K_2\Phi_i(u) \qquad (i = 1, 2) \tag{5.1}$$

for $u \geqslant u_0$. Define $\Phi$ by

$$\Phi(u) = \begin{cases} \Phi_1(u) & \text{if} \quad u \geqslant u_0,\\ \Phi_1(u_0)\Phi_2(u_0)/\Phi_2(u_0^2/u) & \text{if} \quad 0 < u < u_0,\\ 0 & \text{if} \quad u = 0.\end{cases}$$

It is clear that $\Phi \in Y\langle a, b\rangle$. By Lemma 1, Lemma 3 and (5.1), we obtain Theorem 5 immediately.

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# Weak-type multipliers*

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**Abstract.** In this work, we show, for multiplier operators, that weak-type $(p, p)$ for one fixed $p$, $1 < p < 2$, implies type (2,2). We obtain the same result for any fixed $p > 0$ with the a priori condition $m \in L^1_{\mathrm{loc}}$. We also determine a best constant. The last part of the paper is a related counterexample.

**1. Introduction.** Consider a Fourier multiplier operator $Tf(x) = \mathscr{F}^{-1}(m\hat{f})(x)$ on $\boldsymbol{R}^n$. If $T$ is weak-type $(p, p)$ for fixed $p$, $1 < p < 2$, then a duality interpolation argument implies $T$ is strong-type (2,2) hence $m$ is bounded (see [3]). This argument breaks down if the operator is assumed to satisfy the weak-type estimate only for a small subset $S$ of $L^p$.

In this paper we develop a new direct method for studying these questions. In particular we show that if $m \in L^1_{\mathrm{loc}}$ and $T$ is weak-type $(p, p)$ for any $p > 0$, then $m$ is bounded (if $1 < p < 2$, we need not assume $m \in L^1_{\mathrm{loc}}$). We also find appropriate subsets $S$ of $L^p$, depending on $p$ where we assume $T$ is weak-type $(p, p)$, for which the same result holds. We go on to obtain the best constant for the bound on $m$ in terms of the weak-type constant. We also construct a counterexample to show that the result is false if $S$ consists of all characteristic functions of intervals, extending the work of Ash (see [1] and [2]). The counterexample is particularly interesting for the following reason. Stein and Weiss ([3]) show that if one assumes $T$ is weak-type $(p, p)$ against functions which are characteristic functions of measurable sets, then $T$ extends to be weak-type $(p, p)$ on all of $L^p$. The counterexample demonstrates that characteristic functions of intervals is too small a class to obtain the extension.

**2. Positive results.** Notation:

$$\hat{\mathscr{F}}_n = \left\{f:\ \hat{f}(x) = \prod_{j=1}^{n}\left(\frac{1}{\delta^n}\right)\chi_{(-\delta,\delta)}(x_j + k_j\delta),\ x \in \boldsymbol{R}^n, \quad k_j\text{'s integers},\ \delta > 0\right\}.$$

$$\mathscr{F}_n = \left\{f:\ f(x) = \prod_{j=1}^{n}\left(\frac{1}{\delta^n}\right)\chi_{(-\delta,\delta)}(x_j)\,e^{2\pi i x_j k_j/\delta}, \quad x \in \boldsymbol{R}^n,\ k_j\text{'s integers},\ \delta > 0\right\},$$

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Comment: In part (a) of the following theorem, we do not asusme $T$ commutes with translations, rather that it satisfies a certain convolution property. This is necessary since translations of functions in the class $\hat{\mathcal{F}}_n$ do not keep you in $\hat{\mathcal{F}}_n$. However it is easy to show that a weak-type $(p, p)$ operator on $L^p(\mathbf{R}^n)$, for $1 < p < 2$, commutes with translations if and only if it satisfies the convolution property by certain functional analysis considerations. In part (b) of the theorem we deal with $0 < p < \infty$. Because the Fourier transform of an $L^p$ function with $p > 2$ and $p < 1$ is not generally well-defined, certain a priori assumptions on the multiplier will be made.

THEOREM 1. (a) *Let $T$ be an operator satisfying $Tf*g = f*Tg$ for $f \in \hat{\mathcal{F}}_n$ (the convolution property), and $\lambda_{Tf}(\alpha) \leqslant c^p \|f\|_p^p/\alpha^p$ for $f \in \hat{\mathcal{F}}_n$, $1 < p < 2$. Then $T$ is type (2,2) on all of $L^2$.*

(b) *Let $k$ be a non-negative integer. Assume $m \in L^1_{\mathrm{loc}}$. Then if $Tg_k(x) = \mathcal{F}^{-1}(m\hat{g}_k)(x)$ is weak-type $(p, p)$ with constant $c$ for fixed $p$, $0 < p < \infty$, against all functions $g_k$ such that $\hat{g}_k$ is a dilation and translation of functions of the form $(1-x^2)^k \chi_{(-1,1)}(x)$ with $p > 1/(k+1)$, then $T$ is type (2,2) on all of $L^2$.*

Observe that as $p \to 0$ more and more smoothness on $\hat{g}_k$ is demanded since $\|g_k\|_p \to \infty$ as $p \to 1/(k+1)$. Also note that for $k = 0$ (i.e. $p > 1$) part (b) is contained in part (a) of the theorem.

