Damit sind die von Pełczyński in Studia Math. 38 (1970), S. 476 vorgeschlagenen Probleme 38, 39, 40 vollständig gelöst.

Wir haben ferner sogar bewisen, daß es keinen (FS)-Raum gibt, der sämtliche nuklearen (F)-Räume als Quotienten hat (zur Nichtexistenz eines universellen (FS)-Raumes bzgl. Unterräumen s. [8]). Da wir die Konstruktion von $\lambda(A)$ in 3.6 ohne Schwierigkeiten so gestalten können, daß $\lambda(A)$ s-nuklear (s. [6], [9]) oder auch $\Lambda(a)$-nuklear in einem beliebigen Sinne (s. z.B. [3]) ist, folgt, daß auch kein (FS)-Raum (und damit kein nuklearer, s-nuklearer usw. Raum) existiert, der alle diese Räume zu Quotienten hat.

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Tempered nontangential boundedness

by

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Abstract. We prove results on the growth and convergence properties of distributions and their derivatives, with the help of results from our earlier paper [3].

The object of this article is to show that with certain necessary modifications all the standard results on nontangential boundedness of harmonic functions are true when any mollifier is used, not just the Poisson kernel.

Let $u(x, t)$ be a harmonic function of the upper half space $\Omega^* = \{(x, t): x \in \mathbf{R}^n, t > 0\}$. u is said to be *nontangentially bounded at a point* x_0 if there exist constants $A > 0$, $\alpha > 0$, $h > 0$ such that

$$\sup_{\Gamma_\alpha^h(x_0)} |u(x, t)| < \infty$$

where $\Gamma_\alpha^h(x_0) = \{(x, t): |x - x_0| < \alpha t, 0 < t < h\}$. It has been shown that if u is nontangentially bounded at almost every point of a set E, then it is nontangentially convergent at almost every point of that set. In addition, every harmonic conjugate of u is nontangentially bounded almost everywhere in E.

With this in mind we attempt to apply the notion of nontangential boundedness to the study of a distribution f at a point x_0. Let φ be a C^∞ function such that $\int \varphi(x)dx \neq 0$. Define $u(x, t) = f * \varphi_t(x)$. If φ is the Poisson kernel, then the results quoted above are statements about the good behavior of f on the set E. In view of the work of Fefferman and Stein on the real variable theory of H^p spaces [2], it is reasonable to expect that any mollifier could be used, not only the Poisson kernel.

It is therefore surprising to discover that this is not the case. We will see in Chapter II that there exist a tempered distribution $f \in \mathscr{S}'$ and Schwartz functions φ and Φ with mean value one such that $f * \varphi_t(x)$ is nontangentially bounded almost everywhere but $f * \Phi_t(x)$ is nontangentially bounded almost nowhere.

It is possible to strengthen the definition of nontangential boundedness in such a way that we can eliminate this dependence on the mollifier. We do this by defining tempered nontangential boundedness. In Chapter I we will show that this does not depend on the mollifier being used.

Let $f \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$ such that $\int \varphi(x)\,dx \neq 0$. f is said to be *tempered nontangentially bounded at* x_0 if there exist constants A and λ such that

$$\sup_{(x,t)\in\Gamma_\alpha^h(x_0)} |f*\varphi_t(x)| \leqslant A(1+\alpha)^\lambda \quad \text{for all } \alpha > 0.$$

Thus we are requiring that $f*\varphi_t(x)$ is nontangentially bounded and the nontangential bound increases slowly as a function of the aperture α.

This stronger requirement that is forced on us is not however a great restriction. In fact if we have any reasonable control over f then f will be tempered nontangentially bounded; for example, if $f \in H^p$ or if f is locally integrable in some open set.

In the third chapter we review the theory of certain $\mathscr{G}$ functions and the corresponding Banach spaces $\Lambda_\gamma^{p,\lambda}(x_0)$. As in the case of Lusin's area integral, the operations of differentiation and singular integral operators are best handled by using various integral expressions, rather than by tempered nontangential boundedness. The proofs of the results of this chapter can be found in [3].

