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Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for functions harmonic in a Lipschitz domain

by

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Abstract. We prove weighted integral inequalities between the Lusin area integral and the nontangential maximal function of a function harmonic in a Lipschitz domain. These inequalities are extensions to the Lipschitz case of inequalities obtained by Gundy and Wheeden [7] for functions harmonic in a half space.

1. Introduction. In this paper we shall prove integral inequalities between area integrals and nontangential maximal functions for functions harmonic in a Lipschitz domain $\Omega \subset R^n$. That is, we shall assume that to each boundary point $P \in \partial\Omega$ there is associated an open cone $\Gamma(P)$ with vertex at P such that $\Gamma(P) \subset \Omega$. If now u is harmonic in Ω we define

$$A(u, P) = \Big(\int_{\Gamma(P)} |P-Q|^{2-n} |\nabla u(Q)|^2 dm(Q)\Big)^{\frac{1}{2}}$$

and

$$N(u, P) = \sup_{\Gamma(P)} |u(Q)|.$$

Here ∇u denotes the gradient of u and m denotes the Lebesgue measure. Our main result is that if the cones $\Gamma(P)$ satisfy suitable regularity conditions (to be formulated later) then for all harmonic functions u vanishing at a fixed point P^* we have

$$(1.1)\qquad C_1 \int_{\partial\Omega} \Phi\big(A(u)\big)\, d\mu \leqslant \int_{\partial\Omega} \Phi\big(N(u)\big)\, d\mu \leqslant C_2 \int_{\partial\Omega} \Phi\big(A(u)\big)\, d\mu .$$

Here μ is allowed to vary over a wide class of measures which includes the surface measure of $\partial\Omega$ and the harmonic measure. The precise assumption on μ is that μ is positive, nonvanishing on any component of $\partial\Omega$ and that there are numbers $A > 0$ and $\theta > 0$ such that for all $P \in \partial\Omega$ and all $r > 0$ we have that whenever $E \subset \Delta(P, r)$ then

$$(1.2)\qquad \frac{\mu(E)}{\mu(\Delta(P, r))} \leqslant A\left(\frac{\sigma(E)}{\sigma(\Delta(P, r))}\right)^{\theta} \quad\text{and}\quad \mu(\Delta(P, 2r)) \leqslant C\mu(\Delta(P, r)).$$

Here σ denotes the surface measure of $\partial\Omega$ and $A(P,r) = \partial\Omega \cap B(P,r)$, where $B(P,r)$ denotes the ball with center P and radius r. A measure satisfying these assumptions is said to satisfy condition A_∞. (The fact that the harmonic measure satisfies condition A_∞ is a consequence of the results in Dahlberg [4].) The assumption on $\Phi\colon [0,\infty[\to [0,\infty[$ are that Φ is unbounded, nondecreasing, continuous and $\Phi'(2t) \leqslant C\Phi'(t)$.

Inequalities of the type (1.1) have been proved by many authors. For the classical case $(\Phi(t) = t^p,\ p > 1)$ we refer to Stein [13]. When Ω is a halfspace, Fefferman and Stein [6] showed (1.1) for μ being the surface measure and $\Phi(t) = t^p$, $p > 0$. This was generalized by Burkholder and Gundy [2] to include the general class of Φ introduced above. Their results were then extended by Gundy and Wheeden [7] to the general class of weights (1.2) (but still for Ω being a half-space).

The plan of the paper is the following. After some preliminary material in Sections 2 and 3 we prove in Section 4 our basic estimates, which compare the behaviour of A and N on special sets. The key ingredient in the proof of this is a comparison of the harmonic measures of certain sets, which is made possible by the fact that the density of the harmonic measure with respect to the surface measure satisfies a reversed Hölder inequality (see Dahlberg [4]). In Section 5 we prove our main result and we conclude with some final remarks in Section 6.

2. Statement of results. We recall that a bounded domain $\Omega \subset R^n$ is called a Lipschitz domain if $\partial\Omega$ can be covered by finitely many open right circular cylinders whose bases have positive distance from $\partial\Omega$ and corresponding to each cylinder L there is a coordinate system (x, y) with $x \in R^{n-1}$, $y \in R$, with the y-axis parallel to the axis of L, and a function $\varphi\colon R^{n-1} \to R$ satisfying a Lipschitz condition (i.e. $|\varphi(x) - \varphi(z)| \leqslant M|x-z|$) such that $L \cap \Omega = \{(x,y)\colon y > \varphi(x)\} \cap L$ and $L \cap \partial\Omega = \{(x,y)\colon y = \varphi(x)\} \cap L$. (We shall say that Ω is *defined inside* L *by* φ.)

From now on a cone means a non-empty, convex, truncated and circular cone.

Suppose that $\{\Gamma(P)\colon P \in \partial\Omega\}$ is a family of cones such that the vertex of $\Gamma(P)$ is P for all $P \in \partial\Omega$. We say that this family is *regular* with respect to the Lipschitz domain Ω if there is a covering of $\partial\Omega$ with finitely many open sets U such that the following requirements are satisfied: To each set U there is an open, right circular cylinder L such that $\overline{U} \subset L$ and $\partial\Omega$ is defined inside L by a Lipschitz function. Also, to this U there are associated three cones γ_i, $i = 1, 2, 3$, all with its vertex at 0 and its axis along the axis of L such that $\gamma_1 \subset \overline{\gamma}_2 - \{0\} \subset \gamma_3$ and for all $P \in L \cap \partial\Omega$ we have

$$\gamma_1 + P \subset \Gamma(P) \subset \overline{\Gamma(P)} - \{P\} \subset \gamma_2 + P \quad \text{and} \quad \gamma_3 + P \subset \Omega \cap L.$$

We shall put

$$A(u, \Gamma, P) = \Big(\int_{\Gamma(P)} |\nabla u|^2 |P-Q|^{2-n}\, dm(Q)\Big)^{\frac12}$$

and

$$N(u, \Gamma, P) = \sup_{\Gamma(P)} |u(Q)|.$$

We recall that a positive measure μ on $\partial\Omega$ satisfies an A_∞*-condition* if condition (1.2) holds. For a discussion of the properties of such measures we refer to Coifman and Fefferman [3] and Gundy and Wheeden [7].

We can now formulate our main result.

THEOREM 1. *Let $\Omega \subset R^n$, $n \geqslant 2$, be a Lipschitz domain and let Φ: $[0, \infty[\to [0, \infty[$ be an unbounded nondecreasing, continuous function such that $\Phi(0) = 0$ and $\Phi(2\lambda) \leqslant C\Phi(\lambda)$. Let P^* denote a fixed point in Ω. Suppose that $\{\Gamma(P)\colon P \in \partial\Omega\}$ is a regular family of cones. Suppose μ is a positive measure on $\partial\Omega$ satisfying condition A_∞. Then there are positive constants C_1 and C_2 such that if u is harmonic in Ω and vanishes at P^*, then*

$$C_1 \int \Phi(N)\, d\mu \leqslant \int \Phi(A)\, d\mu \leqslant C_2 \int \Phi(N)\, d\mu,$$

where N and A denotes $N(u, \Gamma, \cdot)$ and $A(u, \Gamma, \cdot)$ respectively. The constants C_1 and C_2 are independent of u but depend on Ω, Φ, P^* and the family $\{\Gamma(P)\}$.