(c) *Suppose $T$ satisfies the conditions of part (a) or (b) of this theorem. If $T$ is furthermore weak-type $(p, p)$ with constant $c$ on all of $L^p(\mathbf{R}^n)$, $p > 0$, then $\|m\|_\infty \leqslant c$. This constant is best possible.*

Before proving this theorem, we establish a lemma:

LEMMA 1. *If the assumptions of Theorem 1(a) are satisfied, then*

$$\widehat{Tf}(x) = m(x)\hat{f}(x) \quad \text{with } m \in L^1_{\mathrm{loc}}(\mathbf{R}^n).$$

Proof of Lemma 1. Note that $\|f\|_p < \infty$, $1 < p < 2$, whenever $f \in \hat{\mathcal{F}}_n$. It is well-known that weak $L^p \subseteq L^{p-b} + L^{p+b}$ with $1 < p \pm b < 2$. Since $Tf \in$ weak $L^p$, then by the Hausdorff–Young inequality $\widehat{Tf} \in L^{(p-b)'} + L^{(p+b)'}$ where $2 < (p \pm b)' < \infty$. It is easily seen that

$$\widehat{Tf*g}(x) = \widehat{Tf}(x)\hat{g}(x).$$

Since $(Tf*g)(x) = (f*Tg)(x)$, then

$$\widehat{Tf}(x)\hat{g}(x) = \hat{f}(x)\widehat{Tg}(x).$$

We then have

$$\frac{\widehat{Tf}(x)}{\hat{f}(x)} = \frac{\widehat{Tg}(x)}{\hat{g}(x)} \quad \text{on} \quad \operatorname{supp}\hat{g} \cap \operatorname{supp}\hat{f}.$$



Define $m(x) = \widehat{Tf}(x)/\hat{f}(x)$ on $\operatorname{supp}\hat{f}$. The computations above show that $m$ is well-defined and $m$ will be defined almost everywhere by allowing $\operatorname{supp} f$ to get large.

Thus $\widehat{Tf}(x) = m(x)\hat{f}(x)$ for $f \in \hat{\mathcal{F}}_n$. For any bounded measurable set in $\mathbf{R}^n$ choose $\hat{f}$ such that $\operatorname{supp}\hat{f} \supseteq$ the bounded set. Then $\|m\chi_{\operatorname{supp}\hat{f}}\|_1 < \infty$ since $\operatorname{supp}\hat{f}$ is compact and $Tf \in L^{(p+b)'} + L^{(p-b)'}$. Hence $m \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. ∎

Proof of Theorem 1. First, we prove part (a). Without loss of generality we may assume $c = 1$. First consider the case $m(x) \geqslant 0$ a.e. and $n = 1$. If $\|m\|_\infty = 0$, then we are done. So assume $\|m\|_\infty > 0$. Fix $k > 0$. Fix $\varepsilon$,

$$0 < \varepsilon < k/6. \tag{1}$$

Consider any interval $I$ such that $|E| > 0$ where $E = \{x \in \mathbf{R} \cap I\colon k - \varepsilon \leqslant m(x) \leqslant k\}$. Fix I. $m \in L^1_{\mathrm{loc}}$ by Lemma 1, so by Lebesgue's theorem for differentation, there exists $x_0 \in E$ such that

$$\frac{1}{(2\delta)} \int_{-\delta}^{\delta} m(x_0 + x)\,dx \to m(x_0) \quad \text{as} \quad \delta \to 0. \tag{2}$$

$$k\hat{f}(x) = m(x)\hat{f}(x) + \big(k - m(x)\big)\hat{f}(x).$$

By the Fourier inversion theorem (valid to apply by Lemma 1) one obtains

$$kf(x) = Tf(x) + \mathcal{F}[(k-m)\hat{f}](-x)$$

where $\mathcal{F}$ denotes "Fourier transform". Denote $E_f(x) \equiv \mathcal{F}[(k-m)\hat{f}](-x)$. Just for purposes of clarity, we will carry out a proof which does not yield the best constant the method can produce. At the end, we will describe the modifications necessary to obtain the stated constant.

$$\lambda_{kf}(\alpha) \leqslant \lambda_{Tf}(\alpha/2) + \lambda_{E_f}(\alpha/2). \tag{3}$$

By the hypotheses of the theorem

$$\lambda_{kf}(\alpha) \leqslant 2^p \|f\|_p^p/\alpha^p + \lambda_{E_f}(\alpha/2). \tag{4}$$

Without loss of generality $x_0 = 0$. Choose $\hat{f}(x) = (1/\delta)\chi_{(-\delta,\delta)}(x)$. Then $f(x) = (\sin 2\pi\delta x)/(\pi\delta x)$. We obtain

$$\|E_f\|_\infty = \|\mathcal{F}[(k-m)\hat{f}](-x)\|_\infty \leqslant \|(k-m)\hat{f}\|_1 \tag{5}$$

$$\leqslant 2\cdot\left(\frac{1}{2\delta}\right)\int_{-\delta}^{\delta} |k - m(x)|\,dx \leqslant 2\cdot\frac{3}{2}|k - m(0)| \leqslant 3\varepsilon$$

for $\delta$ small enough by (1) and (2).