The spaces $\Lambda_\gamma^{p,\lambda}(x_0)$ are defined by

$$\Lambda_\gamma^{p,\lambda}(x_0) = \{f \in \mathscr{S}' \colon N_\gamma^{p,\lambda}(f)(x_0) < \infty\}$$

where

$$N_\gamma^{p,\lambda}(f)(x_0) = \mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) + \text{“harmless terms”}.$$

The important parameter is $\gamma \in \boldsymbol{R}$. This gives the order of differentiability of f at x_0. We will recall that differentiation and certain pseudo-differential operators are bounded on these spaces.

In the final chapter we will discuss the connection between tempered nontangential boundedness and the spaces $\Lambda_\gamma^2(E)$. For a set E of positive measure, the space $\Lambda_\gamma^2(E)$ is defined by

$$\Lambda_\gamma^2(E) = \{f \in \mathscr{S}' \colon f \in \Lambda_\gamma^2(x_0) = \bigcup_{\lambda>0} \Lambda_\gamma^{2,\lambda}(x_0) \text{ for almost every } x_0 \in E\}$$

$$= \{f \in \mathscr{S}' \colon \mathscr{G}_{\gamma,k}^{2,\lambda}(f)(x_0) < \infty \text{ for some } \lambda > 0 \text{ and } k \gg \gamma \text{ for a.e. } x_0 \in E\}.$$

We will prove that $u(x,t) = f*\varphi_t(x)$ is tempered nontangentially bounded if and only if $f \in \Lambda_0^2(E)$. Since $f \in \Lambda_0^2(E)$ is equivalent to the finiteness of some Littlewood-Paley g_λ^* function, this result will be our analog of the theorem for harmonic functions that nontangential boundedness a.e. E is equivalent to the finiteness of the area integral a.e. E.

For harmonic functions, if f is nontangentially bounded at every point of a set E, then f is nontangentially convergent almost everywhere in E. By taking advantage of this result and using the $\Lambda_0^2(E)$ spaces we can prove a similar theorem. Let $f \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$ with $\int \varphi(x)\,dx \neq 0$. We will prove that if $f*\varphi_t(x)$ is tempered nontangentially bounded everywhere in E, then it is tempered nontangentially convergent almost everywhere in E.

A function f is said to be *differentiable in the harmonic sense at* x_0 if $u(x,t) = f*\mathscr{P}_t(x)$ and all its first order derivatives are nontangentially convergent at x_0. The corresponding notion of differentiability for tempered nontangential boundedness is closely connected with our spaces $\Lambda_\gamma^2(E)$. For any nonnegative integer k, f is in $\Lambda_k^2(E)$ if and only if all its kth order derivatives are tempered nontangentially convergent at almost every point of E.

If $u_0, \ldots, u_n$ are conjugate harmonic functions and u_0 is nontangentially bounded almost everywhere in E then all of $u_0, \ldots, u_n$ are nontangentially bounded almost everywhere in E. If $f \in L^2(\boldsymbol{R}^n)$ then the problem of finding a harmonic conjugate of $f*\mathscr{P}_t(x)$ is solved by applying a Riesz transform to f. With this in mind we see that our generalization of this theorem will be a result about singular integral operators. By using the boundedness of singular integral operators on the spaces $\Lambda_\gamma^{p,\lambda}(x_0)$ we will be able to prove that if f has compact support and f is tempered nontangentially bounded at almost every point of E, then Tf is tempered nontangentially bounded almost everywhere in E, where T is a singular integral operator.

We have made the necessary extra assumption that our expressions are tempered in some sense. Once this has been done however we see that all the standard results about nontangential boundedness of harmonic functions are true also for any function u of the form $u(x,t) = f*\varphi_t(x)$.

I thank my teacher, Charles Fefferman, for the various comments and suggestions that he has made. This paper and its companion [3] were my doctoral dissertation, written under his direction.

I. TEMPERED NONTANGENTIAL BOUNDEDNESS

§ 1. Change of approximate identity. Let $\mathscr{S}$ denote the set of Schwartz functions on $\boldsymbol{R}^n$ and let $\mathscr{S}_* = \{\varphi \in \mathscr{S} \colon \int \varphi(x)\,dx \neq 0\}$, the set of Schwartz functions whose mean value is nonzero. $\mathscr{S}'$ will then be the space of all tempered distributions on $\boldsymbol{R}^n$.