We would like to remark that in Dahlberg [5] it is proved that if $\infty \geqslant p \geqslant 2$, then

$$\Big(\int_{\partial\Omega} N^p\, d\sigma\Big)^{1/p} \leqslant C_p \Big(\int_{\partial\Omega} |u|^p\, d\sigma\Big)^{1/p} \tag{2.1}$$

and if Ω is assumed to be C^1, then (2.1) holds for $1 < p \leqslant \infty$. Therefore, it follows from (2.1) that if $u(P^*) = 0$, then for the appropriate range of p: s the $L^p(\sigma)$-norm of u : s boundary values and the L^p-norm of $A(u, \Gamma, \cdot)$ are equivalent.

As we shall see in Section 6, there are positive constants C_i, $i = 1, 2$, such that if $u(P^*) = 0$, then

$$C_1 \int A^2\, d\sigma \leqslant \int_\Omega |\nabla u|^2 d(Q)\, dm(Q) \leqslant C_2 \int A^2\, d\sigma,$$

where $d(Q)$ denotes the distance from Q to $\partial\Omega$. Hence it follows from (2.1) that if u is harmonic in Ω and $u(P^*) = 0$, then the L^2-norm of u : s boundary values is equivalent to the L^2-norm of the gradient with respect to the measure $d(Q)\, dm$.

3. Technical preliminaries. In this section we shall establish some technical tools for proving our basic estimates in Section 4.

We start by recalling some concepts. The *Lipschitz constant* of a Lipschitz function φ is the smallest number M such that the inequality

$|\varphi(x)-\varphi(z)| \leqslant M|x-z|$ holds. If Ω is a Lipschitz domain we say that the *Lipschitz constant of Ω is less than M* if the Lipschitz constants of those Lipschitz functions appearing in the definition of Ω can be chosen to be $\leqslant M$.

We recall that a Lipschitz domain Ω is called starshaped with respect to P^* with standard inner cone Γ if $P^* \in \Gamma(P) \subset \Omega$ for all $P \in \partial\Omega$, where $\Gamma(P)$ denotes the cone with vertex P having its axis along the line between P and P^* and being congruent to Γ.

We shall from now on in this section assume that Ω is a Lipschitz domain with Lipschitz constant less than M, being starshaped with respect to 0 with standard inner cone Γ. In the sequel we shall make certain estimates and in those estimates the letters C and K will denote constants which may depend on the dimension n, M and Γ but not on anything else.

We shall let G denote the Green function of Ω and $g = G(\cdot, 0)$. If $P \in \Omega$ we let $\omega(P, \cdot)$ denote the harmonic measure of Ω evaluated at P. If $P = 0$ we write, however, $\omega(0, \cdot) = \omega$. As is easily seen that there is a number $r_0 = r_0(M, \Gamma) > 0$ such that $B(0, 10r_0) \subset \Omega$. We will write $\Omega^* = \Omega - \overline{B(0, r_0)}$. If $P \in \Omega - \{0\}$ we let $\hat{P}$ denote the intersection of $\partial\Omega$ with $\{rP\colon r > 0\}$.

We start by giving the following variant of Dahlberg [4], Lemma 1.

LEMMA 1. *Let $d(P)$ denote the distance from P to $\partial\Omega$. Then there is a number $K > 0$ such that for all $P \in \Omega^*$ we have*

$$K^{-1}d(P)^{n-2}g(P) \leqslant \omega\big(A(\hat{P}, d(P))\big) \leqslant Kd(P)^{n-2}g(P). \tag{3.1}$$

Proof. We shall assume $n \geqslant 3$ and leave the easy modifications for the case $n = 2$ to the reader.

There is a number $K > 0$ such that $0 \notin B\big(P, 2Kd(P)\big) \subset \Gamma(\hat{P})$. Let B denote $B\big(P, Kd(P)\big)$. If $Q \in B$ then

$$q(P, Q) \leqslant G(P, Q) \leqslant |P-Q|^{2-n}, \tag{3.2}$$

where q is the Green function of $B\big(P, 2Kd(P)\big)$. From the explicit representation of q, see Helms [8], follows $\inf\{q(P, Q)\colon Q \in B\} \geqslant Kd(P)^{2-n}$. Choose the number $t \in (0, 1)$ such that $t\hat{P} = P$ and let u denote the function $u(Q) = G(P, tQ)$. Then u is non-negative and superharmonic in Ω. If $Q \in A\big(\hat{P}, Kd(P)\big)$ then $|tQ - t\hat{P}| = |tQ - P| \leqslant Kd(P)$. Hence it follows from the left hand side of (3.2) that $Kd(P)^{n-2}u(Q) \geqslant 1$. Since $v = \omega\big(\cdot, A(\hat{P}, Kd(P))\big)$ is harmonic, $\leqslant 1$ with vanishing boundary values outside $A\big(\hat{P}, Kd(P)\big)$ it follows from the maximum principle that $Kd(P)^{n-2}u(Q) \geqslant v(Q)$ for all $Q \in \Omega$. Taking $Q = 0$ yields that

$$Kd(P)^{n-2}g(P) \geqslant \omega\big(A(\hat{P}, Kd(P))\big).$$

It is now a well known fact that

$$\omega\big(A(P, 2r)\big) \leqslant K\big(A(P, r)\big), \tag{3.3}$$

see Hunt–Wheeden [9], p. 311. This yields the right-hand side of inequality (3.1).

To prove the remaining part of (3.1) we recall that there is a number $C > 0$ such that $\omega\big(Q, A(\hat{P}, d(P))\big) \geqslant C$ for all $Q \in \bar{B}$, see Hunt–Wheeden [9], Lemma 1. From (3.2) follows that if $Q \in \partial B$, then $G(P, Q) \leqslant Cd(P)^{2-n}$. Since the function $G(P, \cdot)$ is harmonic in $\Omega - \bar{B}$ with vanishing boundary values on $\partial\Omega$, it follows from the maximum principle that $CG(P, Q)d(P)^{n-2} \leqslant \omega\big(Q, A(\hat{P}, d(P))\big)$ for all $Q \in \Omega - \bar{B}$. Taking $Q = 0$ yields the remaining part of the lemma.

We shall now study the relationship between ω and σ, where σ denotes the surface measure of $\partial\Omega$. It is known that ω and σ are mutually absolutely continuous (see Dahlberg [4]) and if we write $d\omega = kd\sigma$, then $k \in L^2(\sigma)$ and k satisfies a reversed Hölder condition. What is of decisive importance to us is that the constants only depend on n, M and Γ.

LEMMA 2. *There is a number $K > 0$ such that for all $P \in \partial\Omega$ and all $r > 0$ we have*

$$\left(\frac{1}{\sigma(A)}\int_A k^2\,d\sigma\right)^{1/2} \leqslant K\frac{1}{\sigma(A)}\int_A k\,d\sigma,$$

where $A = A(P, r)$.

Proof. We first observe that there is no loss in generality in assuming $0 < r \leqslant r_0$. Put $v = r\,\partial g/\partial r$. Then it is well known that v is non-negative (see Warschawski [14]), superharmonic in Ω and harmonic in $\Omega - \{0\}$. Hence v has non-tangential boundary values a.e. on $\partial\Omega$ (see Hunt–Wheeden [9]), which we denote by f. Since there are constants $C_i > 0$, $i = 1, 2$, such that $C_1 r^{n-1} \leqslant \sigma\big(A(P, r)\big) \leqslant C_2 r^{n-1}$ for all $P \in \partial\Omega$ it follows from Lemma 1 that

$$C_1 k \leqslant f \leqslant C_2 k \quad \text{a.e. on } \partial\Omega. \tag{3.4}$$

Let q denote the characteristic function of A and let u be the Poisson integral of fq. Then $0 \leqslant u \leqslant v$ and u has vanishing boundary values outside A. If P_r denotes the point on the line segment between P and 0 having distance r to P, then it is well known that

$$u(0) \leqslant Cu(P_r)\,\omega(A), \tag{3.5}$$

see Hunt–Wheeden [10], p. 512. Now there is a number $c > 0$ such that $B(P_r, cr) = B \subset \Omega$. We now recall that if a function h is non-negative and harmonic in a ball $B(Q, \varrho)$, then $|\nabla h(Q)| \leqslant Ch(0)\varrho^{-1}$, where C only depends on n. Since g is non-negative and harmonic in B it follows that $u(P_r) \leqslant v(P_r) \leqslant C|\nabla g(P_r)| \leqslant Cr^{-1}g(P_r)$. Recalling Lemma 1 and (3.3) it follows that

$$u(P_r) \leqslant C\omega(A)r^{1-n} \leqslant C\frac{\omega(A)}{\sigma(A)}.$$

Hence we have from (3.5) that

$$\sigma(A)u(0) \leqslant C(\omega(A))^2.$$

However $\omega(A) = \int_A k\,d\sigma$ and $u(0) = \int_A fk\,d\sigma \geqslant C_1 \int_A k^2 d\sigma$ by (3.4), which yields the lemma.

