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Central limit problem for symmetric case: Convergence to non-Gaussian laws

by

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Abstract. A general theorem is proven giving necessary and sufficient conditions for the row sums of a uniformly infinitesimal symmetric triangular array (with independence in each row) to be conditionally compact. Using this limit theorems are proven in spaces of type p-Rademacher, cotype q-Rademacher and type p-stable. Characterizations of these spaces in terms of these limit theorems are also obtained.

0. Introduction. This paper is devoted to the study of the Central Limit Problem in a real separable Banach space E. We first establish necessary and sufficient conditions for the row sums of a triangular array of uniformly infinitesimal symmetric independent random variables to be stochastically bounded as well as to be compact, in the case that the limit points are non-Gaussian. These results generalize a result in Feller ([5], p. 309) as well as some work of G. Pisier ([26], Theorem 3.1). The main tools are a result of Le Cam ([16], p. 237) and ideas involved in proving some inequalities as in (H–J [8] and Jain [11]). As a consequence of these results we characterize Banach spaces for which the classical conditions hold. In particular we show that the spaces in which classical conditions ([6], p. 116) are necessary and sufficient are isomorphic to Hilbert space and the spaces for which both halves of the domain of attraction problem hold for stable laws of order $p < 2$ are precisely the type p-stable Banach spaces. We also derive from the necessary conditions in the latter problem the existence of the αth moment of the norm with respect to the laws in the domain of normal attraction of the stable law of order p for $\alpha < p$. Our work includes some of the recent work of Woyczynski [29] and Marcus and Woyczynski ([19], [20], [21]).

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1. Preliminaries and notation. Let E be a real separable Banach space with the (topological) dual E' and Borel field $\mathscr{B}(E)$. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space; then X on Ω to E will be called an *E-valued random* *variable* if X is $(\mathscr{B}(E), \mathscr{F})$-measurable. We note that due to the separability of E, X is strongly measurable. The distribution induced by X, namely $P\circ X^{-1}$ will be called *law of* X and written as $\mathscr{L}(X)$. We say that an E-valued random variable X is *symmetric* if $\mathscr{L}(X) = \mathscr{L}(-X)$. We shall be dealing with truncation. For an E-valued random variable X and a subset $A \subset E$ we denote by

$$\tau(X, A) = \begin{cases} X & \text{if } X \in A, \\ 0 & \text{if } X \notin A. \end{cases} \tag{1.1}$$

We denote by

$$\tilde{\tau}(X, A) = \tau(X, A^c) \tag{1.2}$$

and note that $X = \tau(X, A) + \tilde{\tau}(X, A)$.

A family of symmetric E-valued random variables $\{X_{nj},\ j = 1, 2, \ldots, k_n,\ n = 1, 2, \ldots\}$ will be called a *symmetric triangular X-array* if for each n, $\{X_{nj},\ j = 1, 2, \ldots, k_n\}$ is an independent family of random variables. Associated with a triangular array we shall use the following notation:

$$\begin{cases} \text{(a)}\ S_n = \sum_{j=1}^{k_n} X_{nj}, \quad \mu_n = \mathscr{L}(S_n), \\ \text{(b)}\ F_n = \sum_{j=1}^{k_n} \mathscr{L}(X_{nj}),\ F_n^{(\delta)} = \sum_{j=1}^{k_n} \mathscr{L}\big(\tilde{\tau}(X_{nj}, B_\delta)\big), \\ \text{(c)}\ S_n(\delta) = \sum_{j=1}^{k_n} \tau(X_{nj}, B_\delta), \end{cases} \tag{1.3}$$

where τ and $\tilde{\tau}$ are as in (1.2), (1.3) and $B_\delta = \{x|\ \|x\| < \delta\}$.

We conclude the section by recalling some standard results and definitions. We say that a sequence of finite measures $\{\nu_n\}_{n=1}^\infty$ on $(E, \mathscr{B}(E))$ converges weakly to a finite measure ν if $\int g\,d\nu_n \to \int g\,d\nu$ for every bounded continuous function g on E. It is known that $\{\nu_n\}_{n=1}^\infty$ is weakly conditionally compact (for short, compact) iff for every $\varepsilon > 0$ exists a compact set $K(\varepsilon)$ such that $\nu_n(K(\varepsilon)) < \varepsilon$ and $\sup_n \nu_n(E)$ is finite. Given a finite measure ν on $(E, \mathscr{B}(E))$ we denote by $e(\nu)$, the exponential of ν, defined as

$$e(\nu) = \exp\big(-\nu(E)\big)\Big\{\delta_0 + \sum_{n=1}^\infty \frac{\nu^{*n}}{n!}\Big\}$$

where δ_0 is the dirac measure at zero and ν^{*n} is n-fold convolution of ν. If further, $\nu = \mathscr{L}(X)$ for some E-valued random variable X, then $e(\nu) = \mathscr{L}\big(\sum_{j=1}^N X_j\big)$ where $\{X_j\}$ independent E-valued random variables with $\mathscr{L}(X_j) = \mathscr{L}(X)$ and N is Poisson random variable with parameter one, independent of the sequence $\{X_j\}_{j=1}^\infty$. Finally the characteristic function of a cylindrical measure μ ([4]) is defined to be

$$\varphi_\mu(y) = \int_E \exp(i\langle y, x\rangle)\,\mu\,dx \quad \text{for} \quad y \in E',$$

where $\langle\cdot, \cdot\rangle$ denotes the duality function on $E' \times E$.

2. Stochastic boundedness and compactness of row sums. A sequence $\{Y_n\}_{n=1}^\infty$ of E-valued random variables is said to be stochastically bounded if for every $\varepsilon > 0$, $\exists t$ finite such that $P\{\|Y_n\| > t\} < \varepsilon$ for all n. Given a symmetric triangular X-array we get the following extension of Feller's Theorem ([5], p. 309).