Choose $\alpha = k$. Then

(6) $$\lambda_{Ef}(k/2) = |\{x: |E_f(x)| > k/2\}| = 0$$

by (1) and (5).

By (4) and (6) we have

(7) $$\lambda_{kf}(k/2) \leqslant 2^p \|f\|_p^p / k^p.$$

(8) $$\|f\|_p^p = \int_{-\infty}^{\infty} \left|\frac{\sin 2\pi\,\delta x}{(\pi\delta x)}\right|^p dx = \left(\frac{1}{\delta}\right) \int_{-\infty}^{\infty} \left|\frac{\sin 2\pi x}{(\pi x)}\right|^p dx = \left(\frac{1}{\delta}\right) A_1$$

where $A_1 = A_1(p)$ is finite and uniquely determined by $p$.

(9) $$\lambda_{kf}(k/2) = |\{x: |kf(x)| > k/2\}| = |\{x: |f(x)| > 1/2\}|$$
$$= \left|\left\{x: \left|\frac{\sin 2\pi\,\delta x}{(\pi\,\delta x)}\right| > \frac{1}{2}\right\}\right| = \frac{1}{\delta}\left|\left\{x: \left|\frac{\sin 2\pi x}{(2\pi x)}\right| > \frac{1}{4}\right\}\right| = \left(\frac{1}{\delta}\right) A_2$$

where $A_2$ is finite and uniquely determined. Putting (8) and (9) into (7) yields

$$\left(\frac{1}{\delta}\right) A_2 \leqslant 2^p A_1/(\delta k^p).$$

So $k \leqslant 2(A_1/A_2)^{1/p}$, which concludes the main step.

To handle $m(x)$ arbitrary, say $\|m\|_\infty > 0$, consider any $k > 0$, and any interval $I$ such that $|E_0| > 0$ where $E_0 = \{x \in I: k - \varepsilon/2 \leqslant |m(x)| \leqslant k\}$ with $\varepsilon$ as before. Then clearly there exists an angle $\varphi$ and a set $E \subseteq E_0$ with $|E| > 0$ where $E = \{x: |m(x) - ke^{i\varphi}| \leqslant \varepsilon\} \cap E_0$. The rest of the argument proceeds as before. Finally consider the $n$-dimensional case. Use $\hat{f}(x) = \prod_{j=1}^{n} \hat{f}(x_j)$ and $\alpha = A_0^n k$ where $\hat{f}(x_j) = \left(\frac{1}{\delta}\right) \chi_{(-\delta,\delta)}(x_j)$ with $A_0$ as before. Note that

$$\lambda_{kf}(\alpha) = |\{x: |kf(x)| > A_0^n k\}| = |\{x: |f(x)| > A_0^n\}|$$
$$= \left|\left\{x: \prod_{j=1}^{n} |f(x_j)| > A_0^n\right\}\right| \geqslant \prod_{j=1}^{n} |\{x: |f(x_j)| > A_0\}| = A_2^n/\delta^n$$

and that

$$\|f\|_p^p = \prod_{j=1}^{n} \|f(x_j)\|_{p,x_j}^p = A_1^n/\delta^n$$

with $A_1$ and $A_2$ as before. So one gets by the same argument as before

$$A_2^n/\delta^n \leqslant A_1^n/\delta^n (A_0^n k)^p, \qquad k \leqslant [A_1^{1/p}/(A_0 A_2^{1/p})]^n.$$



This concludes part (a) of the theorem. The proof of part (b) follows the same method as part (a).

There is only left to do part (c). Without loss of generality $c = 1$, $m(x) \geqslant 0$ a.e., $n = 1$. If $\|m\|_\infty = 0$, we are done. So assume $\|m\|_\infty > 0$. By part (a) and (b) of this theorem, $\|m\|_\infty \leqslant B$ where $B = B(p) < \infty$ is a constant depending only on $p$. Fix $k$, $1 \leqslant k \leqslant B(p)$. Fix $\varepsilon$, $0 < \varepsilon < k$. As before, consider any interval $I$ on $\boldsymbol{R}$ such that $|E| > 0$ where $E = \{x \in \boldsymbol{R} \cap I: k - \varepsilon \leqslant m(x) \leqslant k\}$. Fix $I$. By Lebesgue's theorem for differentiation there exists an $x_0 \in E$ such that:

(10) $$(1/2\delta) \int_{-\delta}^{\delta} |m(x_0 + x) - k|\, dx \to |m(x_0) - k| \quad \text{as} \quad \delta \to 0.$$

Without loss of generality $x_0 = 0$.

