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Some properties of asymptotic functions

by

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Abstract. The present paper is concerned with some new properties of three types of asymptotic functions which have previously been introduced by J. Marcinkiewicz ([4], [5]), S. Koizumi ([3]) and Y. M. Chen ([1], [2]). We shall prove (Theorem 2) that these three types of asymptotic functions are in a certain sense equivalent and that they are related to convex functions. As an immediate consequence of these results we can generalize two interpolation theorems due to Marcinkiewicz and Koizumi which have various applications in approximation theory.

1. Definitions and notation. Throughout this paper we use K to denote some positive constants which may be different from one occurrence to another. We use $K_1, K_2, \ldots$ to denote some specified positive constants so that $K_i \neq K_j$ if $i \neq j$.

DEFINITION 1. Let $-\infty < a \leqslant b < \infty$ and E a measurable subset of $(-\infty, \infty)$. (For most practical purposes we take $E = (0, \infty)$ or $= [0, \infty)$.)

(a) By $\Phi \sim [a, b]$ we denote the non-negative even functions defined in E, which are not identically zero in E, satisfying

(i) $\Phi(u)/|u|^a$ is non-decreasing as $|u|$ increases;

(ii) $\Phi(u)/|u|^b$ is non-increasing as $|u|$ increases.

The class of all such functions is denoted by $Y[a, b]$.

(b) If $a < b$, by $\Phi \sim \langle a, b\rangle$ we denote the function $\Phi \sim [a+\varepsilon, b-\varepsilon]$ for some small $\varepsilon > 0$. The class of all such functions is denoted by $Y\langle a, b\rangle$.

(c) In a similar way as in (a) and (b), we define $\Phi \in Y\langle a, b]$ and $\Phi \in Y[a, b\rangle$, which means that $\Phi \sim \langle a, b]$ and $\Phi \sim [a, b\rangle$, respectively (cf. [1], pp. 362–363).

Remark 1. If $a > 0$ and $\Phi \sim [a, b]$, then we have $\lim_{u\to 0}\Phi(u) = 0$. In this case we put $\Phi(0) = 0$ so that Φ is continuous in $(-\infty, \infty)$ ([2], Lemma 1).

Remark 2. If $L(u)$ is a slowly varying function in the sense of Hardy and Rogosinski (i.e., given any small $\delta > 0$, $u^{-\delta}L(u)$ is non-increasing and $u^\delta L(u)$ is non-decreasing for large u. See [6], I, p. 186.), then for any $\delta > 0$, $L(u) \sim [-\delta, \delta]$ for large u.

Definition 2. By $M(a, b)$, where $0 \leqslant a < b < \infty$, we mean the class of all continuous non-decreasing functions $\varphi(u)$ defined in $[0, \infty)$, which are not identically zero and satisfty $\varphi(0) = 0$ and the following relations:

$$\int_u^\infty t^{-b-1}\varphi(t)\,dt = O\big(u^{-b}\varphi(u)\big), \tag{1.1}$$

$$\int_1^u t^{-a-1}\varphi(t)\,dt = O\big(u^{-a}\varphi(u)\big), \tag{1.2}$$

as $u \to \infty$.

Definition 3. By $Z(a, b)$, where $0 \leqslant a < b < \infty$, we mean the subclass of $M(a, b)$ such that all functions in $Z(a, b)$ satisfy the following relations:

$$\int_u^1 t^{-b-1}\varphi(t)\,dt = O\big(u^{-b}\varphi(u)\big), \tag{1.3}$$

$$\int_0^u t^{-a-1}\varphi(t)\,dt = O\big(u^{-a}\varphi(u)\big), \tag{1.4}$$

as $u \to +0$.