The following consequence from Lemma 2 is well known (see Coifman and Fefferman [3]): There are positive constants C, K, a, b such that for all $P \in \partial\Omega$ and all $r > 0$ and all subsets E of $A = A(P, r)$ we have

$$\frac{\sigma(E)}{\sigma(A)} \leqslant C\left(\frac{\omega(E)}{\omega(A)}\right)^a \quad \text{and} \quad \frac{\omega(E)}{\omega(A)} \leqslant K\left(\frac{\sigma(E)}{\sigma(A)}\right)^b. \tag{3.6}$$

We shall recall some maximal inequalities that will be useful. First if $f \in L^1(\omega)$ and if

$$f^*(P) = \sup \frac{\int_{B\cap\partial D} |f|\,d\omega}{\omega(B\cap\partial D)}$$

where B runs over all balls with center at ∂D and containing P, then it follows from Besicovitch' theory of differentiation [1] that

$$\omega\{f^* > \lambda\} \leqslant C\lambda^{-1}\int |f|\,d\omega, \tag{3.7}$$

where C only depends on n.

Also, it is known that if $f \in L^1(\omega)$ and Hf denotes the Poisson integral of f, then

$$\sup_{\gamma(P)} |Hf| \leqslant Cf^*(P), \tag{3.8}$$

see Hunt and Wheeden [9], Lemma 4.

We conclude by remarking that the estimates in this section are invariant under a change of scale.

4. Main estimates. We shall in this section compare the boundary behaviour of the Lusin area integral and the non-tangential maximal function.

We let (x, y), $x \in R^{n-1}$, $y \in R$ be a fixed coordinate system. We denote by S the set of all cones with vertex at the origin and having its axis along the y-axis.

We shall work with a Lipschitz function $\varphi\colon R^{n-1} \to R$ having its Lipschitz constant less than M and $\varphi(0) = 0$. We shall assume that $\Gamma \in S$

is a cone such that there is a cone $\Gamma^* \in S$ with the properties that $\Gamma - \{0\} \subset \Gamma^*$ and $\Gamma^* + P \subset D$ for all $P \in \partial D$, where $D = \{(x, y)\colon y > \varphi(x)\}$. A well-known property of Lipschitz functions is that we can find numbers $r_0 = r_0(M, \Gamma)$, $a = a(M, \Gamma)$ $L = L(M, \Gamma)$ and a cone $T \in S$ such that whenever $P = (x_0, y_0) \in \partial D$, $0 < r \leqslant r_0$ and $E \subset \{(x_1\varphi(x))\colon |x - x_0| \leqslant r\}$ then $\left[\bigcup_{Q\in E} \Gamma + Q\right] \cap \{(x, y)\colon y < 2ar\} = D(E, P, r)$ is a starshaped Lipschitz domain with starcenter $P + (0, ar)$ and standard inner cone rT with Lipschitz constant less than L. (Notice that the diamater of $D(E, P, r)$ is $\approx r$.)

We shall assume that $\gamma \in S$ and $\bar{\gamma} - \{0\} \subset \Gamma$ and r_0 is very small compared to the height of γ.

We shall assume that u is harmonic in a neighbourhood of 0 and in the next lemma we shall use the notation

$$A(P) = \left(\int_{\gamma(P)} |\nabla u|^2 |P - Q|^{2-n}\, dm(Q)\right)^{\frac{1}{2}} \quad \text{and} \quad N(P) = \sup_{\Gamma(P)} u(Q),$$

where $\gamma(P) = \gamma + P$ and $\Gamma(P) = \Gamma + P$, $P \in \partial D$.

LEMMA 3. *Let* $V = \{(x, \varphi(x))\colon |x| \leqslant r\}$ *and assume that* $r < r_0$ *and* $A(P) \leqslant \lambda$ *for some point* $P_0 \in V$. *Given* $\alpha > 1$ *and* $\beta > 1$ *there is a number* $\varepsilon > 0$ *which can be taken to depend only on* M, Γ, γ, α *and* β *such that*

$$\alpha\sigma\{P \in V\colon A(P) > \beta\lambda,\ N(P) \leqslant \varepsilon\lambda\} \leqslant \sigma(V),$$

where σ *denotes the surface measure of* ∂D.

Proof. We shall first show that the main contribution to $A(P)$ comes from the behaviour of u "near" P. For $t > 0$ let

$$U = \{Q \in \Gamma(P)\colon |Q - P| > tr\}.$$

Then

$$\int_U |\nabla u|^2 |Q - P|^{2-n}\, dm(Q) = \int_{U\cap\Gamma(P_0)} + \int_{U\setminus\Gamma(P_0)}.$$

Notice that there is a function $a(t) \to 1$ as $t \to \infty$ such that $|Q - P|^{2-n} \leqslant a(t)|Q - P_0|^{2-n}$ for $Q \in U$. Hence

$$\int_{U\cap\Gamma(P_0)} |\nabla u|^2 |Q - P|^{2-n}\, dm(Q) \leqslant a(t)A^2(P_0).$$

Also notice that there are numbers t_0 and C_1 the choice of which only depends on M and γ such that if $t \geqslant t_0$ then the surface area $\nu(s)$ of the set $\{(x, y)\colon y = s\} \cap U - \Gamma(P_0)$ satisfies $\nu(s) \leqslant Crs^{n-2}$. Observing that there is a number $C = C(\Gamma, \gamma)$ such that $|\nabla u(Q)| \leqslant C\varepsilon|P - Q|^{-1}$ for $Q \in \gamma(P)$ we get

$$\int_{U-\Gamma(P_0)} |\nabla u|^2 |Q - P|^{2-n}\, dm(Q) \leqslant C\varepsilon^2\lambda^2 r\int_{\nu}^{\infty} s^{-n}\nu(s)\,ds \leqslant C\varepsilon^2\lambda^2.$$

Hence we can choose t and ε so that given any number $q > 1$ we have

$$\int_U |\nabla u|^2 |Q-P|^{2-n}\, dm(Q) \leqslant q\lambda^2.$$

Also

$$\int_{\gamma(P)\cap\{Q:\ t\tau<|P-Q|<\tau r\}} |\nabla u|^2 |P-Q|^{2-n}\, dm(Q) \leqslant C\varepsilon^2\lambda^2 \log\frac{t}{\tau}.$$

Summing up, this means that given $\tau > 0$ there is an ε, which only depends on β, τ, Γ and γ such that if $P \in E = \{P \in V:\ A(P) > \beta\lambda,\ N(P) \leqslant \varepsilon\lambda\}$, then

$$\int_{\gamma(P)\cap\{Q:\ |P-Q|<\tau r\}} |\nabla u|^2 |P-Q|^{2-n}\, dm(Q) > \left(\frac{\beta+1}{2}\right)\lambda^2.$$

We shall next choose τ.