2.1. THEOREM. *Let S_n be as in* (1.3); *then S_n is stochastically bounded iff*

(a) *for every $\varepsilon > 0$, there exists t large so that* $\sup_n F_n(B_t^c) < \varepsilon$,

(b) *for every $c > 0$* $\sup_n E\|S_n(c)\|^p$ *is finite.*

Proof. Since $X_{nl} = \sum_{j=1}^{l} X_{nj} - \sum_{j=1}^{l-1} X_{nj}$ we have by the triangle inequality $\|X_{nl}\| \leqslant \big\|\sum_{j=1}^{l} X_{nj}\big\| + \big\|\sum_{j=1}^{l-1} X_{nj}\big\|$. Hence

$$P\big(\max_{1\leqslant j\leqslant k_n} \|X_{nj}\| > t\big) \leqslant P\Big(\max_{1\leqslant l\leqslant k_n} \Big\|\sum_{j=1}^{l} X_{nj}\Big\| > \tfrac{1}{2}t\Big). \tag{2.2}$$

By independence,

$$P\big(\max_{1\leqslant j\leqslant k_n} \|X_{nj}\| > t\big) = 1 - \prod_{j=1}^{k_n} \big(1 - P(\|X_{nj}\| > t)\big). \tag{2.3}$$

By the exponential inequality.

$$1 - P(\|X_{nj}\| > t) \leqslant \exp[P(\|X_{nj}\| > t)]. \tag{2.4}$$

From (2.2), (2.3), (2.4) and Lévy inequality we get for all t,

$$P(\|S_n\| > \tfrac{1}{2}t) \geqslant 1 - \exp\big(-F_n(t)\big).$$

Hence we get (a) from stochastic boundedness of $\{S_n\}$. Let now $c > 0$; then following an argument similar to ([11], Lemma 5.3) we get for $t > 0$,

$$P(\|S_n(c)\| > t) \leqslant 2P(\|S_n\| > t).$$

Therefore for every $c > 0$, $\{S_n(c)\}_{n=1}^\infty$ is stochastically bounded. Let c be fixed. Following the proof of Hoffmann-Jørgensen ([8], Theorem 3.1) we get that $E\|S_n(c)\|^p \leqslant 3K$, where $K = 2\cdot 3^p c^p + 8\cdot 3^p t_0^p$. Here t_0 is chosen so that

$$P(\|S_n\| > t_0) < \frac{1}{32\cdot 3^p}.$$

To prove the converse,

$$P(\|S_n\|>2t)\leqslant P(\|S_n(c)\|>t)+P\left(\left\|\sum_{j=1}^{k_n}\tilde{\tau}(X_{nj},B_c)\right\|>t\right)$$
$$\leqslant\frac{1}{t^p}E\|S_n(c)\|^p+\sum_{j=1}^{k_n}P(\|X_{nj}\|>c).$$

Given $\varepsilon>0$, choose c_0 so that $\sup_n F_n^{(c_0)}(E)<\varepsilon$. Since $\sup_n E\|S_n(c_0)\|^p$ is finite we get $\lim_{t\to\infty}\sup_n P(\|S_n\|>2t)<\varepsilon$ giving stochastic boundedness of $\{S_n\}$.

Before we study compactness of $\{S_n\}$, we give some general results.

2.5. Definition. We say that a symmetric X-triangular array is *uniformly infinitesimal* (*UI*) if $\max_{1\leqslant j\leqslant k_n} P\{\|X_{nj}\|>\delta\}\to 0$ for every $\delta>0$.

2.6. Definition. (a) A probability measure μ on E is said to be *infinitely divisible* (*i.d.*), if for each n, there exists a probability measure μ_n on $(E,\mathscr{B}(E))$ such that $\mu=\mu_n^{*n}$. (If μ_n exists, it is unique.)

(b) A probability measure μ on $(E,\mathscr{B}(E))$ is said to be *centered Gaussian* if for each $y\in E'$, $\mu\circ y^{-1}$ is symmetric Gaussian.

2.7. Remark. By the uniqueness of the measures μ_n of Definition 2.6(a) if μ symmetric, the measures μ_n of Definition 2.6(a) are symmetric. It is known ([24], [13]) that each symmetric i.d. measure μ can be written as $\mu=\varrho*\nu$, where ϱ is centered Gaussian and $\nu=\lim_n e(\nu_n)$ where ν_n is an increasing sequence of symmetric finite measures. Furthermore, the measure ϱ and ν are unique. We shall refer to ν above as non-Gaussian i.d.

We note that μ is i.d. implies that $\mu\circ y^{-1}$ is i.d. for $y\in E'^k$. The following result gives a converse in case μ is symmetric.

2.8. Theorem. *Let E be a real separable Banach space and μ a symmetric probability measure on Borel subsets of E. Then μ is i.d. iff $\mu\circ y^{-1}$ is i.d. for all $y\in E'^k$, for all k.*

Proof. The "only if" part of the theorem being obvious, it suffices to prove the "if" part. Let n be a non-negative integer then for each $y\in E'^k$ there exists a symmetric measure $\mu_n(y)$ on R^k, $\mu\circ y^{-1}=\mu_n^{*n}(y)$. Since $\mu\circ y^{-1}$ is i.d. we get for all t, $\varphi_{\mu\circ y^{-1}}(t)>0$ $(t\in R^k)$ giving $\{\mu_n(y)\colon y\in E'^k, k\geqslant 1\}$ is a cylinder measure ([4]). Call it μ_n; then $\varphi_{\mu_n\circ y^{-1}}(t)=\varphi_{\mu_n(y)}(t)$ for all $t\in R^k$. We then get $\varphi_\mu(y)=\varphi^n_{\mu_n(y)}(1)=\varphi^n_{\mu_n}(y)$, giving $\mu=\mu_n^{*n}$. As μ is symmetric this implies ([4]) that μ_n is a measure, giving the result.

For E-valued r.v. Z and $y=(y_1,y_2,\ldots,y_k)\in E'^k$, $\langle y,Z\rangle$ will denote $(\langle y_1,Z\rangle,\langle y_2,Z\rangle,\ldots,\langle y_n,Z\rangle)$.

2.9. Theorem. *The symmetric i.d. laws on E coincide with the limit laws of row sums of UI symmetric triangular array.*

Proof. Let $\{X_{nj}, j=1,2,\ldots,k_n;\ n=1,2,\ldots\}$ be a UI symmetric triangular array. Then for each k and $y\in E'$, $\{\langle y,X_{nj}\rangle, j=1,\ldots,k_n, n=1,2,\ldots\}$ is a UI symmetric triangular array of R^k-valued random variables. Let $S_n=\sum_{j=1}^{k_n}X_{nj}$ and μ be weak limit of $\mathscr{L}(S_n)$, then $\mu\circ y^{-1}$ is the weak limit of $\mathscr{L}\big(\sum_{j=1}^{k_n}\langle y,X_{nj}\rangle\big)$ giving $\mu\circ y^{-1}$ i.d. by a result in [24], p. 199. Now μ being symmetric we get, by Theorem 2.8, that μ is i.d. Suppose μ is symmetric i.d. then for each n, $\mu=\mu_n^{*n}$ where μ_n is a symmetric probability measure on E. Let $\{X_{nj}\}_{j=1,\ldots,n}$ $(n=1,2,\ldots,n)$ be triangular array such that for each n, $\{X_{nj}\}_{j=1,\ldots,n}$ are independent identically distributed with distribution μ_n. Clearly, $\mu=\lim_n\mu_n^{*n}$. It therefore remains to prove $\{X_{nj}\}_{j=1,\ldots,k_n}$ $(n=1,2,\ldots)$ are uniformly infinitesimal. Since μ_n is symmetric and μ_n^{*n} is relatively compact we get by ([24], p. 59) $\{\mu_n, n=1,2,\ldots\}$ is relatively compact. Also we get by the one-dimensional result ([17], p. 297) $\langle y,X_{nj}\rangle\to\delta_0\circ y^{-1}$ for all y. This implies $\mu_n\Rightarrow\delta_0$. Hence

$$\max_{1\leqslant j\leqslant n}P(\|X_{nj}\|>\varepsilon)=\mu_n\{x\colon\|x\|>\varepsilon\}\to 0\quad\text{ for every }\varepsilon>0.$$

We now give conditions for the convergence of row sums to a non-Gaussian i.d.