$\varphi_t(x)$ will be used to denote dilation by t. Thus

$$\varphi_t(x) = \frac{1}{t^n}\varphi\left(\frac{x}{t}\right).$$

In fact we will adopt the convention that whenever t or s appears as a subscript then that subscript indicates dilation.

Our chief tool for studying distributions f will be convolution. If $f \in L^1$ then we form

$$u(x, t) = f * \varphi_t(x) = \int_{\mathbf{R}^n} f(y)\varphi_t(x-y)\,dy.$$

If $\int \varphi(x)\,dx = 1$ then φ_t is an approximate identity and φ will be referred to as a mollifier.

The cone of height h, aperture a, and vertex x_0 is the set

$$\Gamma_a^h(x_0) = \{(x, t) \in \Omega^*\colon\ |x-x_0| < at,\ 0 < t < h\}.$$

A function $u(x, t)$ on Ω^* is said to be *nontangentially bounded at* x_0 if there exist a and h such that the supremum of $|u(x, t)|$ over $\Gamma_a^h(x_0)$ is finite.

This condition involves using only one cone at each point x_0 but, except for a set of x_0 of measure zero, boundedness over one cone implies boundedness over all cones. More precisely,

LEMMA 1 (Stein [5], p. 201). *If u is continuous in Ω^* and u is nontangentially bounded at every point of E, $E \subset \mathbf{R}^n$, then for every $\varepsilon > 0$, $a > 0$, $h > 0$ there exists a compact subset $E_0 \subset E$ such that $|E - E_0| < \varepsilon$ and for all $x_0 \in E_0$ there exists $A = A(a, h, \varepsilon)$ so that*

$$\sup_{(x,t)\in\Gamma_a^h(x_0)} |u(x, t)| \leqslant A.$$

For f in $\mathscr{S}'$ and φ in $\mathscr{S}_*$, consider $u(x, t) = f*\varphi_t(x)$. It turns out that the set E of all x_0 in $\mathbf{R}^n$ where u is nontangentially bounded depends very strongly on φ. We will see an example to illustrate this in the next chapter. In order to eliminate the dependence of E on the mollifier φ what is needed therefore is a stronger type of nontangential boundedness.

DEFINITION. We will say that $u(x, t)$ is *tempered nontangentially bounded* (TNTB) *at* x_0 if there exist A, h, and λ such that

$$\sup_{(x,t)\in\Gamma_a^h(x_0)} |u(x, t)| \leqslant A(1+a)^\lambda \quad \text{for all } a > 0.$$

The extra condition then is that the nontangential bound grows slowly as a function of the aperture. This condition is not very restrictive. Consider any fixed φ in $\mathscr{S}_*$. If f is a tempered L^p function then $f*\varphi_t(x)$ is tempered nontangentially bounded almost everywhere. Similarly if f is an H^p distribution ($p > 0$) then $f*\varphi_t(x)$ is TNTB almost everywhere and λ can be any number $> n/p$ (Fefferman and Stein [2], p. 166). If f is a tempered distribution such that on an open set E it is equal to a locally integrable function then $f*\varphi_t(x)$ is TNTB at almost every point of E.

Also, using Harnack's theorem, it is possible to show that if $u(x, t)$ is a positive harmonic function in Ω^* that is nontangentially bounded at x_0, then $u(x, t)$ is TNTB at x_0. A similar proof shows that if $u(x, t)$ is TNTB below at x_0, then $u(x, t)$ is TNTB above at x_0, and *vice versa*.

The following theorem shows that tempered nontangential boundedness is independent of the mollifier used.

THEOREM 1. *Suppose that*

$$\text{(1)} \qquad \sup_{(x,t)\in\Gamma_a^h(x_0)} |(f*\varphi_t)(x)| \leqslant B_h(1+a)^\lambda \quad \textit{for all } a > 0$$

where φ is a C^∞ function such that $\hat{\varphi}$ is of class C^N in some neighborhood of the origin and $\hat{\varphi}(0) \neq 0$. Here N is some integer greater than $n+\lambda$ and B_h is a nondecreasing function of h.