Remark 3. The original definitons of φ introduced by Marcinkiewicz ([6], II, p. 116) and Koizumi ([3], p. 195) are slightly different from those introduced here. In their definitions, only the case $1 \leqslant a < b < \infty$ was considered. Furthermore, in [6] the additional condition,

$$\varphi(2u) = O\big(\varphi(u)\big) \tag{1.5}$$

as $u \to \infty$, was defined. In fact condition (1.5) is superfluous since it follows from the other conditions. We observe that

$$u^{-b}\varphi(2u) \leqslant K\varphi(2u)\int_{2u}^{4u} t^{-b-1}\,dt \leqslant K\int_u^\infty t^{-b-1}\varphi(t)\,dt \leqslant Ku^{-b}\varphi(u).$$

Similarly, in [3] the additional condition,

$$\varphi(2u) = O\big(\varphi(u)\big) \tag{1.6}$$

as $u \to +0$ may be omitted.

Definition 4. *Quasi-linear operators* (cf. [6], II, p. 111).

(a) Let R be a measure space with measure μ, and let $0 < r < \infty$. Let f be any real- or complex-valued function defined on R. We write

$$N(r, \mu; f) = \Big(\int_R |f|^r d\mu\Big)^{1/r}$$

which coincides with $\|f\|_{r,\mu}$, the (r, μ) norm of f when $r \geqslant 1$.

An operator T defined on the class of functions f is said to be *quasi-linear* if

(i) $T(f_1+f_2)$ is uniquely defined whenever Tf_1 and Tf_2 are defined, and if

(ii) $|T(f_1+f_2)| \leqslant K(|Tf_1|+|Tf_2|)$, where K is independent of f_1 and f_2.

(b) Let $0 < r < \infty$ and $0 < s < \infty$. A quasi-linear operator T which maps functions f defined on the measure space R_1 to functions Tf defined on the measure space R_2, is said to be *of type* (r, s) if

$$N(s, \nu; Tf) \leqslant KN(r, \mu; f),$$

where K is independent of f and μ, ν are measures defined on the measure spaces R_1 and R_2, respectively.

2. Main results. It will be shown that the classes $Y\langle a, b\rangle$ and $Z(a, b)$ are dominated by each other, although $Y\langle a, b\rangle$ is in fact a proper subclass of $Z(a, b)$ (Lemma 1). Indeed, the three classes of asymptotic functions $M(a, b)$, $Z(a, b)$ and $Y\langle a, b\rangle$ are equivalent to each other in some set $E \subset [0, \infty)$ (Theorem 2). Hence the functions in the classes M, Z and Y, in a number of inequalities may be interchanged. Our results give immediate generalizations (in Theorems 3, 4, 5) of two interpolation theorems (Lemmas 2, 3) due to Marcinkiewicz and Koizumi, which have various applications in approximation theory.

Theorem 1. *If* $\Phi \sim [a, b]$, *where* $-\infty < a \leqslant b < \infty$, *then* Φ *is absolutely continuous in* $[\varepsilon, N]$ $(0 < \varepsilon < N < \infty)$.

Remark 4. Since the proof of Theorem 1 follows essentially the same line as that of Lemma 1 in [2] (when $0 \leqslant a \leqslant b < \infty$), we omit the proof here. Here we wish to correct a discrepancy in [1] (p. 363, lines 9 and 10). It was thought that $\Phi \sim \langle -1, 1\rangle$ has no meaning since Φ may not be measurable. But now we find that this is not so. By Theorem 1, $\Phi \sim \langle -1, 1\rangle$ is absolutely continuous in $[\varepsilon, N]$ and is thus measurable in $[\varepsilon, N]$ for any $\varepsilon > 0$. It is therefore measurable in $(0, \infty)$ and also in $(-\infty, \infty)$. So we can define the class $Y[a, b]$ when $-\infty < a \leqslant b < \infty$ instead of the particular case when $0 \leqslant a \leqslant b < \infty$ as in [1] and [2].

Theorem 2. *Let* $0 \leqslant a < b < \infty$.