2.10. Theorem. *Let $\{X_{nj}, j=1,2,\ldots,k_n, n=1,2,\ldots\}$ be a UI symmetric triangular array and S_n, $S_n(\delta)$, $F_n(\delta)$, μ_n etc. be as in (1.3). Then $\{\mu_n\}$ is conditionally compact with all limit points non-Gaussian iff*

(a) *for each $\delta>0$, $F_n^{(\delta)}$ is conditionally compact,*

(b) $\lim_{\delta\to 0}\sup_n E\|S_n(\delta)\|^p=0$ $(0<p<\infty)$.

Proof. Since $\{\mu_n\}$ is conditionally compact we get by ([16], Theorem 2) that (a) holds. Using symmetry, we get as in ([11], Lemma 5.3), $\tau(X_{nj},B_\delta)=\frac{1}{2}(X_{nj}+X'_{nj})$ where X'_{nj} has the same law as X_{nj}. Hence we get that $\{\mathscr{L}(S_n(\delta))\}_{n,\delta}$ is conditionally compact if $\{\mu_n\}$ is. Furthermore, for $y\in E'$ and $F_{nj}=\mathscr{L}(X_{nj})$,

$$E\langle y,S_n(\delta)\rangle^2=\sum_{j=1}^{k_n}E\langle y,\tau(X_{nj},B_\delta)\rangle^2=\sum_{j=1}^{k_n}\int_{\|x\|\leqslant\delta}\langle y,x\rangle^2F_{nj}\,dx.\tag{2.11}$$

By classical conditions ([6], p. 116) for convergence to non-Gaussian i.d. laws we get

$$\lim_{\varepsilon\to 0}\sup_n\sum_{j=1}^{k_n}\int_{\{|\langle y,x\rangle|\leqslant\varepsilon\}}\langle y,x\rangle^2F_{nj}\,dx=0.\tag{2.12}$$

Since $B_\delta \subseteq \{x|\ |\langle y, x\rangle| \leqslant \|y\|\delta\}$ we get from the conditional compactness of $\{(S_n(\delta)) \colon n \geqslant 1,\ \delta > 0\}$, (2.11), (2.12) and Chebychev's inequality, for every $\varepsilon > 0$,

$$P\{\|S_n(\delta)\| > \varepsilon\} \to 0 \quad \text{uniformly in } n \text{ as } \delta \to 0.$$

Given $\eta > 0$ choose δ_0 such that $\forall \delta \leqslant \delta_0$

$$\operatorname*{Sup}_n P\big(\|S_n(\delta)\| > \tfrac{1}{3}\eta^{1/p}(16)^{-1/p}\big) \leqslant \tfrac{1}{16}3^{-p}.$$

Now following the proof of Theorem 3.1 ([8]) we get that

$$\operatorname*{Sup}_n E\|S_n(\delta)\|^p \leqslant 4\cdot 3^p\delta^p + \eta.$$

From this condition (b) follows.

To prove the converse, given $\varepsilon > 0$ choose $\delta > 0$ so that

$$\operatorname*{Sup}_n E\|S_n(\delta)\|^p \leqslant \tfrac{1}{3}\varepsilon^{p+1} \tag{2.13}$$

and $K \subseteq B_\delta^c$, symmetric compact so that

$$F_n^{(\delta)}(K^c) \leqslant \tfrac{1}{3}\varepsilon. \tag{2.14}$$

Choose a simple function $t\colon E \to E$ such that $\|x - t(x)\| \leqslant \eta$ on K, and $t(x) = 0$ off K with $\eta < \delta$ and $\eta \operatorname*{Sup}_n F_n^{(\delta)}(E) < \frac{1}{3}\varepsilon^2$

$$P\Big\{\Big\|S_n - \sum_{j=1}^{k_n} t(X_{nj})\Big\| > 4\varepsilon\Big\} \tag{2.15}$$

$$\leqslant P\Big\{\Big\|\sum_{j=1}^{k_n}\tau\big(X_{nj} - t(X_{nj}), B_\delta\big)\Big\| > 2\varepsilon\Big\} + P\Big\{\Big\|\sum_{j=1}^{k_n}\tilde{\tau}\big(X_{nj} - t(X_{nj}), B_\delta\big)\Big\| > 2\varepsilon\Big\}.$$

Now the second term of the RHS of the above inequality does not exceed

$$\sum_{j=1}^{k_n} P\{\|X_{nj} - t(X_{nj})\| > \delta\} = \sum_{j=1}^{k_n} P\{\|X_{nj} - t(X_{nj})\| > \delta,\ X_{nj} \notin K\}$$

since $\{X_{nj} \in K\} \cap \{\|X_{nj} - t(X_{nj})\| > \delta\} = \emptyset$.

But $t(X_{nj}) = 0$ if $X_{nj} \notin K$, giving

$$P\Big\{\Big\|\sum_{j=1}^{k_n}\tilde{\tau}\big(X_{nj} - t(X_{nj}), B_\delta\big)\Big\| > 2\varepsilon\Big\} \leqslant F_n^{(\delta)}(K^c). \tag{2.16}$$

The first term on the RHS of (2.15) does not exceed

$$P\Big\{\Big\|\sum_{j=1}^{k_n}\tau\big(X_{nj} - t(X_{nj}), B_\delta\big)1(X_{nj} \notin K)\Big\| > \varepsilon\Big\} + \tag{2.17}$$

$$+ P\Big\{\Big\|\sum_{j=1}^{k_n}\big(X_{nj} - t(X_{nj}), B_\delta\big)1(X_{nj} \in K)\Big\| > \varepsilon\Big\}.$$

Since $t(X_{nj}) = 0$ for $X_{nj} \notin K$ and $B_\delta \subseteq K^c$ we get that the first term of (2.17) does not exceed

$$P\Big\{\Big\|\sum_{j=1}^{k_n}\tau(X_{nj}, B_\delta)\Big\| > \varepsilon\Big\} \leqslant \frac{1}{\varepsilon^p}E\|S_n(\delta)\|^p. \tag{2.18}$$

