measure is in $L^1(\Omega)$ it follows that to each compact set $K \subset \Omega$ there is a number $C$ such that
\[ |V_n| \leq C \cdot L \text{ in } K. \]
Hence
\[ \int_{\Omega} F_n^2 d(Q) d\mu(Q) \leq \int_{\Omega} \frac{1}{\mu} \int_{\Gamma} Q(P) d\sigma + C L^2 \int_{\Omega} w^2 d\sigma, \]
which completes the proof of the corollary.

References


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Lifting subnormal double commutants

by

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Abstract. We give a necessary condition for elements of the commutant of a subnormal operator on a Hilbert space to lift to the commutant of the minimal normal extension, and based on this condition we show there exists a subnormal operator having elements in its double commutant that do not lift. Our example is also irreducible and finitely cyclic.

1. Let $T$ be a subnormal operator on a Hilbert space $H$ with minimal normal extension (m.n.e.) $N$ on $K$. Thus, $N$ is a normal operator on a space $K$ containing $H$, $H$ is invariant for $N$, the restriction $N|_H = T$, and $K$, the smallest reducing subspace of $N$ containing $H$, is the closed linear span of \( \{ N^n h : j = 0, 1, \ldots, n \in H \} \). Clearly, for any polynomial $p$, $p(N)|_H = p(T)$ and it is therefore easy to see that every element in the weakly closed algebra generated by $T$ lifts to, i.e. is the restriction to $H$ of, some element in the commutant of $N$. This result is included in the well-known paper of J. Bram, who gave the following necessary and sufficient condition for an element of $\{ T^n \}$, the commutant of $T$, to lift to $\{ N^n \}$:

THEOREM II [2, p. 87]. A necessary and sufficient condition that $A \in \{ T^n \} \subset B(H)$ have an extension $B \in \{ N^n \} \subset B(K)$ is that there exist a positive constant $\epsilon$ such that for every finite set $h_1, \ldots, h_m$ in $H$ we have
\[ \sum_{n=1}^{m} \sum_{k=1}^{n} (T^n A h_k, T^n A h_k) < \epsilon \sum_{n=1}^{m} \sum_{k=1}^{n} (T^n h_k, T^n h_k). \]

If the extension $B$ exists, then it is unique.

Bram pointed out that there exist subnormal operators $T$ such that not every element of $\{ T^n \}$ lifts to $\{ N^n \}$. T. Yoshino, however, showed that every element of $\{ T^n \}$ does lift to $\{ N^n \}$ if $T$ has a cyclic vector [6], p. 49. (Note that there is an error in the proof of this theorem but the result is true.) M.B. Abramovsky and C. Berger raised the question of whether

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for a general subnormal \( T \) every element of \((T')'\), the double commutant of \( T \), lifts to \( (N)' \) [1], Problem 2. Abramson also asked whether every element of \((T')'\) must lift to \( (N)'\) if \( T \) is irreducible [1], Problem 1. We give an example below which provides a negative answer to both questions; we note the R. Ohm and J. Thomson independently discovered an example with the same properties. Although the main idea underlying their example is similar to ours, we present a somewhat different approach to the problem. We conclude by showing that a certain class of subnormal operators considered by Baum [2] is finitely cyclic.

2. Our example is based on the following theorem which describes a situation in which elements of \((T)'\) do not lift. In Section 3 we will construct an operator satisfying the conditions of Theorem 1.

For operators \( T_{1} \in B(H_{1}) \) and \( T_{2} \in B(H_{2}) \), we denote \( I(T_{1}, T_{2}) = \{ X \in B(H_{1}, H_{2}); X T_{1} = T_{2} X \} \). Two subspaces \( \mathcal{H}_{1} \) and \( \mathcal{H}_{2} \) of \( H \) are called complementary if \( \mathcal{H}_{1} \cap \mathcal{H}_{2} = \{0\} \) and \( H = \mathcal{H}_{1} + \mathcal{H}_{2} = \{ m_{1} + m_{2}; m_{1} \in \mathcal{H}_{1}; m_{2} \in \mathcal{H}_{2} \} \) (note that this implies that \( \mathcal{H}_{1} + \mathcal{H}_{2} \) is closed), and we denote such a sum by \( \mathcal{H}_{1} + \mathcal{H}_{2} \).