(i) *Suppose that* $\varphi \in Z(a, b)$ *and that* $\varphi(u) > 0$ *when* $u > 0$. *Then there exists a function* $\Phi \sim \langle a, b\rangle$ *in* $[0, \infty)$, *such that* Φ *is twice continuously differentiable in* $(0, \infty)$, *and that*

$$K_1\Phi(u) \leqslant \varphi(u) \leqslant K_2\Phi(u) \tag{2.1}$$

for $u \geqslant 0$, *where* K_1 *and* K_2 *are independent of* u. *In particular, if* $a \geqslant 1$, *then* Φ *is convex.*

(ii) *Suppose that* $\varphi \in M(a, b)$ *and that* $\varphi(u) > 0$ *when* $u \geqslant u_0 \geqslant 1$. *Then there exists a twice continuously differentiable function* $\Phi \sim \langle a, b\rangle$

defined in $[u_0, \infty)$, *satisfying*

$$K_1\Phi(u) \leqslant \varphi(u) \leqslant K_2\Phi(u)$$

for $u \geqslant u_0$, *where* K_1 *and* K_2 *are independent of* u. *In particular, if* $a \geqslant 1$, *then* Φ *is convex.*

In the following Theorems 3, 4, 5 we let $0 \leqslant a_i < b_i < \infty$ $(i = 1, 2, \ldots, n)$. Let R_1 and R_2 be measure spaces with measures μ and ν, respectively, and T a quasi-linear operator.

Theorem 3. *Let* $a = \sum_{i=1}^{n} a_i > 0$, $b = \sum_{i=1}^{n} b_i$, *and let the operator* T *be of types* (a, a), (b, b).

(i) (a generalization of Lemma 2). *Suppose that* $\mu(R_1) < \infty$, $\nu(R_2) < \infty$, $\varphi_i \in M(a_i, b_i)$ $(i = 1, 2, \ldots, n)$, *and* $\varphi(u) = \prod_{i=1}^{n} \varphi_i(u)$. *Then*

(1) Tf *is defined for every* f *with* $\varphi(|f|)$ *integrable over* R_1, *and*
(2) $\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu + K$, *where* K *is independent of* f.

(ii) (a generalization of Lemma 3). *Suppose that* $\mu(R_1) \leqslant \infty$, $\nu(R_2) \leqslant \infty$, $\varphi_i \in Z(a_i, b_i)$ $(i = 1, 2, \ldots, n)$, *and* $\varphi(u) = \prod_{i=1}^{n} \varphi_i(u)$. *Then*

(1) Tf *is defined for every* f *with* $\varphi(|f|)$ *integrable over* R_1, *and*
(2) $\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu$, *where* K *is independent of* f.

Theorem 4. *Let* $a = \prod_{i=1}^{n} a_i > 0$, $b = \prod_{i=1}^{n} b_i$, *and the operator* T *be of types* (a, a), (b, b).

(i) (a generalization of Lemma 2). *Suppose that* $\mu(R_1) < \infty$, $\nu(R_2) < \infty$, $\varphi_i \in M(a_i, b_i)$ $(i = 1, 2, \ldots, n)$, *and* $\varphi(u) = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1(u)$, *where* $\circ$ *denotes the usual compositon of mappings. Then*

(1) Tf *is defined for every* f *with* $\varphi(|f|)$ *integrable over* R_1, *and*
(2) $\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu + K$, *where* K *is independent of* f.

(ii) (a generalization of Lemma 3). *Suppose* $\mu(R_1) \leqslant \infty$, $\nu(R_2) \leqslant \infty$, $\varphi_i \in Z(a_i, b_i)$ $(i = 1, 2, \ldots, n)$ *and* $\varphi(u) = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1(u)$. *Then*

(1) Tf *is defined for every* f *with* $\varphi(|f|)$ *integrable over* R_1, *and*
(2) $\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu$, *where* K *is independent of* f.