The second term of (2.17) by Chebychev and the triangle inequalities is $\leqslant (1/\varepsilon)E\sum_{j=1}^{k_n}\big\|\tau\big(X_{nj} - t(X_{nj}), B_\delta\big)1(X_{nj} \in K)\big\|$ and hence does not exceed $(\eta/\varepsilon)F_n(K)$ using the fact that for $X_{nj} \in K$, $\|X_{nj} - t(X_{nj})\| \leqslant \eta$ and $\|X_{nj}\| > \delta$. Thus the second term of (2.17) does not exceed

$$\frac{\eta}{\varepsilon}F_n^{(\delta)}(E). \tag{2.19}$$

Using (2.13), (2.14), (2.16), (2.18) and (2.19) we get that $\{\mathscr{L}(S_n)\}$ is flatly concentrated. Now as before, for any $c > 0$ and K and δ as in (2.13) and (2.14)

$$P\big(\|S_n(c)\| > 2\lambda\big)$$

$$\leqslant P\Big(\Big\|\sum_{j=1}^{k_n}\tau(X_{nj}, B_c \cap B_\delta) + \sum_{j=1}^{k_n}\tau(X_{nj}, B_c \cap K)\Big\| > 2\lambda,\ X_{nl} \in B_\delta \cup K \text{ for some } l \leqslant k_n\Big)$$

$$+ \sum_{l=1}^{k_n} P(\|X_{nl}\| > \delta,\ X_{nl} \notin K).$$

Using the fact that $K \subseteq B_\delta^c$, first the argument of (2.15) and then that of (2.18), yields that for all $c > 0$, $\{S_n(c)\}$ is stochastically bounded. Hence $\langle y, S_n(c)\rangle$ is stochastically bounded. Using the proof of Theorem 2.1 and condition (a) implies that $\{\langle y, S_n\rangle\}$ is stochastically bounded. Now ([1], Theorem 3.1) completes the proof.

2.20. Remark. We note that in the sufficiency part of Theorem 2.10, we have not used the UI hypothesis on the triangular array.

3. R-type, cotype and convergence conditions. This section is devoted to the characterization of the Banach spaces E for which classical conditions hold. We start with the definition.

3.1. Definition. A Banach space E is said to be of *R-type p* if for a family $\{X_1, X_2, \ldots, X_n\}$ of symmetric independent random variables with finite pth moment there exists a constant C independent of n and the random variables such that

$$E\|X_1 + \ldots + X_n\|^p \leqslant C\sum_{j=1}^{n} E\|X_j\|^p. \tag{3.2}$$

We note that ([9], p. 589), condition (3.2) is valid for $\{X_1, \ldots, X_n\}$ iff it is valid for $X_i = \varepsilon_i x_i$ $(i = 1, 2, \ldots, n)$, where $\{\varepsilon_i\}_{i=1}^n$ are symmetric independent Bernoulli random variables and for all $\{x_1, \ldots, x_n\} \subset E$. Hence we have the nomenclature R-type (Rademacher type).

3.3. Theorem. *Let $\{X_{nj}, j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots\}$ be a UI symmetric triangular array of E-valued random variables such that $\mathscr{L}(S_n) \Rightarrow \mathscr{L}(Z)$ where $\mathscr{L}(Z)$ is non-Gaussian. Then there exists a σ-finite-measure F on E, F finite outside every neighbourhood of zero such that $F_n^{(\delta)}$ converges weakly to $F^{(\delta)}$ for all $\delta > 0$ such that $F(\partial B_\delta) = 0$.*

Proof. We note that $\mathscr{L}(Z)$ being a non-Gaussian infinitely divisible law, by ([24], p. 103) there exists a unique σ-finite-measure G, finite outside the neighbourhood of zero such that for each $y \in E'$ $\langle y, Z\rangle$ has Lévy measure $G \circ y^{-1}$. Let $\{\delta_k\}_{k=1}^\infty$ be a sequence of positive real numbers converging to zero. By Theorem 2.10, using Cantor's diagonalization procedure we get that there exists a subsequence $\{n'\}$ of $\{n\}$ such that for each k, $F_{n'}^{(\delta_k)}$ converges to a finite measure, say, F_k. Furthermore $F_k \uparrow$. Let us define $F = \lim_k F_k$. Then F is σ-finite. Using the fact that $\mathscr{L}(\langle y, S_n\rangle) \Rightarrow \mathscr{L}(\langle y, Z\rangle)$, gives by classical results and the uniqueness of G that $F \circ y^{-1} = G \circ y^{-1} \forall y \in E'$. Thus $F = G$ and, in particular, $F = G$ outside the neighbourhood of zero. Hence F is the unique limit of every convergent subsequence.

3.4. Corollary ([29], Theorem 4). *The following properties of a Banach space E are equivalent*

(i) *E is R-type p.*

(ii) *For each UI symmetric triangular array $\{X_{nj}, j = 1, 2, \ldots, k_n, n = 1, 2 \ldots\}$ of E-valued random variables and a σ-finite measure F, satisfying*

(a) *$F_n^{(\delta)}$ converges weakly to $F^{(\delta)} = F|_{B_\delta^c}$ where F_n is as in* (1.3), *$B_\delta = \{x|\ \|x\| \leqslant \delta\}$ and $\delta \cdot \ni \cdot F(\partial B_\delta) = 0$; and*

(b) $\lim_{\varepsilon \to 0} \overline{\lim_n} \sum_{j=1}^{k_n} \int_{\|x\| \leqslant \varepsilon} \|x\|^p F_{nj}(dx) = 0$

we have $\mathscr{L}(S_n) \Rightarrow \mathscr{L}(Z)$ where $\mathscr{L}(Z)$ is an i.d. probability measure with characteristic function

$$\exp \int_E [\cos\langle y, x\rangle - 1] F(dx), \quad y \in E'. \tag{3.5}$$

Proof. (i) implies (ii): We first observe that conditions (a) and (b) imply $\int_{\|x\| \leqslant \varepsilon} \|x\|^p F(dx) < \infty$ for some $\varepsilon > 0$. Since, condition (a) gives

$$\int_{\|x\| \leqslant \varepsilon} \|x\|^p F(dx) \leqslant \int_0^{\varepsilon^p} F(\|x\| > t^{1/p})\, dt = \int_0^{\varepsilon^p} \lim_n F_n(\|x\| > t^{1/p})\, dt$$