Let $\Omega = D(E, 0, r)$ and let $P^* = (0, ar)$. Then there is a number η only depending on M and Γ such that $B(P^*, \eta r) \subset \Omega$. We now choose τ only depending on M, γ and Γ such that $\gamma_0(P) \subset D - B(P^*, \eta r)$, where $\gamma_0(P) = \gamma(P) \cap B(P, \tau r)$ and $P \in E$.

Let $P, Z \in E$. If $Q \in \gamma_0(P) \cap \gamma_0(Z)$, then $Z \in Q - \gamma_0 = \{Q - Q':\ Q' \in \gamma_0\}$. Let $\tilde{Q}$ denote the point on $\partial\Omega$ below Q. Because of the construction of Ω we know that $\tilde{Q} - \Gamma$ is contained in the complement of Ω. Hence there is a number C only depending on γ and Γ such that $Z \in B(\tilde{Q}, C|Q-\tilde{Q}|) \cap E$. Since $\Gamma + \tilde{Q} \subset \Omega$ there is a number C such that $|Q - \tilde{Q}| \leqslant C d(Q)$. Let as in Section 3 $\hat{Q}$ denote the point at which a ray originating from P^* and going through Q hits $\partial\Omega$. Then $|\hat{Q} - Q| \leqslant C d(Q)$ and therefore $Z \in A(\hat{Q}, C d(Q))$.

Put $a(P) = \int_{\gamma_0(P)} |\nabla u|^2\ |P-Q|^{2-n}\, dm(Q)$. We get from Fubini's theorem that

$$\int_E (aP)\, d\omega(P) \leqslant \int_W |\nabla u|^2 d(Q)^{2-n} \int_E \chi(P, Q)\, d\omega(P)\, dm(Q),$$

where $W = \bigcup_{P\in E} \gamma(P)$ and ω is the harmonic measure of Ω evaluated at 0. The function χ is defined by $\chi(P, Q) = 1$ if $Q \in \gamma_0(P)$ and zero otherwise. From the above considerations and Lemma 1 follow

$$\int_E \chi(P, Q)\, d\omega(P) \leqslant \omega\big(A(\hat{Q}, C d(Q))\big) \leqslant C d(Q)^{2-n} g(Q),$$

where g denotes the Green function of Ω with pole at P^*. Hence

$$\int_E a(P)\, d\omega(P) \leqslant \int_W |\nabla u|^2 g(Q)\, dm(Q) \leqslant \int_\Omega |\nabla u^2| g(Q)\, dm(Q).$$

Since $\Delta u^2 = 2|\nabla u|^2$ it follows from the Riesz representation theorem that

$$u^2(P^*) = \int_{\partial\Omega} u^2\, d\omega - 2\int |\nabla u|^2 g(Q)\, dm(Q).$$

Hence $\frac{1}{2}(\beta+1)\lambda^2\omega(E) \leqslant \int a(P)\, d\omega(P) \leqslant \varepsilon^2\lambda^2$.

Let μ denote the surface measure of Ω. Then it follows from (3.6) that

$$\frac{\mu(E)}{\mu(V)} \leqslant C\left(\frac{\omega(E)}{\omega(V)}\right)^a.$$

Since there are constants K_1 and K_2 only depending on the Lipschitz constant of Ω and D respectively such that $\mu(V) \leqslant K_1 r^{n-1}$ and $\sigma(V) \geqslant K_2 r^{n-1}$ and $\mu(E) = \sigma(E)$ the lemma follows.

Remark. If the assumption $A(P_0) \leqslant \lambda$ for some $P_0 \in V$ is dropped we see by considering the function a instead, that the following conclusion can be made: There is a number $\tau = \tau(M, \Gamma, \gamma) > 0$ such that if $a(P) = \int_{\gamma(P)\cap B(P,\tau r)} |\nabla u|^2 |P-Q|^{2-n}\, dm(Q)$, then given $\alpha > 1$ $\beta > 1$ there is a number $\varepsilon = \varepsilon(\alpha, \beta, \Gamma, \gamma, M, \tau)$ such that $\alpha\sigma\{P \in V:\ a(P) > \beta^2\lambda^2,\ N(P) \leqslant \varepsilon\lambda\} \leqslant \sigma(V)$. We also remark that it is not necessary to assume that u is harmonic in a neighbourhood of 0. In the case when u is assumed to be harmonic in $\{(x, y):\ |x| < R_1\varphi(x) < y < R_2\}$ then the functions $u_j = u(x, y+1/j)$ fulfil the assumptions of Lemma 3 so the conclusion follows in this case also after letting $j \to \infty$.

In the next lemma we shall switch the role of Γ and γ for the definitions of A and N respectively, i.e. we put now

$$A(P) = \left(\int_{\Gamma(P)} |\nabla u|^2 |P-Q|^{2-n}\, dm(Q)\right)^{\frac{1}{2}},$$

$$D(P) = \sup_{\Gamma(P)} |\nabla u(Q)|\,|P-Q|^{-1}, \quad \text{and} \quad N(P) = \sup_{\gamma(P)} |u(Q)|.$$

We shall let q denote the characteristic function of the set of all $P \in \partial D$ such that $\Gamma(P)$ is in the domain of definition of u and $A(P) > \varepsilon\lambda$. We put

$$q^*(P) = \sup \int_{\frac{\partial D\cap B}{\sigma(B\cap\partial D)}} q\, d\sigma,$$

where B runs over all balls in the center at ∂D containing P.

Lemma 4. *Let $V = \{(x, \varphi(x)):\ |x| \leqslant r\}$ and assume that $r < r_0$ and $N(P_0) \leqslant \lambda$ for some $P_0 \in V$. Given $\alpha > 1$, $\beta > 1$ there are numbers $\varepsilon > 0$ and $\delta > 0$ which can be taken to depend only on $M, \Gamma, \gamma, \alpha$ and β such that*

$$\alpha\sigma\{P \in V:\ N(P) > \beta\lambda,\ D(P) \leqslant \varepsilon\lambda,\ A(P) \leqslant \varepsilon\lambda,\ q^*(P) \leqslant \delta\} \leqslant \sigma(V).$$

Proof. Let $\Omega_1 = D(V^*, 0, 5r)$, where $V^* = \{(x, \varphi(x)): |x| < 5r\}$ and let g_1 denote the Green function of Ω_1. If $Q \in \Omega_1$ let $d_1(Q)$ denote the distance from Q to $\partial\Omega_1$ and let $\bar{Q}$ denote the radial projection of Q onto $\partial\Omega_1$ from the starcenter $P^* = (0, 5ar)$.