Theorem 5 (a generalization of Lemma 3). *Suppose that* $\mu(R_1) \leqslant \infty$, $\nu(R_2) \leqslant \infty$, $\varphi_i \in M(a_i, b_i)$ $(i = 1, 2,)$ *and* φ_1, φ_2 *are positive when* $u \geqslant u_0 > 0$. *Let* $a = \min(a_1, a_2) > 0$, $b = \max(b_1, b_2)$. *If* T *is of types* (a, a), (b, b), *and* φ *is defined by*

$$\varphi(u) = \begin{cases} \varphi_1(u) & \text{if} \quad u \geqslant u_0, \\ \varphi_1(u_0)\varphi_2(u_0)/\varphi_2(u_0^2/u) & \text{if} \quad 0 < u \leqslant u_0, \\ 0 & \text{if} \quad u = 0, \end{cases}$$

then

(1) Tf *is defined for every* f *with* $\varphi(|f|)$ *integrable over* R_1, *and*
(2) $\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu$, *where* K *is independent of* f.

3. Preliminary lemmas.

Lemma 1 (cf. [1], Theorem 1). *Suppose that* $0 \leqslant a < b < \infty$. *We have*

(i) $Y\langle a, b\rangle \subsetneqq Z(a, b)$,
(ii) *if* $\varphi(u) \in M(a, b)$, *then*

$$u^a = o(\varphi(u)) \tag{3.1}$$

as $u \to \infty$. *If, in addition,* $\varphi(u) \in Z(a, b)$, *then besides* (3.1) *we also have*

$$u^b = o(\varphi(u)) \tag{3.2}$$

as $u \to +0$.

Proof. We first prove $Y\langle a, b\rangle \subsetneqq Z(a, b)$ $(0 \leqslant a < b < \infty)$. Let $\varphi \in Y\langle a, b\rangle$. It is clear that φ is continuous, non-decreasing, $\varphi(0) = 0$ (cf. Remark 1), and $\varphi(2u) \leqslant 2^b\varphi(u)$ when $u \in [0, \infty)$. Since $\varphi(u)/u^{a+\varepsilon}$ is non-decreasing in $(0, \infty)$ for some $\varepsilon > 0$, we have

$$\int_0^u t^{-a-1}\varphi(t)\,dt = \int_0^u t^{-a-\varepsilon}\varphi(t)t^{-1+\varepsilon}dt \leqslant u^{-a-\varepsilon}\varphi(u)\int_0^u t^{-1+\varepsilon}dt \leqslant Ku^{-a}\varphi(u).$$

Hence (1.2) and (1.4) are satisfied. By similar method we can show that (1.1) and (1.3) are satisfied. Thus $\varphi \in Z(a, b)$. Since conditions in Definition 1(b) are pointwise conditions but (1.1)–(1.4) are asymptotic relations, it is not difficult to construct a function $\varphi \in Z(a, b) \setminus Y\langle a, b\rangle$.

In order to show that (3.1) is true as $u \to \infty$, we observe that from (1.2) we have $u^a \leqslant K\varphi(u)$ when $u > B$, which is large. Then from (1.2) we have

$$Ku^{-a}\varphi(u) \geqslant \int_1^u t^{-a}\varphi(t)t^{-1}\,dt \geqslant \int_B^u Kt^{-1}\,dt \to \infty,$$

as $u \to \infty$. Hence (3.1) follows (3.2) is proved in a similar way.

Lemma 2 (Marcinkiewicz's interpolation theorem [6], II, p. 116). *Suppose that* $\mu(R_1) < \infty$, $\nu(R_2) < \infty$, *where* μ *and* ν *are measures of the spaces* R_1 *and* R_2, *respectively, and that the quasi-linear operator* T *is of types* (a, a), (b, b), *where* $0 < a < b < \infty$, *and that* $\varphi \in M(a, b)$. *Then we have*

(i) Tf *is defined for every* f *with* $\varphi(|f|)$ *integrable over* R_1, *and*
(ii) $\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu + K$, *where* K *is independent of* f.