We now construct an $f$ such that $\|f\|_p \cong \|f\|_{\mathrm{weak}L^p}$ and such that $\hat{f}$ dies off exponentially. One then dilates this $\hat{f}$ by $\delta$ with the same purpose in mind as in the proof of the first part of the theorem.

Let

$$f(x) = \frac{1}{\sqrt{2\pi}\,\delta_0} e^{-(x/\delta_0)^2} * \chi_{(-1,1)}(x), \qquad 0 < \delta_0 < 1.$$
$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} e^{-(\delta_0 x)^2} \frac{\sin 2\pi x}{\pi x}.$$

Define $\hat{f}_\delta(x) = \frac{1}{\delta}\hat{f}\left(\frac{x}{\delta}\right)$. Fix $\varepsilon_0 > 0$. Clearly one can choose $\delta_0$ and $\varepsilon_1 > 0$ small enough so that

(11) $$\left|(1-\varepsilon_1)^p |\{x: |f(x)| > (1-\varepsilon_1)\}| - 2\right| < \varepsilon_0,$$
$$\left|\|f\|_p^p - 2\right| < \varepsilon_0, \quad \text{and} \quad \frac{1}{\delta_0} e^{-1/\delta_0} < \frac{\varepsilon_0}{4B}.$$

Fix $\varepsilon_2 > 0$ small, but such that $k(1-\varepsilon_1)\varepsilon_2 > 2\varepsilon_0$. Following the lines of the proof of part (a) one arrives at

(12) $$\lambda_{kf_\delta}(k(1-\varepsilon_1)) \leqslant \lambda_{Tf_\delta}(k(1-\varepsilon_1)(1-\varepsilon_2)) + E_f$$

where

$$E_f = |\{x: |\mathscr{F}^{-1}[(k-m)\hat{f}_\delta](x)| > k(1-\varepsilon_1)\varepsilon_2\}|.$$

As before

$$E_f \leqslant \frac{1}{\delta} \int_{-\infty}^{\infty} |k - m(x)|\, e^{-(\delta_0 x/\delta)^2} \left|\frac{\sin(2\pi x/\delta)}{(\pi x/\delta)}\right| dx$$

and now

$$\leqslant \frac{1}{\delta}\int_{-\infty}^{\infty}|k-m(x)|\,e^{-(\delta_0 x/\delta)^2}\,dx$$

$$=\frac{1}{\delta}\Big(\int_{-\delta/\delta_0^2}^{\delta/\delta_0^2}+\int_{\delta/\delta_0^2}^{\infty}+\int_{-\infty}^{-\delta/\delta_0^2}\Big)|k-m(x)|\,e^{-(\delta_0 x/\delta)^2}\,dx$$

$$\leqslant (2/\delta_0^2)\left(\frac{1}{(2\delta/\delta_0^2)}\right)\int_{-\delta/\delta_0^2}^{\delta/\delta_0^2}|k-m(x)|\,dx+\frac{4B}{\delta}\int_{\delta/\delta_0^2}^{\infty}e^{-(\delta_0 x/\delta)^2}dx$$

$$=\mathrm{I}+\mathrm{II}.$$

By (10), $\mathrm{I}\leqslant\varepsilon_0$.

After a change of variables, one obtains

$$\mathrm{II}=\frac{4B}{\delta_0}\int_{1/\delta_0}^{\infty}e^{-x^2}dx\leqslant\frac{4B}{\delta_0}\int_{1/\delta_0}^{\infty}e^{-x}dx=\frac{4B}{\delta_0}e^{-1/\delta_0}<\varepsilon_0 \quad \text{by (11).}$$

Following the same method as in part (a) and using (11), (12) reduces to

$$\big|\{|kf_\delta(x)|>k(1-\varepsilon_1)\}\big|\leqslant\|f_\delta\|_p^p/[k(1-\varepsilon_1)(1-\varepsilon_2)]^p.$$

Continuing to argue in the manner of part (a) one finally gets

$$(2-\varepsilon_0)/\delta_0\leqslant(2+\varepsilon_0)/\delta k^p(1-\varepsilon_2)^p.$$

Hence

$$k\leqslant\left[\left(1+\frac{\varepsilon_0}{2}\right)\Big/\left(1-\frac{\varepsilon_0}{2}\right)\right]^{1/p}\Big/(1-\varepsilon_2).$$

Since $\varepsilon_0$ and $\varepsilon_2$ were arbitrary: $k\leqslant 1$.