We remark that we may assume that r_0 has been chosen so small that $2Mr_0 < a$ and if $Q = (x, y) \in \Omega_1$ and $y < \frac{1}{2}ar$, then $\{tQ + (1-t)P^*: t \geqslant 1\} \subset Q - \Gamma'$, where Γ' is an unbounded cone with opening angle strictly less than the opening angle of Γ. Put $\Omega_2 = \{Q = (x, y) \in \Omega_1: y < \frac{1}{2}ar\}$. Then there are positive numbers η and τ such that if $Q \in \Omega_2 \cap \Gamma(Z)$ for some $Z \in \{(x, \varphi(x)): |x| \leqslant 2r\}$, then

$$A(Z, \tau d_1(Q)) \subset \{P \in \partial\Omega_1: Q \in \Gamma(P)\} \text{ and } |Q - Z| \leqslant \eta d_1(Q). \tag{4.1}$$

Put $E = \{P \in V: N(P) > \beta\lambda,\ q^*(P) \leqslant \delta,\ A(P) \leqslant \varepsilon\lambda\}$, $F = \{P \in V: q^*(P) \leqslant \delta\}$ and $H = \{P \in V: A(P) \leqslant \varepsilon\lambda\}$. Let ω_1 denote the harmonic measure of Ω_1 evaluated at P^*. We notice that if $P \in V$ and $Q \in \gamma(P) \cap \Omega_2$, then $d_1(Q) \leqslant C|P - Q|$. Hence, we get from Fubini's theorem that

$$\varepsilon^2\lambda^2 \geqslant \int_H A^2(P)\, d\omega_1(P) \geqslant C \int_{\Omega_2} |\nabla u|^2 d_1(Q)^{2-n} f(Q)\, dm(Q), \tag{4.2}$$

where $f(Q) = \int_{H(Q)} d\omega_1(P)$ and $H(Q) = \{P \in H: Q \in \Gamma(P)\}$. Let $W = \Omega_2 \cap \cap [\bigcup_{P \in F} (P)]$.

From (4.1) follows that if $Q \in W$, then there is a point $Z \in F$ such that

$$f(Q) \geqslant \omega_1(H \cap A),$$

where $A = A(Z, \tau d_1(Q))$. If $B = A(\bar{Q}, d_1(Q))$, then it follows from Lemma 1 that

$$d_1(Q)^{2-n} g_1(Q)^{-1} f(Q) \geqslant C \frac{\omega_1(H \cap A)}{\omega_1(B)}.$$

However, there is by (4.1) a number C such that $B \subset A^* = A(Z, Cd_1(Q))$. Since $\omega_1(B) \leqslant \omega_1(A^*)$ we get from (3.3) that $\omega_1(B) \leqslant C\omega_1(A)$. Therefore we get

$$d_1(Q)^{2-n} g_1(Q)^{-1} f(Q) \geqslant C \frac{\omega_1(H \cap A)}{\omega_1(A)}.$$

From (3.6) follows that

$$\frac{\omega_1(H \cap A)}{\omega_1(A)} \geqslant 1 - C(q^*(Z))^b \geqslant 1 - C\delta^b.$$

Hence it follows from (4.2) that δ may be chosen so small that

$$\int_W |\nabla u|^2 g_1(Q)\, dm(Q) \leqslant C\varepsilon^2\lambda^2. \tag{4.3}$$

Let now $\Omega = D(E, 0, 5r)$. Then for $Q \in \Omega - W$ we have $|\nabla u(Q)| \leqslant C\varepsilon\lambda r^{-1}$. Hence

$$\int_{\Omega - W} |\nabla u|^2 g_1(Q)\, dm(Q) \leqslant C\varepsilon^2\lambda^2 r^{-2} \int g_1(Q)\, dm(Q)$$
$$\leqslant C\varepsilon^2 r^{-2}\lambda^2 \int_{B(P^*, Cr)} |P - P^*|^{2-n}\, dm(P) \leqslant C\varepsilon^2\lambda^2.$$

If g denotes the Green function of Ω with pole at P^*, then $g(P) \leqslant g_1(P)$ so it follows from (4.3) that

$$\int_\Omega |\nabla u|^2 g(Q)\, dm(Q) \leqslant C\varepsilon^2\lambda^2. \tag{4.4}$$

Also notice that if $Q \in \gamma(P) - W$ for some $P \in F$, then there is a point $Q_1 \in \gamma(P_0)$ such that $|Q - Q_1| \leqslant Cr$ and $|\nabla u| \leqslant C\varepsilon\lambda r^{-1}$ on the line segment between Q and Q_1. Hence $|u(Q)| \leqslant (1 + C\varepsilon)\lambda$. Consequently $|u(P^*)| \leqslant (1 + C\varepsilon)\lambda$ and therefore we can choose δ so small

$$\sup\{|V(Q)|: Q \in \gamma(P) \supset W\} \geqslant \tfrac{1}{2}(\beta - 1)\lambda, \quad P \in E, \quad \text{where } V = u - u(P^*). \tag{4.5}$$

From (4.4) follows that if ω denotes the harmonic measure of Ω evaluated at P^*, then

$$\int_{\partial\Omega} V^2 d\omega = 2\int |\nabla u|^2 (gQ)\, dm(Q) \leqslant C\varepsilon^2\lambda^2.$$

For $P \in \partial\Omega$ let $MV(P) = \sup\{|V(tP + (1-t)P^*)|: 0 \leqslant t < 1\}$. From (3.7), (3.8) and the Marcinkiewicz interpolation theorem (see Stein [13], p. 272) follows

$$\int_{\partial\Omega} (MV)^2 d\omega \leqslant C\varepsilon^2\lambda^2. \tag{4.6}$$

If $P \in E$ and $Q \in \gamma(P) \cap W$, then there is a point $Q_1 \in \Gamma(P)$ such that $|Q - Q_1| \leqslant C|P - Q|$ and $|\nabla u| \leqslant C\varepsilon\lambda|P - Q|^{-1}$ on the line segment between Q and Q_1. Hence we deduce from (4.5) that

$$MV(P) \geqslant (\tfrac{1}{2}(\beta - 1) - C\varepsilon)\lambda \geqslant \tfrac{1}{4}(\beta - 1)\lambda \quad \text{for} \quad P \in E,$$

if ε is chosen sufficiently small.

From (4.6) follows that

$$(\tfrac{1}{4}(\beta - 1))^2\lambda^2\omega(E) \leqslant C\varepsilon^2\lambda^2.$$

Arguing as in the conclusion of the proof of Lemma 3 yields the Lemma.

Remark. If the assumption $N(P_0) \leqslant \lambda$ for some $P_0 \in V$ is dropped, we see by considering the function V that the following conclusion can made: There is a number $\tau = \tau(M, \Gamma, \gamma)$ such that if $\alpha > 1$, $\beta > 1$ are given and if $|u(P^*)| \leqslant \frac{1}{2}(\beta+1)\lambda$ then there are numbers $\varepsilon > 0$ and $\delta > 0$ depending on $\alpha, \beta, \Gamma, \gamma$ and M such that

$$\alpha\sigma\{P \in V: N'(P) > \beta\lambda,\ D(P) \leqslant \varepsilon\lambda,\ A(P) \leqslant \varepsilon\lambda,\ q^* \leqslant \delta\} \leqslant \sigma(V),$$

where $N'(P) = \sup\{|u(Q)|: Q t\gamma(P) \cap B(P, \tau r)\}$.

5. Proof of Theorem 1. We let $\Omega = R^n$, $n \geqslant 2$, be a Lipschitz domain. Before proving the Theorem we need some preliminary estimates. If Φ is as in the Theorem we put $\Phi^{-1}(t) = \sup\{x: \Phi(x) = t\}$.

LEMMA 5. *Assume $\{\Gamma(P)\}$ is a family of cones which is regular with respect to Ω. Let u be harmonic in Ω and put $N(P) = \sup_{\Gamma(P)}|u|$ and suppose that $\int_{\partial\Omega} \Phi(N)\,d\mu = L < \infty$ for some measure μ satisfying an A_∞-condition, where Φ is as in the Theorem. Then given any compact set $K \subset \Omega$ there is a number C independent of u such that*

$$|u| + |\nabla u| \leqslant C\Phi^{-1}(CL) \quad \text{in } K.$$

Proof. For $t > 0$ let $\Omega_t = \{P \in \Omega: d(P) > t\}$.