which implies the desired result by condition (b) and Fatou's Lemma. Now E is of R-type p, F is finite outside every neighbourhood of zero and for some $\varepsilon > 0$, $\int_{\|x\| \leqslant \varepsilon} \|x\|^p F\, dx$ is finite implies (3.5) is the characteristic function of a measure μ on E ([7]). Let Z be such that $\mathscr{L}(Z) = \mu$. Also by (3.2) condition (b) implies the condition (b) of Theorem 2.10 giving $\mathscr{L}(S_n)$ compact. Now by using the given conditions and UI (see [13], pp. 145–146) we get $\mathscr{L}(\langle y, S_n\rangle) \Rightarrow \mathscr{L}(\langle y, Z\rangle)$ for each $y \in E'$ giving $\mathscr{L}(S_n) \Rightarrow \mathscr{L}(Z)$. To prove the converse assume $\{x_j\}_{j=1}^\infty \subset E$ is such that $\sum_{j=1}^\infty \|x_j\|^p$ converges and $\sum_{j=1}^\infty \varepsilon_j x_j$ does not converge a.s. Then by ([10], p. 40) there exists subsequence $\{l_n\}_{n=1}^\infty$, $\{k_n\}_{n=1}^\infty$ $(l_n, k_n \to \infty)$ such that $\mathscr{L}\big(\sum_{j=l_n+1}^{l_n+k_n} \varepsilon_j x_j\big) \not\Rightarrow \delta_0$, the measure degenerate at zero. Define $X_{nj} = \varepsilon_{l_n+j} x_{l_n+j}$ $(1 \leqslant j \leqslant k_n)$. Now $F_{nj} = \frac{1}{2}(\delta_{x_{l_n+j}} + \delta_{-x_{l_n+j}})$ and hence $F_n = \sum_{j=l_n+1}^{k_n+l_n} \frac{1}{2}(\delta_{x_j} + \delta_{-x_j})$.

Clearly $\{X_{nj}, j = 1, \ldots, k_n, n = 1, 2\}$ is a UI symmetric triangular array. Furthermore, condition $\sum_{j=1}^\infty \|x_j\|^p$, finite implies $F_n^{(t)} \Rightarrow 0$ for all $t > 0$ and since

$$\int_{\|x\| \leqslant \varepsilon} \|x\|^p F_n\, dx \leqslant \sum_{\{j: \|x_j\| \leqslant \varepsilon\}} \|x_j\|^p,$$

also condition (b), giving $\mathscr{L}\big(\sum_{j=l_n+1}^{l_n+k_n} \varepsilon_j x_j\big)$ converges to zero since $F \equiv 0$ in this case. This proves that $\sum_j \|x_j\|^p < \infty$ implies $\sum_{j=1}^\infty \varepsilon_j x_j$ converges in distribution. Hence by ([10], p. 40) a.s. But this gives E is of R-type p.

3.6. Definition. We say that E is of *cotype q* if for a family $\{X_1, \ldots, X_n\}$ of symmetric independent random variables with finite qth moment there exists a constant $\tilde{C}$ independent of n (and the random variables) such that

$$E\|X_1 + \ldots + X_n\|^q \geqslant \tilde{C} \sum_{j=1}^n E\|X_j\|^q. \tag{3.7}$$

We note as before that the condition (3.2) is valid iff for any sequence $\{x_j\}_{j=1}^\infty \subset E$, $\sum \varepsilon_j x_j$ converges in distribution implies $\sum \|x_j\|^q$ is finite ([9]).

3.8. Corollary. *The following properties of a Banach space E are equivalent*

(i) *E is of cotype q.*

(ii) *For each UI symmetric triangular array $\{X_{nj}, j = 1, 2, \ldots, k_n, n = 1, 2, \ldots\}$ of E-valued random variables $\mathscr{L}(S_n) \Rightarrow \mathscr{L}(Z)$, Z non-Gaussian implies*

$$\lim_{\varepsilon \to 0} \overline{\lim_n} \sum_{j=1}^{k_n} \int_{\|x\| < \varepsilon} \|x\|^q F_{nj}(dx) = 0. \tag{3.9}$$

Here $F_{nj} = \mathcal{L}(X_{nj})$ *and* $S_n = \sum_{j=1}^{k_n} X_{nj}$.

Proof. In view of Theorem 3.3, it suffices to show that condition (b) of Theorem 2.10 implies (3.9). But this is a consequence of E being of cotype q and (3.7). To prove the converse, suppose $\sum \varepsilon_j x_j$ converges in distribution but $\sum \|x_j\|^q = \infty$. Then there exists $\{l_n\}_{n=1}^{\infty}$ and $\{k_n\}_{n=1}^{\infty}$ $(l_n, k_n \to \infty)$ such that $\lim_n \sum_{l_n+1}^{l_n+k_n} \|x_j\|^q \nrightarrow 0$. Define now $X_{nj} = \varepsilon_{j+l_n} x_{j+l_n}$, $j = 1, 2, \ldots, k_n$. Then $\{X_{nj}, j = 1, 2, \ldots, k_n, n = 1, 2, \ldots\}$ is a UI symmetric triangular array and

$$\sum_{j=1}^{k_n} X_{nj} = \sum_{l_n+1}^{l_n+k_n} \varepsilon_j x_j \to 0.$$

Hence by (ii)

$$\lim_{\varepsilon\to 0} \overline{\lim_n} \sum_{\substack{\|x_j\|\leqslant\varepsilon \\ \{l_n<j\leqslant l_n+k_n\}}} \|x_j\|^q = 0.$$

But $x_j \to 0$ since $\sum_j \varepsilon_j x_j$ converges. Hence for n sufficiently large $\lim_n \sum_{l_n+1}^{l_n+k_n} \|x_j\|^q = 0$ contradicts the assumption giving E is of cotype q.

From Corollaries 3.4, 3.8, Theorem 3.3 and Kwapień's Theorem; we get

3.10. Corollary. *A Banach space E is isomorphic to a Hilbert space iff for every UI symmetric triangular array of E-valued random variables the following are equivalent with the notation* (1.3)

(i) $\mathcal{L}(S_n) \Rightarrow \mu$, *$\mu$ non-Gaussian i.d.*

(ii) (a) *There exists a σ-finite measure F on E such that $F_n^{(\delta)}$ converges weakly to $F^{(\delta)} = F|B_\delta^c$ for each $\delta > 0$ with $F(\partial B_\delta) = 0$.*

(b) $\lim_{\varepsilon\to 0} \overline{\lim_n} \sum_{j=1}^{k_n} \int_{\|x\|\leqslant\varepsilon} \|x\|^2 F_{nj}(dx) = 0$.

Furthermore, in either case F is the Lévy measure associated with μ.

3.11. Definition. A Banach space E is said to be of *type p-stable* if for $\{x_i\}_{i=1}^{\infty} \subset E$ with $\sum \|x_i\|^p$ finite we get $\sum x_i \eta_i$ converges a.e. if $\{\eta_i\}$ are independent, identically distributed symmetric stable random variables with $\varphi_{\mathcal{L}(\eta_1)}(t) = \exp(-|t|^p)$.