This is clearly best possible by simply considering the identity operator. ∎

THEOREM 2. *Let $Tf*g=f*Tg$ for all $f,g\in\mathscr{F}_n$, and let $T$ satisfy*

$$\lambda_{Tf}(\alpha)\leqslant C\|f\|_p^p/\alpha^p \quad \text{for} \quad 1<p<2.$$

*Then $T$ is type $(2,2)$ on all of $L^2$.*

Proof. Without loss of generality $c=1$, $m(x)\geqslant 0$ a.e., $\|m\|_\infty>0$, and $n=1$. One handles $m$ complex-valued and $n$ general in the same way as in the proof of Theorem 1. As in the proof of Lemma 1, $\widehat{Tf}(x)$ $=m(x)\hat f(x)$ with $m\in L^1_{\mathrm{loc}}$. As in the proof of Theorem 1, let $E=\{x\in\boldsymbol{R}\cap I\colon k-\varepsilon\leqslant m(x)\leqslant k\}$ with $k>0$ and $|E|>0$. Again we have

$$(1/2\delta)\int_{-\delta}^{\delta}m(x_0+x)\,dx\to m(x_0) \quad \text{as} \quad \delta\to 0 \text{ for some } x_0,$$

and without loss of generality we may choose $x_0=0$. Choose $\hat f(x)$ $=\sin(2\pi x/\delta)/(\pi x)$ so that $f\in\mathscr{F}_1$. Let $\hat h(x)=\chi_{(-\delta,\delta)}(x)$.

$$k\hat h(x)\hat f(x)=\hat h(x)m(x)\hat f(x)+\big(k-m(x)\big)\hat h(x)\hat f(x).$$

Define

$$H(f)(x)=\mathscr{F}^{-1}[\hat h\hat f\,](x).$$

By the Fourier inversion theorem (applicable by the proof of Lemma 1), we have

$$kH(f)(x)=H\big(T(f)\big)(x)+\mathscr{F}[(k-m)\hat h\hat f\,](-x).$$

By a similar argument as in Theorem 1, we reduce to

$$(1)\qquad \lambda_{kHf}(\alpha)\leqslant\lambda_{HTf}(\alpha).$$

It is well-known (see [3]) that for the Hilbert transform $H_0$

$$H_0\colon \text{weak } L^p\to\text{weak } L^p \quad \text{for} \quad 1<p\leqslant 2.$$

However it is easy to see that

$$\hat h(x)=\left(\frac{1}{2i}\right)[\hat H_0(x+\delta)-\hat H_0(x-\delta)].$$

From this it is clear that

$$H\colon \text{weak } L^p\to\text{weak } L^p \quad \text{for} \quad 1<p\leqslant 2.$$

This yields the following equation:

$$(2)\qquad \sup_\alpha[\alpha^p\lambda_{HTf}(\alpha)]\leqslant L_0(p)\sup_\alpha[\alpha^p\lambda_{Tf}(\alpha)].$$

But $T$ is weak-type $(p,p)$ against $f$. So this means that

$$(3)\qquad \sup_\alpha[\alpha^p\lambda_{Tf}(\alpha)]\leqslant\|f\|_p^p.$$

Combining (2) and (3) and substituting into (1), one arrives at

$$(4)\qquad \lambda_{kHf}(\alpha)\leqslant L_0(p)\|f\|_p^p/\alpha^p.$$

It is trivial to verify

$$(5)\qquad \|f\|_p^p=2/\delta.$$

Choose $\alpha=L_1k$, $B_1$ chosen in a similar way as in Theorem 1. (4), (5), and the choice of $a$ yield:

(6) $$\lambda_{kHf}(L_1 k) \leqslant 2B_0/(L_1^p k^p).$$

(7) $$\begin{aligned}\lambda_{kHf}(L_1 k) &= |\{x:\ |kHf(x)| > kL_1\}| \\ &= |\{x:\ |Hf(x)| > L_1\}| \\ &= |\{x:\ |(\sin 2\pi x/\pi x) * \chi_{(-1,1)}(\delta x)| > L_1\}| \\ &= \left(\frac{1}{\delta}\right)\left|\left\{x:\ \left|\int_{-1}^{1} \sin 2\pi(x-y)/2\pi(x-y)\,dy\right| > L_1/2\right\}\right| = \left(\frac{1}{\delta}\right)L_2\end{aligned}$$

where $L_2$ is a constant depending only on $L_1$. Substituting (7) into (6), we finally have

$$\left(\frac{1}{\delta}\right)L_2 \leqslant 2L_0/(\delta L_1^p k^p).$$

So $k \leqslant [2L_0/(L_2 L_1^p)]^{1/p}$. ∎

**3. Counterexample.** From these and Ash's results (see [1]) the question arises as to what would be the "smallest class" against which weak-type $(p, p)$ implies type $(2, 2)$. The following counterexample shows that if the class consists of $f$ such that $f(x) = \chi_{(-\delta,\delta)}(x+N)$, $N, x \in \boldsymbol{R}$; $\delta > 0$, then the class is too small for $1 < p < \infty$. Previously there was only a counterexample for $p = 2$ against this class (see [2]).