Since $\{\Gamma(R)\}$ is regular there is a number $t_0 > 0$ such that if $0 < t < t_0$, then there is a partition $\{U_j\}$ of $\partial\Omega$ into (small) nonempty open subsets such that if $P_j \in U_j$, then $\partial\Omega_i \subset U\Gamma(P_j)$. Notice that in order to show the lemma, it is sufficient to show that $|u| \leqslant \varphi^{-1}(C_t L)$ in each domain Ω_t, $0 < t < t_0$. It is a well-known property of functions satisfying an A_∞-condition that the density function is nonvanishing a.e. Hence $\inf \mu(U_j) = \varepsilon > 0$. Put $\lambda = 2L/\varepsilon$. Then $\mu\{P \in \partial\Omega: \varphi(N(P)) > \lambda\} \leqslant \frac{1}{2}\varepsilon$. Hence to each j there is a point $P_j \in U_j$ such that $N(P_j) < \lambda$. Since $\partial\Omega_t \subset U\Gamma(P_j)$ it follows from the maximum principle that $|u| \leqslant \lambda$ in Ω_t, which proves the Lemma.

We shall now introduce a concept that will be useful to us. Suppose that $F_j = \{\Gamma_j(P)\}$, $j = 1, 2$ are two families of cones both being regular for Ω. We shall say that F_1 is *strictly in side* F_2 if there is a finite family of open sets $\{U\}$ which covers $\partial\Omega$ and to each U there is associated an open set V and an open, right circular cylinder L such that $U \subset \overline{U} \subset V \subset L \cap \partial\Omega$ and $\Gamma_1(P) \subset \gamma_1 + P \subset \gamma_2 + P \subset \Gamma_2(P)$, where γ_1 and γ_2 are cones with vertex 0 and axis parallel to the axis of L such that $\overline{\gamma}_1 - \{0\} \subset \gamma_2$.

LEMMA 6. *Assume $\{\Gamma(P)\}$ is a family of cones which is regular with respect to Ω. Let $P^* \in \Omega$ and assume u is a function harmonic in Ω vanishing at P^*. Put*

$$A(P) = \Big(\int_{\Gamma(P)} |\nabla u|^2 |P-Q|^{2-n}\,dm(Q)\Big)^{\frac{1}{2}}$$

and suppose that $\int_{\partial\Omega} \Phi(A)\,d\mu = L < \infty$ for a measure μ satisfying an A_∞-condition, where Φ is as in the Theorem. Then given any compact set $K \subset \Omega$ there is a number C independent of u such that

$$|u| + |\nabla u| \leqslant C\varphi^{-1}(CL).$$

Proof. Let $\{\Gamma'(P)\}$ be a family of cones, which is regular for and is strictly inside $\{\Gamma(P)\}$. Then it is well known, see Stein [12, Lemma 5] that

$$d(Q)\,|\nabla u(Q)| \leqslant CA(P), \qquad Q \in \Gamma'(P).$$

Arguing as in the proof of Lemma 5 it follows that we can find points $P_j \in \partial\Omega$ such that $\partial\Omega_t \subset \bigcup \Gamma'(P_j)$ and $A(P_j) \leqslant \varphi^{-1}(CL)$. Hence $|\nabla u| \leqslant Ct^{-1}\varphi^{-1}(CL)$ in Ω_t and since $u(P^*) = 0$ the lemma follows by integration.

The next lemma is an adaption of Lemma 1 in Gundy and Wheeden [7].

LEMMA 7. *Let $F_j = \{\Gamma_j(P)\}$, $j = 1, 2$ be families of cones, which are both regular for Ω. Assume that F_1 is strictly inside F_2 and μ is a measure on $\partial\Omega$ satisfying condition A_∞. Let u be harmonic in Ω and put $N_1(P) = \sup_{\Gamma_1(P)} |u|$. Then there are positive numbers h and C such that*

$$\mu\{N_2 > \lambda\} \leqslant C\mu(N_1 > \lambda),$$

where $N_2(P) = \sup\{|u(Q)|: Q \in \Gamma_2(P) \cap B(P, h)\}$.

Proof. Let f denote the characteristic function of $\{P \in \partial\Omega: N_1(P) > \lambda\} = E$. If $N_2(P) > \lambda$ there is a $Q \in \Gamma_2(P) \cap B(P, h)$ such that $|u(Q)| > \lambda$. If h is chosen sufficiently small there is a $P_1 \in E$ such that $A = A(P_1, cd(Q)) \subset E$ and $|P - P_1| \leqslant Cd(Q)$. Put $f^*(P) = \sup \dfrac{\mu(E \cap B)}{\mu(B)}$, where B runs over all balls with center on $\partial\Omega$ containing P.

We choose the number K such that

$$A \subset A_1 = A(P, KD(Q)) \subset A(P_1, 2Kd(Q)) = A^*.$$

Since $\mu(A^*) \leqslant C\mu(A)$, we have

$$f^*(P) \geqslant \frac{\mu(E \cap A_1)}{\mu(A_1)} \geqslant C\frac{\mu(A)}{\mu(A)} = C > 0.$$

From Besicovitch' theory of differentiation follows $\mu(N_2 > \lambda) \leqslant \mu(f^* > C) \leqslant C\mu(N_1 > \lambda)$, which proves the lemma.

Suppose now that $\{U\}$ is finite covering with open sets of $\partial\Omega$ such that to each U there is an open set V and an open, right circular cylinder L such that $\overline{U} \subset V \subset L \cap \partial\Omega$ and $\partial\Omega$ is defined inside L by a Lipschitz function φ. We now observe that given $K > 0$ it is possible to find a number $r_0 > 0$ such that for all $P \in \partial\Omega$ there is a U such that $A(P, Kr_0) \subset U$ and if $P = (\varepsilon, \varphi(\xi))$ then for suitable a and b it holds $V(P, ar) \subset A(P, r) \subset V(P, br)$, where $V(P, r) = \{x\varphi(x_1): |x - \xi| \leqslant r\}$ and $0 < r < r_0$.

LEMMA 8. *Let $F_j = \{\Gamma_j(P)\}$, $j = 1, 2$, be two families of cones, both being regular for Ω. Suppose that F_1 is strictly inside F_2. Suppose μ is a measure on $\partial\Omega$ which satisfies condition A_∞. Let u be harmonic in Ω and put*

$$N(P) = \sup_{\Gamma_2(P)} |u|, \qquad A(P) = \Big(\int_{\Gamma_1(P)} |\nabla u|^2 |P-Q|^{2-n} dm(Q)\Big)^{2}.$$

Assume $\int_{\partial\Omega} \Phi(N)\,d\mu = L < \infty$, where Φ is as in the Theorem. Then there is a number C independent of u such that if $\lambda > C\varphi^{-1}(CL)$, then given $\alpha > 1$, $\beta > 1$ there is an $\varepsilon > 0$ such that $\alpha\mu\{A > \beta\lambda,\ N \leqslant \varepsilon\lambda\} \leqslant \mu(A > \lambda)$.