From Definition 3.11, Corollary 3.4 and ([22], Proposition 2.1) we get the following corollary with

$$c_p = p \int_0^{\infty} (\cos u - 1) \frac{1}{u^{1+p}} du.$$

3.12. Corollary. *Let $p < 2$. Then the following conditions are equivalent*

(1) *E is of type p-stable.*

(2) *There exists $q > p$ such that for every UI symmetric triangular array $\{X_{nj}, j = 1, 2, \ldots, k_n, n = 1, 2, \ldots\}$ and a finite Borel measure λ on $\Sigma = \{x| \ \|x\| = 1\}$ we have with the notation* (1.3),

$$\lim_{n\to\infty} F_n\left\{x| \ \|x\| > t \text{ and } \frac{x}{\|x\|} \in A\right\} = t^{-p}\lambda(A) \tag{3.13}$$

for each $t > 0$ and λ-continuity set A; and

$$\lim_{\varepsilon\to 0} \overline{\lim_n} \int_{\|x\|\leqslant\varepsilon} \|x\|^q F_n(dx) = 0 \tag{3.14}$$

implies $\mathcal{L}(S_n) \Rightarrow \mathcal{L}(Z)$ where Z is an E-valued stable random variable ∋

$$\varphi_{\mathcal{L}(Z)}(y) = \exp\left(-\int_{\Sigma} |\langle y, u\rangle|^p \Gamma(du)\right), \quad y \in E'; \quad \text{where } \Gamma = c_p\lambda. \tag{3.15}$$

Furthermore, (3.13) *is necessary for $\mathcal{L}(S_n) \Rightarrow \mathcal{L}(Z)$.*

Proof. The only fact that remains to be proved is that if E is p-stable then (3.15) is a characteristic functional of a measure μ on E. But this is known ([2], [23], [18]). The last statement follows by Theorem 3.3.

4. Spaces of stable type and the domain of attraction. We say that an E-valued random variable X is in the domain of attraction of a E-valued random variable Y if there exist $b_n > 0$ and $a_n \in E$ $(n = 1, 2, \ldots)$ such that $\mathcal{L}\left(\frac{X_1 + \ldots + X_n}{b_n} - a_n\right)$ converges weakly to $\mathcal{L}(Y)$. It is shown in [15] that Y has non-empty domain of attraction iff $\mathcal{L}(Y)$ is stable. In case $b_n = n^{1/p}$ we say that X is in the domain of normal attraction of Y. Recently, ([23], [28] see also [2], [18]) it was shown that only on stable type spaces E, the Lévy representation of non-Gaussian stable laws can be completely determined. The problem we shall study in this section is to determine properties of the distribution of X. In case E is a Hilbert space the problem was completely solved in ([14]). In Banach spaces E of stable type partial results on this problem were obtained in ([19], [20], [21], [29]). Our methods are different from all these as we only use Corollary 3.12 and techniques developed in [14] using the work of Feller ([5]).

Remark. We note that X lies in the domain of attraction of Z iff X is in the domain of attraction of tZ for $0 < t < \infty$.

4.1. Theorem. *The following conditions are equivalent for $p < 2$.*

(i) *E is of type p-stable.*

(ii) *A symmetric random variable X lies in the domain of attraction* *of a symmetric stable E-valued random variable Y with* $\varphi_{\mathscr{L}(Y)}(y) = \exp\left[-\int_{\Sigma}|\langle y, u\rangle|^p\Gamma(du)\right]$ *and* $\Gamma(\Sigma) > 0$ *iff*

(4.2.1) $Z(t) = P(\|X\| > t)$ *is regularly varying with exponent* $(-p)$

(see [5], p. 276) *and for every Γ-continuity set A*,

$$\frac{P(\|X\|>t, X/\|X\|\in A)}{P(\|X\|>t)} \to \frac{\Gamma(A)}{\Gamma(\Sigma)}. \tag{4.2.2}$$

(iii) $t^pP(\|X\|>t)\to 0$ *as* $t\to\infty$ *iff X lies in the domain of normal attraction of* $Y \equiv 0$.

Proof. (i) ⇒ (ii): We first note that

$$U(t) \equiv \int_{\|X\|\leqslant t}\|X\|^q dP = qZ_{q-1}(t) - t^qZ(t), \tag{4.3}$$

where $Z_{q-1}(t) = \int_0^t u^{q-1}Z(u)du$. From (4.3) and ([5], Theorem 1, p. 281) we get for $q > p$

$$\lim_{t\to\infty}\frac{t^qZ(t)}{U(t)} = \frac{q-p}{p}. \tag{4.4}$$

Using an argument of ([5], p. 314) we get $\{b_n\}$ $b_n\to\infty$, $\frac{b_n}{b_{n+1}}\to 1$ such that

$$\lim_n nb_n^{-q}U(tb_n) = t^{q-p}. \tag{4.5}$$

From (4.5) (putting $t = \varepsilon$) we get (3.14). Now (4.4) and (4.5) imply

$$nP(\|X\|>b_nt)\to\frac{q-p}{p}t^{-p}.$$

Hence by (4.2.2) we get for a Γ-continuity set A,

$$\lim_n nP\left(\|X\|>b_nt, \frac{X}{\|X\|}\in A\right) = \frac{q-p}{p}t^{-p}\frac{\Gamma(A)}{\Gamma(\Sigma)} = \lambda(A)t^{-p} \text{ (say)}. \tag{4.6}$$

This gives (3.13). By Corollary 3.12 we get that X lies in the domain of attraction of Z; where

$$\varphi_{\mathscr{L}(Z)}(y) = \exp\left[-c_p\int_{\Sigma}|\langle y, u\rangle|^p\lambda du\right] \quad \text{with} \quad c_p = p\int_0^\infty(\cos u - 1)\frac{1}{u^{p+1}}du.$$

By the remark preceding Theorem 4.6 we get X lies in the domain of attraction of Y. Conversely, by (3.12) we get that if X is in the domain of attraction of the stable law $L(Y)$ then (3.13) is satisfied. But (3.13) implies (4.5) using the fact that the b_n involved satisfy $b_n\to\infty$ and $\frac{b_n}{b_{n+1}}\to 1$ ([15], p. 136) as in ([14], p. 160–161). In the above proof $\{X_j/b_n, j = 1, 2, \ldots, n\}$ is clearly UI.

(ii) ⇒ (iii): Assume that X and θ are independent and symmetric, X E-valued and θ real-valued, satisfying $t^pP(\|X\|>t)\to 0$ as $t\to\infty$ and $Ee^{it\theta} = e^{-|t|^p}$. Choose $e\in E$ with $\|e\| = 1$ and define $Y = X+\theta e$. We first show that Y satisfies the hypotheses of (ii).

As in Feller ([5], p. 271, 1st edition) for $t>0$ and $\varepsilon>0$,

$$P[\|Y\|>t]\geqslant P[\|X\|>t(1+\varepsilon)]P[|\theta|<t\varepsilon]+P[|\theta|>t(1+\varepsilon)]P[\|X\|<t\varepsilon],$$

$$P[\|Y\|>t]\geqslant P[\|X\|>t(1-\varepsilon)]+P[|\theta|>t(1-\varepsilon)]+ \\ +P[\|X\|>t\varepsilon]P[|\theta|>t\varepsilon].$$

Hence $\hat{Z}(t) = P(\|Y\|>t)$ is regularly varying of exponent $(-p)$.