**Theorem 1.** Suppose \( T_{1} \), a subnormal operator on \( H \) having m.n.c. \( N \) on \( K \), has a pair of complementary invariant subspaces \( \mathcal{H}_{1} \) and \( \mathcal{H}_{2} \). For \( i = 1, 2 \), denote \( T_{i} = T_{i} \vert_{\mathcal{H}_{i}} \) and let \( K_{i} \) be the closed linear span of \( \bigoplus_{j=0}^{N} \{ b_{j} \in \mathcal{H}_{j}; b_{j} \in \mathcal{H}_{j} \} \). Suppose \( I(T_{1}, T_{2}) = \{0\} \) and \( I(T_{2}, T_{1}) = \{0\} \). Suppose further that \( K_{1} \cap K_{2} = \{0\} \). Then there exist elements of \((T)'\) that do not lift to \((N)'.\)

**Proof.** With respect to the decomposition \( H = \mathcal{H}_{1} + \mathcal{H}_{2} \), we write \( T = T_{1} + T_{2} = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \).

Matrix multiplication shows that \( (T)' = \begin{bmatrix} A_{2} & X \\ Y & A_{1} \end{bmatrix} : A_{i} \in (T_{i})', X \in I(T_{1}, T_{2}), Y \in I(T_{2}, T_{1}) \), so under our hypotheses on the interwining spaces,

\( (T)' = (T_{1})' + (T_{2})' = \begin{bmatrix} A_{1} & 0 \\ 0 & A_{2} \end{bmatrix} : A_{1} \in (T_{1})' \).

It then follows that \( (T)' = (T_{1})' + (T_{2})' = \begin{bmatrix} A_{1} & 0 \\ 0 & A_{2} \end{bmatrix} : A_{1} \in (T_{1})' \).

Let \( A = A_{1} + A_{2} \in \{T\} \subset B(H) \) and suppose \( A \) lifts to \( (N)' \). Then there exists \( B \in \{N\}' \subset B(K) \) such that \( B |_{K} = A \). Note that \( K_{1} \) reduces \( N \) and \( K_{1} = N |_{K_{1}} \) is the m.n.c. of \( T_{1} \). Then \( B_{1} = B |_{K_{1}} \) is an extension of \( A_{1} \)

and since

\[
B_{1}N_{t} = B_{1}N_{t} |_{K_{1}} = B_{1}N |_{K_{1}} = N_{t}B |_{K_{1}} = N_{t}R_{t}B_{1},
\]

we see that \( A_{1} \) lifts to \( \{N_{t}\}' \) and we denote that lifting by \( \bar{A}_{1} \). Since the lifting is unique by Theorem 1, we have that \( A_{1} = B_{1} \), \( t = 1, 2 \).

\[
\bar{A}_{1} |_{K_{1} \cap K_{2}} = B_{1} |_{K_{1} \cap K_{2}} = B_{1} |_{K_{1} \cap K_{2}} = B_{1} |_{K_{1} \cap K_{2}} = \bar{A}_{1} |_{K_{1} \cap K_{2}} \]

i.e., a necessary condition for \( A_{1} \mid A_{2} \) to lift to \( (N)' \) is that \( A_{1} \) and \( A_{2} \) each lift to its respective m.n.c. space \( K_{1} \) and \( K_{2} \) and that the liftings agree on \( K_{1} \cap K_{2} \). This condition is clearly also sufficient.

Now, for the case in hand, let \( A_{1} = 0 |_{\mathcal{H}_{1}} \) and \( A_{2} = I |_{\mathcal{H}_{2}} \), so \( A_{1} + A_{2} \in \{T\}' \). Using the above notation, \( \bar{A}_{1} = 0 |_{\mathcal{H}_{1}} \) and \( \bar{A}_{2} = I |_{\mathcal{H}_{2}} \) and since \( (K_{1} \cap K_{2}) \) is non-trivial, \( A \) does not lift to \( (N)' \) and the theorem is proved.

