measure is in $L^2(\sigma)$ it follows that to each compact set $K \subset \Omega$ there is a number C such that

$$|\nabla u| \leqslant CL \quad \text{in } K.$$

Hence

$$\int_{\Omega} |\nabla u|^2 d(Q)\, dm(Q) \leqslant \int_{U} + \int_{\Omega-U} \leqslant C \int Q(P)\, d\sigma + CL^2 \int_{\Omega-u} dm \leqslant C \int_{L^2_{\partial\Omega}} u^2 d\sigma,$$

which completes the proof of the corollary.

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Lifting subnormal double commutants*

by

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Abstract. We give a necessary condition for elements of the commutant of a subnormal operator on Hilbert space to lift to the commutant of the minimal normal extension, and based on this condition we show there exists a subnormal operator having elements in its double commutant that do not lift. Our example is also irreducible and finitely cyclic.

1. Let T be a subnormal operator on a Hilbert space H with minimal normal extension (m.n.e.) N on K. Thus, N is a normal operator on a space K containing H, H is invariant for N, the restriction $N|_H = T$, and K, the smallest reducing subspace of N containing H, is the closed linear span of $\{N^{*j}h\colon j = 0, 1, \ldots, h \in H\}$. Clearly, for any polynomial p, $p(N)|_H = p(T)$ and it is therefore easy to see that every element in the weakly closed algebra generated by T lifts to, i.e. is the restriction to H of, some element in the commutant of N. This result is included in the well-known paper of J. Bram, who gave the following necessary and sufficient condition for an element of $\{T\}'$, the commutant of T, to lift to $\{N\}'$:

THEOREM B [2, p. 87]. *A necessary and sufficient condition that $A \in \{T\}' \subset B(H)$ have an extension $B \in \{N\}' \subset B(K)$ is that there exist a positive constant c such that for every finite set $h_0, \ldots, h_r$ in H we have*

$$\sum_{m,n=0}^{r} (T^m Ah_n, T^n Ah_m) \leqslant c \sum_{m,n=0}^{r} (T^m h_n, T^n h_m).$$

If the extension B exists, then it is unique.

Bram pointed out that there exist subnormal operators T such that not every element of $\{T\}'$ lifts to $\{N\}'$. T. Yoshino, however, showed that every element of $\{T\}'$ does lift to $\{N\}'$ if T has a cyclic vector [6], p. 49. (Note that there is an error in the proof of this theorem but the result is true.) M.B. Abrahamse and C. Berger raised the question of whether

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for a general subnormal T every element of $\{T\}''$, the double commutant of T, lifts to $\{N\}'$ [1], Problem 2. Abrahamse also asked whether every element of $\{T\}'$ must lift to $\{N\}'$ if T is irreducible [1], Problem 1. We give an example below which provides a negative answer to both questions; we note the R. Olin and J. Thomson have independently discovered an example with the same properties. Although the main idea underlying their example is similar to ours, we present a somewhat different approach to the problem. We conclude by showing that a certain class of subnormal operators considered by Bram [2] is finitely cyclic.

2. Our example is based on the following theorem which describes a situation in which elements of $\{T\}''$ do not lift. In Section 3 we will construct an operator satisfying the conditions of Theorem 1.

For operators $T_1 \in B(H_1)$ and $T_2 \in B(H_2)$, we denote $I(T_1, T_2) = \{X \in B(H_1, H_2): XT_1 = T_2X\}$. Two subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of H are called complementary if $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$ and $H = \mathcal{M}_1 + \mathcal{M}_2 = \{m_1 + m_2: m_i \in \mathcal{M}_j\}$ (Note that this implies that $\mathcal{M}_1 + \mathcal{M}_2$ is closed.) and we denote such a sum by $\mathcal{M}_1 \dotplus \mathcal{M}_2$.

THEOREM 1. *Suppose T, a subnormal operator on H having m.n.e. N on K, has a pair of complementary invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$. For $i = 1, 2$, denote $T_i = T|_{\mathcal{M}_i}$ and let K_i be the closed linear span of $\{N^{*j} h_i: j = 0, 1, \ldots, h_i \in \mathcal{M}_i\}$. Suppose $I(T_1, T_2) = \{0\}$ and $I(T_2, T_1) = \{0\}$. Suppose further that $K_1 \cap K_2 \neq \{0\}$. Then there exist elements of $\{T\}''$ that do not lift to $\{N\}'$.*

Proof. With respect to the decomposition $H = \mathcal{M}_1 \dotplus \mathcal{M}_2$, we write

$$T = T_1 \dotplus T_2 = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}.$$

Matrix multiplication shows that

$$\{T\}' = \left\{\begin{bmatrix} A_1 & X \\ Y & A_2 \end{bmatrix}: A_i \in \{T_i\}',\ X \in I(T_2, T_1),\ Y \in I(T_1, T_2)\right\},$$

so under our hypotheses on the intertwining spaces,

$$\{T\}' = \{T_1\}' \dotplus \{T_2\}' = \left\{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}: A_i \in \{T_i\}'\right\}.$$