By ([6], p. 116)

$$nP\left(\frac{\theta}{n^{1/p}}\in\cdot\right)\Rightarrow d\Gamma\times\frac{dr}{r^{1+p}}(\cdot),$$

where Γ is supported on $\{\pm 1\}$, $\Gamma\{\pm 1\}>0$ and Γ is symmetric. Hence for every $\lambda>0$ there exists a symmetric closed interval J such that interior $(J)\supseteq[-\lambda, \lambda]$ and a $\delta>0$ such that

$$(J^c)^\delta\subseteq[-\lambda, \lambda], \quad \text{and} \quad nP\left(\frac{\theta}{n^{1/p}}\in(J^c)^\delta\right)<\varepsilon.$$

Now choose $\delta_0>0$ such that $[(Je)^c]^{\delta_0}\cap \boldsymbol{R}e\subseteq(J^c)^\delta e$, where the δ_0-ball is computed with respect to the norm on E. Then since $t^pP(\|X\|>t)\to 0$, there exists $n_0 = n_0(\varepsilon, \delta_0)$ such that $n\geqslant n_0$ implies $nP(\|X\|>\delta_0n^{1/p})<\varepsilon$. Therefore,

$$nP\left(\frac{Y}{n^{1/p}}\notin Je\right)\leqslant nP\left(\frac{Y}{n^{1/p}}\notin Je, \|X\|\leqslant\delta_0n^{1/p}\right)+nP(\|X\|>\delta_0n^{1/p})$$
$$\leqslant nP\left(\frac{\theta}{n^{1/p}}e\in[(Je)^c]^{\delta_0}\right)+\varepsilon$$
$$\leqslant nP\left(\frac{\theta}{n^{1/p}}\in(J^c)^\delta\right)+\varepsilon<2\varepsilon.$$

Hence $\left\{nP\left(\frac{Y}{n^{1/p}}\in\cdot\right)\right\}$ is conditionally compact outside each neighborhood of $0\in E$.

On the other hand the conditions on the tail of $\|X\|$ imply that for each $f\in E'$, $f\left(\sum_{j=1}^n X_j\right)/n^{1/p}\xrightarrow{P}0$. Therefore

$$\mathscr{L}\left[f\left(\sum_{i=1}^n\frac{(X_i+\theta_ie)}{n^{1/p}}\right)\right]\Rightarrow\mathscr{L}(f(\theta e)) \quad \text{for each } f\in E'.$$

This implies by the one-dimensional result that $nP\big(f(Y)/n^{1/p}\in\cdot\big)\Rightarrow F\cdot f^{-1}(\cdot)$, where $dF = d\hat{\Gamma}\times\frac{dr}{r^{1+p}}(\cdot)$ and $\hat{\Gamma}$ is supported on $\{\pm e\}$ and $\hat{\Gamma}\{e\}=\hat{\Gamma}\{-e\}=\Gamma(\{1\})$. We then obtain that

$$nP\left(\frac{Y}{n^{1/p}}\in\cdot\right)\Rightarrow F.$$

This yields

$$\frac{P(\|Y\|>t,\ Y/\|Y\|\in\cdot)}{P(\|Y\|>t)}\Rightarrow\frac{F\{\|y\|>t,\ y/\|y\|\in\cdot\}}{F\{\|y\|>t\}}.$$

The hypotheses now imply

$$\sum_{j=1}^{n}\frac{X_j+\theta_j e}{n^{1/p}}\Rightarrow\theta e.$$

By stability of θ we then have $\sum_{j=1}^{n}X_j/n^{1/p}+\theta e\Rightarrow\theta e$. Finally this implies $\sum_{j=1}^{n}X_j/n^{1/p}\xrightarrow{P}0$.

(iii) $\Rightarrow$ (i): First we show that (iii) is a super-property. Let $L_0^{p,\infty}=\{X\colon\ \Omega\to E\mid c^pP(\|X\|>c)\to 0$ as $c\to\infty\}$ and define the quasi-norm $\Lambda_p(\cdot)$ on $L_0^{p,\infty}$ by $\Lambda_p(X)=[\sup_{c>0}c^pP(\|X\|>c)]^{1/p}$. Now let $\mathrm{CL}(p,r)=\{X\colon\ \Omega\to E\mid\|X\|_{p,r}^r\equiv\sup_n E\|\tilde{S}_n/n^{1/p}\|^r<\infty\}$, where $\tilde{S}_n$ is the sum of the symmetrized X_j's. By (iii) (see Proposition 2.1, [26], and also [11], Theorem 5.7) for $r<p$, we may define the inclusion map $T\colon(L_0^{p,\infty},\Lambda_p)\to(\mathrm{CL}(p,r),\|\cdot\|_{p,r})$. By a trivial application of the closed graph theorem there exists a constant $B<\infty$ such that $\|X\|_{p,r}\leqslant B\Lambda_p(X)$. It is also easy to see that $X\in L_0^{p,\infty}$ if and only if X can be approximated in $\Lambda_p(\cdot)$ norm by simple functions. Now if Y is a simple function then by the finite-dimensional central limit theorem $\lim_n E\|\sum_{j=1}^{n}Y_j/n^{1/p}\|^r=0$, since $2>p$. Hence the range of T is included in the set of X's such that

$$\lim_n E\Big\|\sum_{j=1}^{n}X_j/n^{1/p}\Big\|^r=0.$$

Hence (iii) is a super-property.

By the Maurey–Pisier–Krivine Theorem ([22], Theorem 2.3 and [30]) it now suffices to show that (iii) does not hold in l^p. For this purpose let $\{\varepsilon_j\}$ and $\{N_j\}$ be independent sequences of i.i.d. random variables with

$$P(\varepsilon_j=1)=P(\varepsilon_j=-1)=\tfrac{1}{2},$$

$$P(N_j\geqslant n)=\frac{1}{n\,\mathrm{LL}\,n},\quad\text{where}\quad \mathrm{LL}\,n=\begin{cases}\ln(\ln n), & n\geqslant 27,\\ 1 & \text{otherwise},\end{cases}$$

and $P(N_j\in\{1,2,\dots\})=1$. Now let

$$X_j=\varepsilon_j\sum_{N_j^2-N_j<r\leqslant N_j^2+N_j}e_r=\varepsilon_j\sum_{r=1}^{\infty}\varphi_{jr}e_r,$$

where $\{e_r\}$ is the natural basis for l^p and

$$\varphi_{jr}=I[N_j^2-N_j<r\leqslant N_j^2+N_j].$$

Then

$$nP\big(\|X\|_p>(2n)^{1/p}\big)=nP(N>n)=\frac{n}{(n+1)\,\mathrm{LL}(n+1)}\to 0.$$