Proof. We may without loss of generality assume that $\lim_{P'\to P} \chi_{P'}(Q) = \chi_P(Q)$ for all $Q \in R^n$, where χ_P denotes the characteristic function of $\Gamma_1(P)$. Hence it follows from Fatou's theorem that A is lower semicontinuous on $\partial\Omega$. Hence the set $S = \{A > \lambda\}$ is open in $\partial\Omega$. For $P \in S$ let $r(P) = \sup\{t \leqslant r_0\colon A(P, t) \subset S\}$. Then $r(P) > 0$ and we can find countably many $P_j \in S$ such that $\bigcup B_j = \bigcup B(P, r(P))$, where $B_j = B(P_j, r_j)$, $r_j = r(P_j)$ and no point in R^n lies in more than $C = C(n)$ balls B_j, see Landkof [11]. To each j there is one out of finitely many cylinders L such that $B_j \cap \partial\Omega \subset L \cap \partial\Omega$ and $\partial\Omega$ is defined inside L by a Lipschitz function φ. If $P_j = (x_j, \varphi(x_j))$ let V_j be the smallest set of the form $\{(x, \varphi(x))\colon |x - x_j| \leqslant r\}$ which contains $B_j \cap \partial\Omega$. Then $\sigma(V_j) \leqslant C\sigma(B_j)$. If $r_j < r_0$ it follows from the maximality of R_j that $A(P_0) \leqslant \lambda$ for some $P_0 \in V_j$. If $E = \{A > \beta\lambda,\ N \leqslant \varepsilon\lambda\}$, then it follows from Lemma 3 that

$$\alpha\sigma(E \cap B_j) \leqslant \sigma(B_j).$$

(Here we used that $\sigma(V_j) \leqslant C\sigma(B_j)$.) If $r_j = r_0$ it follows from the remark after Lemma 3 that if $\beta' > 1$ is given, then ε can be chosen so that

$$\alpha\sigma(E' \cap B_j) \leqslant \sigma(B_j),$$

where $E' = \{A' > \beta'\lambda,\ N \leqslant \varepsilon\lambda\}$, $A'(P) = \Big(\int_{\Gamma_3(P)} |\nabla u|^2 |P-Q|^{2-n} dm(Q)\Big)$, and $\Gamma_3(P) = \Gamma_1(P) \cap B(P, \eta r_0)$.

It follows from Lemma 5 that $A(P) \leqslant A'(P) + C\varphi^{-1}(CL)$. Hence if $\lambda > C\varphi^{-1}(CL)$ where C is chosen sufficiently large we have that $E \cap B_j \subset E' \cap B_j$, where we have chosen $\beta' = \frac{1}{2}(\beta + 1)$. Hence

$$\alpha\sigma(E \cap B_j) \leqslant \sigma(B_j) \quad \text{for all } j.$$

Using (1.2) we find

$$\mu(E) \leqslant \sum \mu(E \cap B_j) \leqslant A\alpha^{-\theta} \sum \mu(B_j \cap \partial\Omega) \leqslant A\alpha^{-\theta} C(n)\,\mu\big[(\bigcup B_j) \cap \partial\Omega\big] = A\alpha^{-\theta} C(n)\,\mu\{A > \lambda\},$$

which yields the lemma.

LEMMA 9. *Let $P^* \in \Omega$ be a fixed point and let $F_j = \{\Gamma_j(P)\}$, $j = 1, 2$, be two families of cones, both being regular for Ω. Suppose that F_1 is strictly inside F_2. Suppose μ is a measure on $\partial\Omega$ satisfying condition A_∞. Let u be harmonic in Ω, vanishing at P^*, and put*

$$N(P) = \sup_{\Gamma_1(P)} |u|, \qquad D(P) = \sup_{\Gamma_2(P)} |\nabla u(Q)|\,|P-Q|^{-1} \quad \text{and}$$

$$A(P) = \Big(\int_{\Gamma_2(P)} |\nabla u|^2 |P-Q|^{2-n} dm(Q)\Big)^{\frac{1}{2}}.$$

Assume $\int_{\partial\Omega} \Phi(A)\,d\mu = L < \infty$, where Φ is as in the Theorem. Then there is a number C independent of u such that if $\lambda > C\varphi^{-1}(CL)$, then given $\alpha > 1$, $\beta > 1$ there is an $\varepsilon > 0$ and a $\delta > 0$ such that

$$\alpha\mu\{N > \beta\lambda,\ A \leqslant \varepsilon\lambda,\ D \leqslant \varepsilon\lambda,\ q^* \leqslant \delta\} \leqslant \sigma\{N > \lambda\},$$

where q^* is defined by

$$q^*(P) = \sup \frac{\sigma(F \cap B)}{\sigma(B)},$$

where $F = \{A > \lambda\}$ and B runs over all balls with center at $\partial\Omega$ containing P.

Proof. Since the proof is patterned on the proof of Lemma 8 we content ourselves with a sketch. First there is no loss in generality in assuming N is lower semicontinuous. Choose $P_j \in S = \{N > \lambda\}$ and $r_j \in (0, r_0]$ such that $S = \big(\bigcup B_j\big) \cap \partial\Omega$, where $B_j = B(P_j, r_j)$ and $B_j \cap (\partial\Omega - S) \neq \emptyset$ if $r_j < r_0$ and no point lies in more than $C(n)$ of the B_j's. Put $E = \{N > \beta\lambda,\ A \leqslant \varepsilon\lambda,\ D \leqslant \varepsilon\lambda,\ q^* \leqslant \delta\}$. If $r_j < r_0$, then Lemma 4 yields $\alpha\sigma(E \cap B_j) \leqslant \sigma(B_j)$, for suitable ε and δ. If $r_j = r_0$, then it follows from Lemma 6 and the remark following Lemma 4 that $\alpha\sigma(E \cap B_j) \leqslant \sigma(B_j)$ also holds in this case. The rest of the proof carried out as in the proof of Lemma 8, which yields Lemma 9.

We can now prove our main result.

Proof of Theorem 1. First we suppose that $\int_{\partial\Omega} \Phi(N)\,d\mu = L < \infty$. Choose a family of cones $\{\Gamma'(P)\}$ regular for Ω such that $\{\Gamma(P)\}$ is strictly inside $\{\Gamma'(P)\}$. From Lemmas 5 and 7 follow that if C is chosen large enough, then

$$\mu(N' > \lambda) \leqslant C\mu(N > \lambda), \quad \lambda \geqslant C\Phi^{-1}(CL), \tag{5.1}$$

where $N'(P) = \sup_{\Gamma'(P)} |u|$. From (5.1) follows

$$\int_{\partial\Omega} \Phi(N')\,d\mu = \int_0^\infty \mu(N' > \lambda)\,\Phi'(\lambda)\,d\lambda \leqslant CL. \tag{5.2}$$

Let T be a large positive number. Then

$$(5.3)\quad \int_0^T \mu(A>\lambda)\Phi'(\lambda)\,d\lambda = 2\int_0^{T/2}\mu(A>2\lambda)\Phi'(2\lambda)\,d\lambda \leqslant C\int_0^T \mu(A>2\lambda)\Phi'(\lambda)\,d\lambda.$$

Put $\tau = C\Phi^{-1}(CL)$, where C is chosen sufficiently large. Then $\Phi(\tau)\leqslant CL$. From Lemma 8 follows that if $a>1$ is given and if $\varepsilon>0$ is chosen sufficiently small, then

$$\int_\tau^T \mu(A>2\lambda)\Phi'(\lambda)\,d\lambda \leqslant C\int_0^\infty \Phi(N')\,d + \frac{1}{a}\int_0^T \mu(A>\lambda)\Phi'(\lambda)\,d\lambda.$$

Hence it follows from (5.2) and (5.3) that if a is chosen sufficiently large, then

$$\int_0^T \mu(A>\lambda)\Phi'(\lambda)\,d\lambda \leqslant CL + \tfrac{1}{2}\int_0^T \mu(A>\lambda)\Phi'(\lambda)\,d\lambda,$$

which yields $\int_0^\infty \mu(A>\lambda)\Phi'(\lambda)\,dt = \int_{\partial\Omega}\Phi(A)\,d\mu \leqslant CL$.