It then follows that

$$\{T\}'' = \{T_1\}'' \dotplus \{T_2\}'' = \left\{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}: A_i \in \{T_i\}''\right\}.$$

Let $A = A_1 \dotplus A_2 \in \{T\}' \subset B(H)$ and suppose A lifts to $\{N\}'$. Then there exists $B \in \{N\}' \subset B(K)$ such that $B|_H = A$. Note that K_i reduces N and $N_i = N|_{K_i}$ is the m.n.e. of T_i. Then $B_i = B|_{K_i}$ is an extension of A^i

and since

$$B_i N_i = B_i N|_{K_i} = BN|_{K_i} = NB|_{K_i} = N_i B_i,$$

we see that A_i lifts to $\{N_i\}'$ and we denote that lifting by $\tilde{A}_i$. Since the lifting is unique by Theorem B, we have that $\tilde{A}_i = B_i$, $i = 1, 2$. Thus,

$$\tilde{A}_1|_{(K_1\cap K_2)} = B_1|_{(K_1\cap K_2)} = B|_{(K_1\cap K_2)} = B_2|_{(K_1\cap K_2)} = \tilde{A}_2|_{(K_1\cap K_2)},$$

i.e. a necessary condition for $A_1 \dotplus A_2$ to lift to $\{N\}'$ is that A_1 and A_2 each lift to its respective m.n.e. space K_1 and K_2 and that the liftings agree on $K_1 \cap K_2$. This condition is clearly also sufficient.

Now, for the case in hand, let $A_1 = 0|_{\mathcal{M}_1}$ and $A_2 = I|_{\mathcal{M}_2}$, so $A = A_1 \dotplus A_2 \in \{T\}''$. Using the above notation, $\tilde{A}_1 = 0|_{K_1}$ and $\tilde{A}_2 = I|_{K_2}$ and since $(K_1 \cap K_2)$ is non-trivial, A does not lift to $\{N\}'$ and the theorem is proved.

3. Let μ be Lebesgue area measure on the annulus $A = \{\frac{1}{3} \leqslant |z| \leqslant 3\}$ and consider the overlapping subannuli $A_1 = \{\frac{1}{3} \leqslant |z| \leqslant 2\}$ and $A_2 = \{\frac{1}{2} \leqslant |z| \leqslant 3\}$ with characteristic functions χ_1 and χ_2 respectively. (The specific radii are chosen to simplify a following computation and are not essential to the example.) For $i = 1, 2$, let $\mathcal{M}_i = \{\chi_i f(z) \in L^2(\mu): f$ is holomorphic on $A_i\}$. Then, $\{\chi_i z^n: n = 0, \pm 1, \pm 2, \ldots\}$ is an orthogonal basis for $\mathcal{M}_i$ and $\chi_1 z^n$ is orthogonal to $\chi_2 z^m$ for $n \neq m$; on each $\mathcal{M}_i$, multiplication by z is a bilateral weighted shift.

We clearly have $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$. We claim that $\mathcal{M}_1 + \mathcal{M}_2$ is closed in $L^2(\mu)$. To see this, let $f(z) = \chi_1 \sum_{-\infty}^{\infty} f_n z^n \in \mathcal{M}_1$ and $g(z) = \chi_2 \sum_{-\infty}^{\infty} g_n z^n \in \mathcal{M}_2$. Then

$$(1/\pi)|(f, g)_{L^2(\mu)}| = (1/\pi)\Big|\sum_{n,m} f_n \bar{g}_m \int_A \chi_1 \chi_2 z^n \bar{z}^m \, d\mu\Big|$$

$$= \Big|\sum_n f_n \bar{g}_n \int_{\frac{1}{2}}^{2} 2r^{2n+1}\, dr\Big|$$

$$\leqslant \Big[\sum_{n \neq -1} |f_n g_n| (4^{n+1} - 4^{-(n+1)})/(n+1)\Big] + |f_{-1} g_{-1}| \ln 4$$

$$\leqslant K\Big(\Big[\sum_{n \neq -1} |f_n g_n| (6^{n+1} - 6^{-(n+1)})/(n+1)\Big] + |f_{-1} g_{-1}| \ln 6\Big)$$

$$= K\Big(\Big[\sum_{n \neq -1} |f_n| [(4^{n+1} - 9^{-(n+1)})/(n+1)]^{1/2} \cdot |g_n| [9^{n+1} - 4^{-(n+1)}/(n+1)]^{1/2}\Big] +$$

$$+ |f_{-1}| (\ln 6)^{1/2} |g_{-1}| (\ln 6)^{1/2}\Big)$$

$$\leqslant K\Big(\sum_{-\infty}^{\infty} |f_n|^2 \int_{\frac{1}{3}}^{2} 2r^{2n+1}\, dr\Big)^{\frac{1}{2}} \Big(\sum_{-\infty}^{\infty} |g_n|^2 \int_{\frac{1}{2}}^{3} 2r^{2n+1}\, dr\Big)^{\frac{1}{2}}$$

$$= (1/\pi) K \|f\| \, \|g\|, \quad \text{where } K < 1.$$

Thus, the angle between the subspaces $\mathscr{M}_1$ and $\mathscr{M}_2$ is strictly positive and hence $\mathscr{M}_1+\mathscr{M}_2$ is closed [4], p. 339. We note that this can also be shown directly.