3. Let \( \mu \) be Lebesgue area measure on the annulus \( A = \{ \frac{1}{2} \leq |z| \leq 3 \} \) and consider the overlapping subannuli \( A_{1} = \{ \frac{1}{2} \leq |z| \leq 2 \} \) and \( A_{2} = \{ \frac{1}{2} \leq |z| \leq 3 \} \) with characteristic functions \( \chi_{1} \) and \( \chi_{2} \) respectively. (The specific radii are chosen to simplify a following computation and are not essential to the example.) For \( \varepsilon = 1, 2 \), let \( \mathcal{H}_{\varepsilon} = (\chi_{\varepsilon}(z) \mathcal{D}(\mathbb{Z}); f \text{ is holomorphic on } A_{\varepsilon}). \)

Then \( \langle g_{\varepsilon} \rangle_{\varepsilon} = \{ g_{\varepsilon} \in \mathcal{D}(\mathbb{Z}); f \text{ is holomorphic on } A_{\varepsilon} \} \) is an orthogonal basis for \( \mathcal{H}_{\varepsilon} \) and \( \chi_{1} \rho_{1} \) is orthogonal to \( \chi_{2} \rho_{2} \) for \( \rho_{1}, \rho_{2} \) on \( \mathcal{H}_{\varepsilon} \). Thus \( \chi_{1} \rho_{1} \) and \( \chi_{2} \rho_{2} \) are orthogonal to \( \chi_{1} \rho_{1} \) and \( \chi_{2} \rho_{2} \) on \( \mathcal{H}_{\varepsilon} \).

We clearly have \( \mathcal{H}_{\varepsilon} \cap \mathcal{H}_{\bar{\varepsilon}} = \{0\} \). We claim that \( \mathcal{H}_{1} + \mathcal{H}_{2} \) is closed in \( L^{2}(\mu) \).

To see this, let \( f(z) = \sum_{n=0}^{\infty} f_{n} \chi_{1} \rho_{1} \in \mathcal{H}_{1} \) and \( g(z) = \sum_{n=0}^{\infty} g_{n} \chi_{2} \rho_{2} \in \mathcal{H}_{2} \).

Then

\[
\frac{1}{2\pi i} \langle f; g \rangle_{\lambda, \mu} = \int_{\mu} \sum_{n=0}^{\infty} f_{n} \overline{g_{n}} \, d\mu = \sum_{n=0}^{\infty} \int_{\mu} f_{n} \overline{g_{n}} \, d\mu = \sum_{n=0}^{\infty} \int_{\mu} f_{n} \overline{g_{n}} \, d\mu = \sum_{n=0}^{\infty} \int_{\mu} f_{n} \overline{g_{n}} \, d\mu = \sum_{n=0}^{\infty} \int_{\mu} f_{n} \overline{g_{n}} \, d\mu.
\]
Thus, the angle between the subspaces \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) is strictly positive and hence \( \mathcal{N}_1 + \mathcal{N}_2 \) is closed [4], p. 339. We note that this can also be shown directly.

Thus, using the notation of Theorem 1, we let \( H \) be the Hilbert space \( \mathcal{N}_1 + \mathcal{N}_2 \), and \( T \) be multiplication by \( s \) on \( H \). \( T \) is subnormal with m.n.e. equal to multiplication by \( s \) on \( L^2(\mu) \), and since \( T^* = T_{\mathcal{N}_2} \), we have \( K_1 = L^2(\mu_{\mathcal{N}_1}) = \{ \chi f \in L^2(\mu) \} \). Thus, \( K_1 \cap K_2 = \{ \chi f \in L^2(\mu) \} = L^2(\mu_{\mathcal{N}_1 \cap \mathcal{N}_2}) \neq 0 \). It remains to show that the intertwining spaces are trivial.