COUNTEREXAMPLE. *If* $m(x) = \sum_{n=2}^{\infty} n\chi_{(-1,1)}(x - n^a)$ *with* $p \geqslant a/(a-2)$, *then* $m$ *is a weak-type* $(p, p)$ *multiplier operator against the class* $f(x) = \chi_{(-\delta,\delta)}(x+N)$, *in fact (strong)-type* $(p, p)$ *against the class, but is not type* $(2, 2)$.

Proof. $f(x) = \chi_{(-\delta,\delta)}(x+N)$ $\|f\|_p = (2\delta)1/p$. Hence

$$\hat{f}(x) = \frac{(\sin 2\pi\delta x)e^{2\pi iNx}}{(\pi x)}.$$

For convenience we will only do the case $p \geqslant 3/2$ which is general.

$$m(x) = \sum_{n=2}^{\infty} n\chi_{(-1,1)}(x - n^6).$$

Clearly $\|m\|_\infty = \infty$ so $T$ is not type $(2, 2)$ where $Tf(x) = \mathscr{F}^{-1}(m\hat{f})(x)$. Then

$$Tf(x) = \sum_{n=2}^{\infty}\int_{n^6-1}^{n^6+1} e^{-2\pi ixy} n(\sin 2\pi\delta y)e^{2\pi iNy}/(\pi y)\,dy$$

by the Fourier inversion formula.

$$Tf(x) = \sum_{n=2}^{\infty} 2n\delta e^{2\pi i(N-x)n^6}\int_{-1}^{1}\frac{\sin 2\pi\delta(y+n^6)e^{2\pi i(N-x)y}}{[2\pi\delta(y+n^6)]}\,dy$$

after change of variables.



LEMMA 2. *If* $\tilde{g}(x) = \int_{-1}^{1} g(y)e^{2\pi ixy}dy$, *then*

$$|\tilde{g}(x)| \leqslant \frac{1}{\pi|x|}(\|g\|_\infty + \|g'\|_\infty)$$

*where the* $\infty$*-norm takes place only on the interval* $[-1, 1]$.

Proof. Integrating by parts is all that is needed to obtain the result as we now show:

$$\begin{aligned}|\tilde{g}(x)| &= \left|\int_{-1}^{1} g(y)e^{2\pi ixy}dy\right| \\ &= \left|\left(\frac{1}{2\pi ix}\right)e^{2\pi ixy}g(y)]_{-1}^{1} - \left(\frac{1}{2\pi ix}\right)\int_{-1}^{1} g'(y)e^{2\pi ixy}dy\right| \\ &\leqslant \left(\frac{1}{2\pi|x|}\right)\cdot 2\|g\|_\infty + \left(\frac{1}{2\pi|x|}\right)\|g'\|_\infty\cdot 2 \\ &= \left(\frac{1}{\pi x}\right)[\|g\|_\infty + \|g'\|_\infty]. \ ∎\end{aligned}$$

We now return to the proof of the counterexample. Define

$$I \equiv \int_{-1}^{1}[\sin 2\pi\delta(y+n^6)/2\pi\delta(y+n^6)]e^{2\pi i(N-x)y}dy.$$

Then

(1) $$|Tf(x)| \leqslant \sum_{n=2}^{\infty} 2n\delta|I|.$$

Let $g(y) = [\sin 2\pi\delta(y+n^6)]/2\pi\delta(y+n^6)$. Then $I = \tilde{g}(1/(N-x))$. First consider the case $\delta < 1$ and $n^6 \leqslant 1/\delta$. Clearly $\|g\|_\infty \leqslant 1$ and

$$|g'(y)| = \left|\left(\frac{1}{y+n^6}\right)\left[\cos 2\pi\delta(y+n^6) - \frac{\sin 2\pi\delta(y+n^6)}{2\pi\delta(y+n^6)}\right]\right| \leqslant 2$$

for all $n \geqslant 2$.

Hence by Lemma 2 we have

(2) $$|I| \leqslant (1+2)/\pi|N-x| = 3/\pi|N-x| \quad \text{for} \quad |N-x| \geqslant 1.$$

If $|N-x| \leqslant 1$, then one obtains

(3) $$|I| \leqslant \int_{-1}^{1} |[\sin 2\pi\delta(y+n^6)/2\pi\delta(y+n^6)]e^{2\pi i(N-x)y}|dy \leqslant 2.$$

Now we handle the case $\delta \leqslant 1$ and $n^6 > 1/\delta$.