On the other hand, if $\frac{S_n}{n^{1/p}}\to 0$ in probability, then $\frac{S_n'}{n^{1/p}}\to 0$ in probability, where $S_n'=\sum_{j=1}^{n}X_{nj}$ and $X_{nj}=\varepsilon_j\sum_{r=1}^{n^2+n}\varphi_{jr}e_r$. From the proof of Theorem 3.1 in [8] we have

$$E\left\|\frac{S_n'}{n^{1/p}}\right\|_p^p\leqslant B+2\cdot 3^pE\left[\max_{1\leqslant j<n}\frac{\|X_{nj}\|_p^p}{n}\right].$$

But $\|X_{nj}\|_p^p=2N_jI[N_j\leqslant n]$. Hence

$$\sup_n E\left\|\frac{S_n'}{n^{1/p}}\right\|_p^p<\infty.$$

Now by Khintchine's inequality for the Rademacher functions, there exists $K_p<\infty$, such that

$$K_p^pE\left\|\frac{S_n'}{n^{1/p}}\right\|_p^p=K_p^p\sum_{r=1}^{n^2+n}E\left|\sum_{j=1}^{n}\frac{\varepsilon_j\varphi_{jr}}{n^{1/p}}\right|^p\geqslant\sum_{r=1}^{n^2+n}E\left[\sum_{j=1}^{n}\frac{\varphi_{jr}}{n^{2/p}}\right]^{p/2}$$
$$\geqslant\frac{1}{n}\sum_{r=1}^{n^2+n}P\left(\sum_{j=1}^{n}\varphi_{jr}\geqslant 1\right)=\frac{1}{n}\sum_{r=1}^{n^2+n}[1-(1-p_r)^n]$$
$$\geqslant\frac{1}{n}\sum_{r=n}^{n^2+n}(1-p_r)^{n-1}np_r=A_n,$$

where

$$p_r=P(N^2-N<r\leqslant N^2+N)=P\left(\frac{-1+\sqrt{1+4r}}{2}\leqslant N<\frac{1+\sqrt{1+4r}}{2}\right).$$

Now $\frac{\delta}{r\,\mathrm{LL}r} \leqslant p_r \leqslant \frac{C}{r\,\mathrm{LL}r}$ for some $0 < \delta$, $C < \infty$ and $p_1 \geqslant p_2 \geqslant \ldots$

Therefore

$$(1-p_r)^{n-1} \geqslant (1-p_n)^{n-1} \quad (\text{since } r \geqslant n)$$
$$\geqslant \left(1-\frac{C}{n\,\mathrm{LL}n}\right)^{n-1}.$$

Then

$$A_n \geqslant \delta\left(1-\frac{C}{n\,\mathrm{LL}n}\right)^{n-1} \sum_{r=n}^{n^2+n} \frac{1}{r\,\mathrm{LL}r} \geqslant \delta\left(1-\frac{C}{n\,\mathrm{LL}n}\right)^{n-1} \left(\sum_{r=n}^{n^2+n} \frac{1}{r\,\mathrm{L}r}\right)\left(\frac{\mathrm{L}n}{\mathrm{LL}n}\right)$$
$$\geqslant \delta\mathrm{L}(2)\left(1-\frac{C}{n\,\mathrm{LL}n}\right)^{n-1}\left(\frac{\mathrm{L}n}{\mathrm{LL}n}\right) \to \infty.$$

4.7. Remarks. After this work was completed we received some work of A. Araujo and E. Giné. It contains different conditions for the general domain of attracion problem similar to those in [14]. However, our condition as well as proof are simple and follow easily from our main Theorem 2.10. Thus our methods are entirely different.

We have also received work [3] of de Acosta, Araujo and Giné which contains conditions for convergence to i.d. laws. However, again their conditions and methods are entirely different. In both works, they consider the general (non-symmetric) case.

4.8. Theorem. *If X is in the domain of attraction of a symmetric stable law of index $p < \lambda$ on any real separable Banach space E, then*

$$P(\|X\| > t) \sim \frac{L(t)}{t^p}$$

as $t \to \infty$, where $L(t)$ is a slowly varying function ([5], p. 276). *In particular, for any symmetric stable random variable on E we get $E\|X\|^q$ is finite for $0 \leqslant q < p$.*

We note that in the latter part one only uses the fact that a symmetric stable random variable is in its own domain of normal attraction ([15], p. 139).

Note added in proof: We thank Professor S.A. Chobanian for pointing out an error in the original version of Theorem 2.8.

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Weighted norm inequalities for the Lusin area integral and the nontangential maximal functions for functions harmonic in a Lipschitz domain

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Abstract. We prove weighted integral inequalities between the Lusin area integral and the nontangential maximal function of a function harmonic in a Lipschitz domain. These inequalities are extensions to the Lipschitz case of inequalities obtained by Gundy and Wheeden [7] for functions harmonic in a half space.

1. Introduction. In this paper we shall prove integral inequalities between area integrals and nontangential maximal functions for functions harmonic in a Lipschitz domain $\Omega \subset R^n$. That is, we shall assume that to each boundary point $P \in \partial\Omega$ there is associated an open cone $\Gamma(P)$ with vertex at P such that $\Gamma(P) \subset \Omega$. If now u is harmonic in Ω we define

$$A(u, P) = \Big(\int_{\Gamma(P)} |P-Q|^{2-n} |\nabla u(Q)|^2 dm(Q)\Big)^{\frac{1}{2}}$$

and

$$N(u, P) = \sup_{\Gamma(P)} |u(Q)|.$$

Here ∇u denotes the gradient of u and m denotes the Lebesgue measure. Our main result is that if the cones $\Gamma(P)$ satisfy suitable regularity conditions (to be formulated later) then for all harmonic functions u vanishing at a fixed point P^* we have

$$C_1 \int_{\partial\Omega} \Phi\big(A(u)\big)\, d\mu \leqslant \int_{\partial\Omega} \Phi\big(N(u)\big)\, d\mu \leqslant C_2 \int_{\partial\Omega} \Phi\big(A(u)\big)\, d\mu. \tag{1.1}$$

Here μ is allowed to vary over a wide class of measures which includes the surface measure of $\partial\Omega$ and the harmonic measure. The precise assumption on μ is that μ is positive, nonvanishing on any component of $\partial\Omega$ and that there are numbers $A > 0$ and $\theta > 0$ such that for all $P \in \partial\Omega$ and all $r > 0$ we have that whenever $E \subset \Delta(P, r)$ then

$$\frac{\mu(E)}{\mu(\Delta(P, r))} \leqslant A\left(\frac{\sigma(E)}{\sigma(\Delta(P, r))}\right)^{\theta} \quad \text{and} \quad \mu\big(\Delta(P, 2r)\big) \leqslant C\mu\big(\Delta(P, r)\big). \tag{1.2}$$