Suppose \( X \in B(\mathcal{N}_1, \mathcal{N}_2) \) with \( T_X X = X T \). For \( \chi f \in \mathcal{N}_1 \), let \( X(\chi f) = \chi g \) with \( g \) holomorphic in \( \mathcal{A}_1 \). Since \( T^* \mathcal{N}_1 = \mathcal{N}_1 \mathcal{N}_2 \), for \( n = 1, 2, \ldots \), we have \( X(\chi f) = \chi g \in \mathcal{A}_n \mathcal{N}_2 \). We have

\[
\|X \xi f \|^2 = \int_{\mathcal{A}_1} |\xi f|^2 \, d\mu = \int_{\mathcal{A}_1} \phi_n |f|^2 \, d\mu < 4^n \|\chi f\|^2
\]

and

\[
\|X \xi g \|^2 = \int_{\mathcal{A}_2} |\xi g|^2 \, d\mu = \Gamma_{\mathcal{A}_2} \int_{\mathcal{A}_2} |\xi g|^2 \, d\mu = \int_{\mathcal{A}_2} \phi_n |g|^2 \, d\mu
\]

\[
\geq (\frac{1}{2^n})^2 \int_{\mathcal{A}_2} |\xi g|^2 \, d\mu = (\frac{1}{2^n})^2 \|\chi g\|^2,
\]

where \( A_{22} = \{ \frac{1}{2} < s < 1 \} \subset A_1 \), since \( X \) is bounded, we have

\[
\|X\| \geq \|X \xi g\| / \|X \xi f\| \geq 4^n \|\xi g\| / \|\xi f\|
\]

for all \( n \); hence \( X \xi g = 0 \) and therefore, since \( g \) is holomorphic, \( g = 0 \).

Thus, \( X = 0 \).

A similar argument works for \( Y \mathcal{N}_2 = \mathcal{N}_2 \) since \( Y(\chi g \Phi^\nu g) = \sigma^\nu \chi f \) for all \( n \). Hence, our example satisfies the conditions of Theorem 1 and we have:

**Theorem 2.** There exists a subnormal operator \( T \) such that not every element of \( \{ T \}' \) lifts to \( \{ N \}' \).

Since in our example each \( T \) is an injective bilateral weighted shift, we know that [5], p. 62,

\[
\{ T \}' = \{ T \}_r + \{ M_{\xi} \in B(\mathcal{N}_2) : \}
\]

\[
M_{\xi} f = \psi f, \quad \psi(s) = \sum_{n=0}^\infty \varphi(n)s^n, \quad \text{and} \quad \varphi, \varphi^* \in \mathcal{A}_2.
\]

Hence, \( S = M_{\xi} + M_{\xi}^* \in \{ T \}' \) lifts to \( \{ N \}' \) if and only if \( \psi(s) = \psi^*(s) \) for \( s \leq 2 \). I.e., \( \varphi(s) = \chi \varphi(s) \) for some \( \varphi(s) \) holomorphic on \( A \). Thus, we have [5], p. 91.

**Corollary 3.** Every element of \( \{ T \}' \) that lifts to \( \{ N \}' \) is the limit in the strong operator topology of a sequence of Laurent polynomials in \( T \).

Our description of \( \{ T \}' \) also yields the following, which answers the second question of Abrahamian.

**Corollary 4.** \( T \) is irreducible.

Proof. \( T \) has a reducing subspace if and only if there is an orthogonal projection in \( \{ T \}' \). Since the only idempotent holomorphic function on a connected domain is 0 or 1, we see the only idempotents in \( \{ T \}' \) are \( 1 + 0 \) and \( 0 + 1 \). However, neither of these is self-adjoint.

By Yoshino's results [6], it follows that \( T \) is not cyclic, Bram, in fact, showed that each \( T \) is not cyclic although its m.n.e. multiplication by \( s \) on \( L^2(\mu) \) is clearly cyclic. In fact, each \( T \) is 2-cyclic and hence \( T \) is finitely cyclic. This result follows from the more general.

**Theorem 5.** Let \( \{ a_n \}_{n=0} \) be a weight sequence for the bilateral weighted shift \( T_n = a_n a_{n+1} \), \( a_n = 0, \pm 1, \ldots \) Then \( T \) is 2-cyclic.

Proof. Let \( T \) project into \( H' \), the closed linear span of \( \{ a_n \} \) \( n \leq 0 \) and consider \( S = \Phi T H' \), the compression of \( T \) to \( H' \). Clearly, \( S \) is unitarily equivalent to a unilateral shift and by [3, p. 82], \( S \) has a cyclic vector \( f \). Then \( \langle f, e_n \rangle \) is cyclic for \( T \) so the theorem follows.

**References**


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