$$I = \int_{-1}^{1} [\sin 2\pi\delta(y+n^6)/2\pi\delta(y+n^6)] e^{2\pi i(N-x)y} dy$$

$$= \cos(2\pi\delta n^6) \int_{-1}^{1} [\sin 2\pi\delta y/2\pi\delta(y+n^6)] e^{2\pi i(N-x)y} dy +$$

$$+\sin(2\pi\delta n^6) \int_{-1}^{1} [\cos 2\pi\delta y/2\pi\delta(y+n^6)] e^{2\pi i(N-x)y} dy$$

$$= (\cos 2\pi\delta n^6/n^6) \int_{-1}^{1} [\sin 2\pi\delta y/2\pi\delta(1+y/n^6)] e^{2\pi i(N-x)y} dy + \text{2nd term}$$

$$\equiv I_1 + I_2.$$

Let $g(y) = \sin 2\pi\delta y/2\pi\delta(1+y/n^6)$.

$$\|g\|_\infty \leqslant 2 \quad \text{for} \quad y \in (-1, 1] \text{ and } n \geqslant 2.$$

$$|g'(y)| = \left|\left(\frac{1}{1+y/n^6}\right)(\cos 2\pi\delta y - \sin 2\pi\delta y/2\pi\delta(y+n^6))\right| \leqslant 2$$

$$\text{for } y \in [-1, 1] \text{ and } n \geqslant 2.$$

Hence by Lemma 2 we have

$$|I_1| \leqslant (2+2)/\pi n^6 |N-x| = 4/\pi n^6 |N-x| \quad \text{for} \quad |N-x| \geqslant 1.$$

To handle $I_2$ let $g(y) = \cos 2\pi\delta y/2\pi\delta(1+y/n^6)$. $\|g\|_\infty \leqslant 1/\delta$ and it is easy to check that $\|g'\|_\infty \leqslant 2/\delta$. Again by Lemma 2 we obtain

$$|I_2| \leqslant 6/\delta\pi n^6 |N-x| \quad \text{for} \quad |N-x| \geqslant 1.$$

So this means

(4) $$|I| \leqslant 10/\pi\delta n^6 |N-x| \quad \text{for} \quad |N-x| \geqslant 1 \text{ since } \delta \leqslant 1.$$

If $|N-x| \leqslant 1$, then we have

$$|I_1| \leqslant (1/n^6) \int_{-1}^{1} |[\sin 2\pi\delta y/2\pi\delta(1+y/n^6)] e^{2\pi i(N-x)y}| dy \leqslant 2/n^6,$$

and

$$|I_2| \leqslant (1/\delta n^6) \int_{-1}^{1} |[\cos 2\pi\delta y/2\pi(1+y/n^6)] e^{2\pi i(N-x)y}| dy \leqslant 2/\delta n^6.$$

This gives

(5) $$|I| \leqslant 4/\delta n^6 \quad \text{for} \quad |N-x| \leqslant 1.$$

We may further break up inequality (1) to obtain

$$|Tf(x)| \leqslant \sum_{n=2}^{\infty} 2n\delta |I| \leqslant \sum_{n=2}^{1/\delta^{1/6}} 2n\delta|I| + \sum_{n=1/\delta^{1/6}}^{\infty} 2n\delta|I|.$$



(For convenience we are assuming $1/\delta^{1/6}$ is an integer.) For $|N-x| \geqslant 1$, one has

$$|Tf(x)| \leqslant \sum_{n=2}^{1/\delta^{1/6}} 2n\delta \cdot 3/\pi|N-x| + \sum_{n=1/\delta^{1/6}}^{\infty} 2n\delta \cdot 10/\pi\delta n^6 |N-x|$$

by (2) and (4).

$$|Tf(x)| \leqslant (6\delta/\pi|N-x|)(1/\delta^{1/6})(1/\delta^{1/6}+1)/2 + 20/\pi|N-x| \sum_{n=1/\delta^{1/6}}^{\infty} 1/n^5$$

$$\leqslant 6\delta^{2/3}/\pi|N-x| + (20/\pi|N-x|)/2(1/\delta^{1/6})^4$$

$$= 16\delta^{2/3}/\pi|N-x|.$$

For $|N-x| \leqslant 1$, one has

$$|Tf(x)| \leqslant \sum_{n=2}^{1/\delta^{1/6}} 2n\delta \cdot 2 + \sum_{n=1/\delta^{1/6}}^{\infty} 2n\delta \cdot 4/\delta n^6 \quad \text{by (3) and (5),}$$

$$\leqslant 4\delta^{2/3} + 4\delta^{2/3} \quad \text{in the same way as above,}$$

$$= 8\delta^{2/3}.$$

We conclude that

(6) $$\|Tf\|_p \leqslant 40\delta^{2/3} \quad \text{for} \quad p \geqslant 3/2 \text{ and } \delta \leqslant 1$$

as can be easily calculated.