To prove the remaining part of Theorem 1 we assume now that $u(P^*)=0$ and $\int_{\partial\Omega}\Phi(A)\,d\mu = L<\infty$. Choose two families of cones $F_j = \{\Gamma_j(P)\}$, $j=1,2$, such that both are regular with respect to Ω and F_2 is strictly inside F_1 and F_2 is strictly inside $\{\Gamma(P)\}$. Put now

$$N'(P) = \sup_{\Gamma_2(P)}|u|,\qquad D(P) = \sup_{\Gamma_1(P)}|\nabla u(Q)|\,|P-Q|^{-1},\qquad \text{and}$$

$$q^*(P) = \sup\frac{\sigma(\{A>\varepsilon\lambda\}\cap B)}{\sigma(B)},$$

where B runs over all balls with center on $\partial\Omega$ containing P. Since it follows from [12], Lemma 5, that $D\leqslant CA$, it follows from Lemma 9 that if $\lambda\geqslant\tau = C\varphi^{-1}(CL)$, where C is chosen sufficiently small, then given $a>1$ there are positive numbers ε and δ such that

$$a\mu(N>2\lambda,\ A\leqslant\varepsilon\lambda,\ q^*\leqslant\delta)\leqslant\mu(N>\lambda).$$

Arguing as above it follows that if a is chosen sufficiently large, then

$$(5.4)\quad \int_0^T \mu(N>\lambda)\varphi(\lambda)\,dt \leqslant CL + \int_0^\infty \mu(q^*>\delta)\varphi(\lambda)\,dt + \tfrac{1}{2}\int_0^T \mu(N>\lambda)\varphi(\lambda)\,dt.$$

Since μ satisfies condition A_∞ there are numbers $C>0$ and $a>0$ such that if $P\in\partial D$ and $E\subset A(P,r)=A$, then

$$C\left(\frac{\sigma(E)}{\sigma(A)}\right)^a \leqslant \frac{\mu(E)}{\mu(A)}.$$

Putting

$$f(P) = \sup\frac{\mu\{A>\varepsilon\lambda\}\cap B}{\mu(B)},$$

where B runs over all balls with center on $\partial\Omega$ containing P, we see that

$$\mu(q^*>\delta)\leqslant\mu(f>C\delta^a)\leqslant C\mu(A>\varepsilon\lambda).$$

Considering (5.4) now yields

$$\int_{\partial\Omega}\Phi(N')\,d\mu = \int_0^\infty \mu(N>\lambda)\Phi'(\lambda)\,dt \leqslant CL.$$

Arguing as in the proof of (5.2) we infer

$$\int_{\partial\Omega}\Phi(N)\,d\mu\leqslant C\int_{\partial\Omega}\Phi(N')\,d\mu\leqslant CL,$$

which completes the proof of Theorem 1.

6. A corollary. In this section we shall prove the following corollary of Theorem 1.

COROLLARY. *Let $\Omega\subset R^n$, $n\geqslant 2$, be a Lipschitz domain. Put $d(P) = \operatorname{dist}(P,\partial\Omega)$. Fix a point $P^*\in D$ and suppose u is harmonic in Ω and vanishing at P^*. Then there is a constant $C>0$ such that*

$$(6.1)\quad C^{-1}\int_{\partial\Omega}u^2\,d\sigma\leqslant\int|\nabla u|^2 d(Q)\,dm(Q)\leqslant C\int_{\partial\Omega}u^2\,d\sigma,$$

where σ denotes the surface measure of $\partial\Omega$.

Proof. Let $\{\Gamma(P)\}$ be a regular family of cones and put $a(P) = \int|\nabla u|^2 d(Q)^{2-n}\,dm(Q)$. Since $C|P-Q|\leqslant d(Q)\leqslant|P-Q|$ for $Q\in\Gamma(P)$ it follows from Theorem 1 that

$$C^{-1}\int u^2\,d\sigma\leqslant\int_{\partial\Omega}a(P)\,d\sigma\leqslant C\int u^2\,d\sigma.$$

It follows from Fubini's theorem that

$$\int_{\partial\Omega}a(P)\,d\sigma = \int_\Omega|\nabla u|^2 d(Q)^{2-n}f(Q)\,dm(Q),$$

where $f(Q) = \sigma\{P\in\partial\Omega\colon Q\in\Gamma(P)\}$. If the height of the cones $\Gamma(P)$ is sufficiently small then $\{P\in\partial\Omega\colon Q\in\Gamma(P)\}\subset A(\tilde P, Cd(Q))$ for some $\tilde P\in\partial\Omega$ which implies $C^{-1}\int_{\partial\Omega}u^2\,d\sigma\leqslant\int_{\partial\Omega}a(P)\,d\sigma\leqslant C\int_\Omega|\nabla u|^2 d(Q)\,dm(Q)$, which yields the left-hand side of (6.1). To prove the right-hand side inequality we assume $\int_{\partial\Omega}u^2\,d\sigma = L^2<\infty$. We can find a neighbourhood V of $\partial\Omega$ such that if $Q\in U = V\cap\Omega$, then for some $\tilde P\subset\partial\Omega$ we have

$$A(\tilde P, cd(Q))\subset\{P\in\partial\Omega\colon Q\in\Gamma(P)\}.$$

Hence $\int a(P)\,d\sigma\geqslant c\int_U|\nabla u|^2 d(Q)\,dm(Q)$. Since the density of the harmonic measure is in $L^2(\sigma)$ it follows that to each compact set $K \subset \Omega$ there is a number C such that

$$|\nabla u| \leqslant CL \quad \text{in } K.$$

Hence

$$\int_{\Omega} |\nabla u|^2 d(Q)\, dm(Q) \leqslant \int_{U} + \int_{\Omega - U} \leqslant C \int Q(P)\, d\sigma + CL^2 \int_{\Omega - u} dm \leqslant C \int_{\substack{L^2 \\ \partial\Omega}} u^2 d\sigma,$$

which completes the proof of the corollary.

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Lifting subnormal double commutants*

by

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Abstract. We give a necessary condition for elements of the commutant of a subnormal operator on Hilbert space to lift to the commutant of the minimal normal extension, and based on this condition we show there exists a subnormal operator having elements in its double commutant that do not lift. Our example is also irreducible and finitely cyclic.

1. Let T be a subnormal operator on a Hilbert space H with minimal normal extension (m.n.e.) N on K. Thus, N is a normal operator on a space K containing H, H is invariant for N, the restriction $N|_H = T$, and K, the smallest reducing subspace of N containing H, is the closed linear span of $\{N^{*j}h\colon j = 0, 1, \ldots, h \in H\}$. Clearly, for any polynomial p, $p(N)|_H = p(T)$ and it is therefore easy to see that every element in the weakly closed algebra generated by T lifts to, i.e. is the restriction to H of, some element in the commutant of N. This result is included in the well-known paper of J. Bram, who gave the following necessary and sufficient condition for an element of $\{T\}'$, the commutant of T, to lift to $\{N\}'$:

THEOREM B [2, p. 87]. *A necessary and sufficient condition that $A \in \{T\}' \subset B(H)$ have an extension $B \in \{N\}' \subset B(K)$ is that there exist a positive constant c such that for every finite set $h_0, \ldots, h_r$ in H we have*

$$\sum_{m,n=0}^{r} (T^m A h_n, T^n A h_m) \leqslant c \sum_{m,n=0}^{r} (T^m h_n, T^n h_m).$$

If the extension B exists, then it is unique.

Bram pointed out that there exist subnormal operators T such that not every element of $\{T\}'$ lifts to $\{N\}'$. T. Yoshino, however, showed that every element of $\{T\}'$ does lift to $\{N\}'$ if T has a cyclic vector [6], p. 49. (Note that there is an error in the proof of this theorem but the result is true.) M.B. Abrahamse and C. Berger raised the question of whether

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