Lemma 3 (Theorem 4 in [3]). *Suppose that* $\mu(R_1) \leqslant \infty$, $\nu(R_2) \leqslant \infty$. *If* T *is of types* (a, a), (b, b), *where* $0 < a < b < \infty$ *and* $\varphi \in Z(a, b)$, *then*

(ii) *in Lemma* 2 *can be replaced by*

$$\int_{R_2} \varphi(|Tf|)\,d\nu \leqslant K \int_{R_1} \varphi(|f|)\,d\mu,$$

where K is independent of f.

Remark 6. Lemma 2 is a generalization of the Marcinkiewicz interpolation theorem ([6], II, p. 116) in which $1 \leqslant a < b < \infty$ is assumed. Lemma 3 is a generalization of Theorem 4 in [3] in which $\mu(R_1) = \infty$, $\nu(R_2) = \infty$ and $1 \leqslant a < b < \infty$ are assumed. In fact, following all the original proofs in [6] and [3] without any essential alteration we can prove these generalizations.

4. Proof of Theorem 2. Suppose that $\varphi \in Z(a, b)$. By definition, φ is continuous, non-decreasing, $\varphi(0) = 0$, and φ satisfies the following relations:

$$\varphi(2u) = O\big(\varphi(u)\big), \tag{1.5}$$

$$\int_u^\infty t^{-b-1}\varphi(t)\,dt = O\big(u^{-b}\varphi(u)\big), \tag{1.1}$$

$$\int_1^u t^{-a-1}\varphi(t)\,dt = O\big(u^{-a}\varphi(u)\big), \tag{1.2}$$

as $u \to +\infty$;

$$\varphi(2u) = O\big(\varphi(u)\big), \tag{1.6}$$

$$\int_u^1 t^{-b-1}\varphi(t)\,dt = O\big(u^{-b}\varphi(u)\big), \tag{1.3}$$

$$\int_0^u t^{-a-1}\varphi(t)\,dt = O\big(u^{-a}\varphi(u)\big), \tag{1.4}$$

as $u \to +0$.

Put

$$g(u) = \int_u^\infty t^{-b-1}\varphi(t)\,dt \quad \text{and} \quad h(u) = \int_0^u t^{-a-1}\varphi(t)\,dt.$$

It follows from the above relations that $g(u)$ and $h(u)$ are both finite when $u > 0$. Write $\Phi(u) = u^b g(u) + u^a h(u)$ $(u > 0)$ and let $\Phi(0) = 0$. Straightforward calculation shows that when $u > 0$,

$$d^2\Phi(u)/du^2 = u^{-2}\{b(b-1)u^b g(u) + a(a-1)u^a h(u) - (b-a)\varphi(u)\}$$

which is continuous in $(0, \infty)$. If $a \geqslant 1$, we have

$$d^2\Phi(u)/du^2 \geqslant u^{-2}\Big\{b(b-1)u^b\varphi(u)\int_u^\infty t^{-b-1}dt - (b-a)\varphi(u)\Big\} \geqslant 0,$$

and then Φ is convex.

Next, we shall prove that

$$K_1\Phi(u) \leqslant \varphi(u) \leqslant K_2\Phi(u) \tag{2.1}$$

when $u \in [0, \infty)$. Since the function $u^{-b}\varphi(u)$ is continuous in $[\varepsilon, \infty)$ for any $\varepsilon > 0$, and from hypothesis $\varphi(u) > 0$ $(u > 0)$, it follows from (1.1), (1.3) and (3.2) that

$$g(u) = \int_u^\infty t^{-b-1}\varphi(t)\,dt \leqslant Ku^{-b}\varphi(u) \tag{4.1}$$

for $u > 0$, where K is independent of u. Similarly, from (1.2), (1.4) and (3.1) we have

$$h(u) = \int_0^u t^{-a-1}\varphi(t)\,dt \leqslant Ku^{-a}\varphi(u) \tag{4.2}$$

for $u > 0$, where K is independent of u. On the other hand, since φ is non-decreasing in $[0, \infty)$, we have