Now consider the case $\delta > 1$. We begin with the following break-up:

$$I = \int_{-1}^{1} [\sin 2\pi\delta(y+n^6)/2\pi\delta(y+n^6)] e^{2\pi i(N-x)y} dy$$

$$= (\cos 2\pi\delta n^6/n^6) \int_{-1}^{1} \left[(e^{2\pi i\delta y} - e^{-2\pi i\delta y})/\left(2i \cdot 2\pi\delta(1+y/n^6)\right)\right] e^{2\pi i(N-x)y} dy +$$

$$+ (\sin 2\pi\delta n^6/n^6) \int_{-1}^{1} \left[(e^{2\pi i\delta y} + e^{-2\pi i\delta y})/\left(2 \cdot 2\pi\delta(1+y/n^6)\right)\right] e^{2\pi i(N-x)y} dy$$

$$\equiv I_1 + I_2.$$

One can express $I_1$ in the following manner:

$$I_1 = \frac{\cos 2\pi\delta n^6}{4\pi n^6 i\delta} \int_{-1}^{1} 1\Big/\left(1+\frac{y}{n^6}\right)[e^{2\pi i(N-x+\delta)y} - e^{2\pi i(N-x-\delta)y}] dy.$$

Do a similar break-up for $I_2$.

By the same method used to handle $\delta < 1$ one arrives at

$$(7)\qquad |I| \leqslant \begin{cases} 4/\pi n^6 \delta & \text{if} \quad |N-x\pm\delta| \leqslant 1, \\ 3/\pi^2 n^6 \delta |N-x\pm\delta| & \text{if} \quad |N-x\pm\delta| \geqslant 1 \text{ when } \delta > 1. \end{cases}$$

By (1) we had

$$|Tf(x)| \leqslant \sum_{n=2}^{\infty} 2n\delta |I|.$$

So by (7) we see

$$|Tf(x)| \leqslant \begin{cases} 4/\pi & \text{if} \quad |N-x\pm\delta| \leqslant 1, \\ 6/\pi^2 |N-x\pm\delta| & \text{if} \quad |N-x\pm\delta| > 1 \text{ when } \delta > 1. \end{cases}$$

Hence we obtain the following bound:

$$(8)\qquad \|Tf\|_p \leqslant 28/\pi \quad \text{if} \quad p \geqslant 3/2 \text{ for } \delta > 1$$

as can be verified by a straightforward computation.

To sum up, for $p \geqslant 3/2$, $\|f\|_p = (2\delta)^{1/p}$, $\|Tf\|_p \leqslant 28/\pi$ for $\delta > 1$ by (8) and $\|Tf\|_p \leqslant 40\delta^{2/3}$ by (6). Hence $\|Tf\|_p \leqslant 40/2^{1/p}\|f\|_p$ whenever $p \geqslant 3/2$. ■

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# Analytic formulae for determinant systems in Banach spaces

by

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**Abstract.** Formulae are proved for the determinant systems of linear mappings $A = S+T$ where $S$ is a fixed Fredholm mapping from a Banach space $X$ into another one $Y$, and $T$ is a quasi-nuclear (or nuclear) mapping from $X$ into $Y$.

The theory of determinants in an arbitrary Banach space $X$ was first created for linear endomorphisms $A = I+T$ where $I$ is the identity mapping in $X$, and $T$ is a nuclear or quasi-nuclear endomorphism in $X$. (Grothendieck [2], Leżański [3], Ruston [4], see also Sikorski [5].) The theory yields analytic formulae for the determinant system of $A$, considered as a function of the quasi-nucleus (or nucleus) $F$ of $T$. Buraczewski [1] generalized the theory to the case of endomorphsisms of the form $A = S+T$ where $S$ is a Fredholm endomorphism in $X$, and $T$ is a quasi-nuclear endomorphism in $X$. He also formulated analytic formulae for the determinant system of $A = S+T$ (when considered as a function of the quasi-nucleus $F$ of $T$), but only under the additional hypothesis that $S$ is right-hand or left-hand invertible (see Buraczewski [1], Theorem (xiv)). The subject of the present paper is to generalize the formulae to the case of $A = S+T$ where $S$ is any fixed Fredholm mapping of a Banach space $X$ into another one $Y$, and $T$ is a quasi-nuclear mapping from $X$ into $Y$. It is not assumed that $S$ is right-hand or left-hand invertible. It is not assumed that $Y = X$, i.e. it is not assumed that $A$, $S$ and $T$ are endomorphisms. The theory of determinant systems, developped in this paper, is formulated in terms of the category of isomorphically conjugate pairs of Banach spaces. The object of the category are the pairs of Banach spaces just mentioned (for definition, see Section 1). The morphisms are operators defined in Section 2. The main theorem of the paper is Theorem 7.1.

Note that the terminology in our earlier papers on determinant systems differ sometimes from that in the present paper.

**1. Pairs of conjugate Banach spaces.** In this paper, either all linear spaces are real, or all are complex. If $E$ denotes a normed space, then $E^*$ denotes the Banach space of all continuous linear functionals on $E$.