Thus, using the notation of Theorem 1, we let H be the Hilbert space $\mathscr{M}_1+\mathscr{M}_2$ and T be multiplication by z on H. T is subnormal with m.n.e. equal to multiplication by z on $L^2(\mu)$, and since $T_i = T|_{\mathscr{M}_i}$, we have $K_i = L^2(\mu|_{A_i}) = \{\chi_i f \in L^2(\mu)\}$. Thus, $K_1 \cap K_2 = \{\chi_1\chi_2 f \in L^2(\mu)\} = L^2(\mu|_{A_1\cap A_2}) \neq \{0\}$. It remains to show that the intertwining spaces are trivial.

Suppose $X \in B(\mathscr{M}_1, \mathscr{M}_2)$ with $T_2X = XT_1$. For $\chi_1 f \in \mathscr{M}_1$, let $X(\chi_1 f) = \chi_2 g$ with g holomorphic in A_2. Since $T_2^n X = XT_1^n$ for $n = 1, 2, \ldots$, we have $X(\chi_1 z^n f) = \chi_2 z^n g$. We have

$$\|\chi_1 z^n f\|^2 = \int_{A_1} |z^n f|^2\, d\mu = \int_{A_1} r^{2n}|f|^2\, d\mu \leqslant 4^n \|\chi_1 f\|^2$$

and

$$\|\chi_2 z^n g\|^2 = \int_{A_2} |z^n g|^2\, d\mu \geqslant \int_{A_{22}} \|z^n g\|^2 d\mu = \int_{A_{22}} r^{2n}\|g\|^2\, d\mu$$
$$\geqslant (\tfrac{5}{2})^{2n} \int_{A_{22}} |g|^2\, d\mu = (\tfrac{5}{2})^{2n}\|\chi_{22} g\|^2,$$

where $A_{22} = \{\frac{5}{2} \leqslant z \leqslant 3\} \subset A_2$. Since X is bounded, we have

$$\|X\| \geqslant \|\chi_2 z^n g\|/\|\chi_1 z^n f\| \geqslant 5^n \|\chi_{22} g\|/\|\chi_1 f\|$$

for all n; hence $\chi_{22} g = 0$ and therefore, since g is holomorphic, $g = 0$. Thus, $X = 0$.

A similar argument works for $YT_2 = T_1 Y$ since $Y(\chi_2 z^{-n} g) = z^{-n}\chi_1 f$ for all n. Hence, our example satisfies the conditions of Theorem 1 and we have:

THEOREM 2. *There exists a subnormal operator T such that not every element of $\{T\}''$ lifts to $\{N\}'$.*

Since in our example each T_i is an injective bilateral weighted shift, we know that [5], p. 62,

$$\{T_i\}' = \{T_i\}'' = \{M_{\varphi_i} \in B(\mathscr{M}_i):$$
$$M_{\varphi_i} f = \varphi_i f,\ \varphi_i(z) = \sum_{n=-\infty}^{\infty} \varphi_i(n) z^n, \text{ and } \varphi_i\mathscr{M}_i \subset \mathscr{M}_i\}.$$

Hence, $S = M_{\varphi_1} + M_{\varphi_2} \in \{T\}'$ lifts to $\{N\}'$ if and only if $\varphi_1(z) = \varphi_2(z)$ for $\frac{1}{2} \leqslant |z| \leqslant 2$, i.e. $\varphi_i(z) = \chi_i\varphi(z)$ for some $\varphi(z)$ holomorphic on A. Thus, we have [5], p. 91.

COROLLARY 3. *Every element of $\{T\}'$ that lifts to $\{N\}'$ is the limit in the strong operator topology of a sequence of Laurent polynomials in T.*

Our description of $\{T\}'$ also yields the following, which answers the second question of Abrahamse.

COROLLARY 4. *T is irreducible.*

Proof. T has a reducing subspace if and only if there is an orthogonal projection in $\{T\}'$. Since the only idempotent holomorphic function on a connected domain is 0 or 1, we see the only idempotents in $\{T\}'$ are $I+0$ and $0+I$. However, neither of these is self-adjoint.

By Yoshino's results [6], it follows that T is not cyclic, Bram, in fact, showed that each T_i is not cyclic although its m.n.e., multiplication by z on $L^2(d\mu)$ is clearly cyclic. In fact, each T_i is 2-cyclic and hence T is finitely cyclic. This results follows from the more general.

THEOREM 5. *Let $\{a_n:\ n = 0, \pm 1, \ldots\}$ be a weight sequence for the bilateral weighted shift $Te_n = a_n e_{n+1}$, $n = 0, \pm 1, \ldots$ Then T is 2-cyclic.*

Proof. Let P project into H^-, the closed linear span of $\{e_n:\ n \leqslant 0\}$ and consider $S = PT|_{H^-}$ the compression of T to H^-. Clearly, S is unitarily equivalent to a unilateral left shift and by [3, p. 82], S has a cyclic vector f. Then $\{f, e_0\}$ is cyclic for T so the theorem follows.

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