$$\varphi(u)u^{-b} \leqslant K\varphi(u)\int_u^\infty t^{-b-1}dt \leqslant Kg(u) \tag{4.3}$$

for $u \in (0, \infty)$. Since the function $\varphi(u)/\varphi(u/2)$ is continuous in $[\varepsilon, N]$ for $0 < \varepsilon < N < \infty$, it follows from (1.5) and (1.6) that $\varphi(u) \leqslant K\varphi(u/2)$ in $(0, \infty)$. Since $a \geqslant 0$ and φ is non-decreasing, we have

$$\varphi(u)u^{-a} \leqslant K\varphi(u/2)\int_{u/2}^u t^{-a-1}dt \leqslant K\int_{u/2}^u t^{-a-1}\varphi(t)\,dt \leqslant Kh(u). \tag{4.4}$$

Combining (4.1) and (4.3) we find

$$K_3u^b g(u) \leqslant \varphi(u) \leqslant K_4u^b g(u) \tag{4.5}$$

for all $u \in (0, \infty)$. Similarly, from (4.2) and (4.4) we obtain

$$K_5u^a h(u) \leqslant \varphi(u) \leqslant K_6u^a h(u) \tag{4.6}$$

for all $u \in (0, \infty)$. Adding up (4.5) and (4.6) we obtain (2.1) for $u > 0$. Since $\Phi(0) = 0 = \varphi(0)$, (2.1) is also satisfied at $u = 0$.

Finally we shall prove that $\Phi \sim \langle a, b\rangle$ $(0 \leqslant a < b < \infty)$. Let $\varepsilon > 0$ be small such that $b - a - \varepsilon > 0$. When $u > 0$ we have

$$\begin{aligned}\frac{d}{du}u^{-a-\varepsilon}\Phi(u) &= (b-a-\varepsilon)u^{b-a-\varepsilon-1}\int_u^\infty t^{-b-1}\varphi(t)\,dt - \varepsilon u^{-\varepsilon-1}\int_0^u t^{-a-1}\varphi(t)\,dt \\ &= I, \text{ say.}\end{aligned}$$

It follows from (4.3) and (4.2) that

$$\begin{aligned}I &\geqslant (b-a-\varepsilon)Ku^{b-a-\varepsilon-1}u^{-b}\varphi(u) - \varepsilon Ku^{-\varepsilon-1}u^{-a}\varphi(u) \\ &= \big((b-a-\varepsilon)K - \varepsilon K\big)u^{-a-\varepsilon-1}\varphi(u),\end{aligned}$$

which is positive if $\varepsilon > 0$ is sufficiently small. Thus $u^{-a-\varepsilon}\Phi(u)$ is non-decreasing for some small $\varepsilon > 0$. Similarly, form (4.1), (4.4) we have $\frac{d}{du}\big(u^{-b+\varepsilon}\Phi(u)\big) < 0$ if $\varepsilon > 0$ is small. Hence $\Phi \sim [a+\varepsilon, b-\varepsilon]$ for sufficiently small $\varepsilon > 0$. This completes the proof of Theorem 2 (i). We omit the proof of Theorem 2 (ii) as the argument is similar to that of part (i), except that we now put $h(u) = \int_1^u t^{-a-1}\varphi(t)\,dt$ for $u \geqslant u_0 \geqslant 1$.

5. Proofs of Theorems 3, 4 and 5. It is obvious that if $\Phi_i \in \Upsilon\langle a_i, b_i\rangle$ ($a_i \geqslant 0$, $i = 1, 2$), then $\Phi_1\Phi_2 \in \Upsilon\langle a_1+a_2, b_1+b_2\rangle$. Since

$$\frac{\Phi_2\circ\Phi_1(u)}{u^{p_1p_2}} = \frac{\Phi_2\big(\Phi_1(u)\big)}{\big(\Phi_1(u)\big)^{p_2}}\left(\frac{\Phi_1(u)}{u^{p_1}}\right)^{p_2} \qquad (p_1, p_2 \geqslant 0),$$

we have $\Phi_2\circ\Phi_1 \in \Upsilon\langle a_1a_2, b_1b_2\rangle$. Theorems 3 and 4 follow readily from Theorem 2 and Lemmas 1, 2, 3.

We come now to prove Theorem 5. Since $\varphi_i \in M(a_i, b_i)$ ($i = 1, 2$), by Theorem 2 there exist $\Phi_i \in \Upsilon\langle a_i, b_i\rangle$ ($i = 1, 2$) in $[u_0, \infty)$ such that

$$K_1\Phi_i(u) \leqslant \varphi_i(u) \leqslant K_2\Phi_i(u) \qquad (i = 1, 2) \tag{5.1}$$

for $u \geqslant u_0$. Define Φ by

$$\Phi(u) = \begin{cases} \Phi_1(u) & \text{if} \quad u \geqslant u_0, \\ \Phi_1(u_0)\Phi_2(u_0)/\Phi_2(u_0^2/u) & \text{if} \quad 0 < u < u_0, \\ 0 & \text{if} \quad u = 0. \end{cases}$$

It is clear that $\Phi \in \Upsilon\langle a, b\rangle$. By Lemma 1, Lemma 3 and (5.1), we obtain Theorem 5 immediately.

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Weak-type multipliers*

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Abstract. In this work, we show, for multiplier operators, that weak-type (p, p) for one fixed p, $1 < p < 2$, implies type (2,2). We obtain the same result for any fixed $p > 0$ with the a priori condition $m \in L^1_{\text{loc}}$. We also determine a best constant. The last part of the paper is a related counterexample.

1. Introduction. Consider a Fourier multiplier operator $Tf(x) = \mathscr{F}^{-1}(m\hat{f})(x)$ on $\boldsymbol{R}^n$. If T is weak-type (p, p) for fixed p, $1 < p < 2$, then a duality interpolation argument implies T is strong-type (2,2) hence m is bounded (see [3]). This argument breaks down if the operator is assumed to satisfy the weak-type estimate only for a small subset S of L^p.

In this paper we develop a new direct method for studying these questions. In particular we show that if $m \in L^1_{\text{loc}}$ and T is weak-type (p, p) for any $p > 0$, then m is bounded (if $1 < p < 2$, we need not assume $m \in L^1_{\text{loc}}$). We also find appropriate subsets S of L^p, depending on p where we assume T is weak-type (p, p), for which the same result holds. We go on to obtain the best constant for the bound on m in terms of the weak-type constant. We also construct a counterexample to show that the result is false if S consists of all characteristic functions of intervals, extending the work of Ash (see [1] and [2]). The counterexample is particularly interesting for the following reason. Stein and Weiss ([3]) show that if one assumes T is weak-type (p, p) against functions which are characteristic functions of measurable sets, then T extends to be weak-type (p, p) on all of L^p. The counterexample demonstrates that characteristic functions of intervals is too small a class to obtain the extension.

2. Positive results. Notation:

$$\hat{\mathscr{F}}_n = \left\{f\colon \hat{f}(x) = \prod_{j=1}^{n}\left(\frac{1}{\delta^n}\right)\chi_{(-\delta,\delta)}(x_j+k_j\delta),\ x \in \boldsymbol{R}^n, \quad k_j\text{'s integers},\ \delta > 0\right\},$$

$$\mathscr{F}_n = \left\{f\colon f(x) = \prod_{j=1}^{n}\left(\frac{1}{\delta^n}\right)\chi_{(-\delta,\delta)}(x_j)\,e^{2\pi i x_j k_j/\delta}, \quad x \in \boldsymbol{R}^n,\ k_j\text{'s integers},\ \delta > 0\right\},$$

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* The work presented here is contained in the author's Ph.D. thesis written under Professor Robert Strichartz at Cornell University.

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