Then

$$\sup_{(x,t)\in\Gamma_a^h(x_0)} |(f*\Phi_t)(x)| \leqslant AB_{Kh}\|\Phi\|(1+a)^\lambda \quad \textit{for all } a > 0$$

where $\|\Phi\| = \sum_{|\gamma|\leqslant N} \int (1+|y|)^\lambda |D^\gamma \hat{\Phi}(y)|\,dy$ and K is a constant depending only on φ. The following norm may also be used:

$$|||\Phi||| = \sum_{|\gamma|\leqslant N} \int (1+|y|)^N |D^\gamma \Phi(y)|\,dy.$$

Proof. If φ is dilated or translated the corresponding function in the upper half plane remains nontangentially bounded. In fact, if ψ is a smooth function with compact support and

$$\text{(2)} \qquad \Phi(x) = \int_{\mathbf{R}^n} \varphi_s(x-y)\psi(y)\,dy = (\varphi_s*\psi)(x),$$

then

$$\text{(3)} \qquad f*\Phi_t(x) = \int (f*\varphi_{st})(x-ty)\psi(y)\,dy.$$

If (x, t) is in a cone with vertex at x_0 and y is in the support of ψ, then $(x-ty, st)$ is in a slightly larger cone with vertex x_0. Hence $f*\Phi_t$ is nontangentially bounded at x_0 and an estimate like (1) holds. The strategy of the proof will be to express Φ as a sum of expressions like (2) and then to use estimate (1).

Let $N_0 = \{x \in \mathbf{R}^n\colon\ |x| < 1\}$, $N_j = \{x \in \mathbf{R}^n\colon\ \frac{1}{4}2^j < |x| < 2^j\}$ for $j = 1, 2, \ldots$ Let η_j be a C^∞ partition of unity subordinate to this open covering of $\mathbf{R}^n$ such that $\eta_j \geqslant 0$ for all j and

$$\text{(4)} \qquad |D^\gamma \eta_j(x)| \leqslant A_\gamma 2^{-j|\gamma|} \quad \text{for} \quad j = 1, 2, \ldots$$

Since $\hat{\varphi}(0) = \int \varphi(x)\,dx \neq 0$, there exists a positive constant K such that if $|z| \leqslant K$ then $|\hat{\varphi}(z)| \geqslant m > 0$ for some fixed m and $\hat{\varphi}$ is C^N in this small neighborhood of the origin.

Now we break Φ into pieces. Let $\hat{\Phi}_j = \eta_j \hat{\Phi}$ and let ψ_j be such that $\hat{\psi}_j(z) = \hat{\Phi}_j(z)/\hat{\varphi}(Kz2^{-j})$. Note that if z is in the support of $\hat{\Phi}_j$, then $|z| < 2^j$ and $|Kz2^{-j}| < K$. Hence $\hat{\varphi}(Kz2^{-j}) > m$ and so $\hat{\psi}_j$ is smooth. Finally define $\phi_{jk} = \eta_k \psi_j$. This gives the following decomposition of Φ:

$$\Phi(x) = \sum_j \Phi_j(x) = \sum_j \varphi_{K2^{-j}} * \psi_j(x) = \sum_j \sum_k (\varphi_{K2^{-j}} * \phi_{jk})(x)$$
$$= \sum_j \sum_k \int \varphi_{K2^{-j}}(x-y)\phi_{jk}(y)\,dy.$$

Comparing this with (2) and (3) we see that

$$(5)\qquad (f*\Phi_t)(x) = \sum_j \sum_k \int (f*\varphi_{Kt2^{-j}})(x-ty)\phi_{jk}(y)\,dy.$$

If (x, t) is in $\Gamma_a^h(x_0)$ and y is in the support of $\phi_{jk} \subset N_k$, then

$$|(x-ty)-x_0| \leqslant at + |y|t \leqslant (a+2^k)t = 2^j(a+2^k)K^{-1}\cdot Kt2^{-j}.$$

So $(x-ty, Kt2^{-j})$ is in the cone of aperture $2^j(a+2^k)K^{-1}$ and height $Kh2^{-j}$. Therefore, using (1),

$$(6)\qquad |(f*\Phi_t)(x)| \leqslant \sum_j \sum_k \int |(f*\varphi_{Kt2^{-j}})(x-ty)|\,|\phi_{jk}(y)|\,dy$$
$$\leqslant \sum_j \sum_k \int A\,(1+2^j(a+2^k)K^{-1})^\lambda |\phi_{jk}(y)|\,dy.$$

If y is in the support of ϕ_{jk}, then $|y| \geqslant \frac{1}{4}2^k$. Thus

$$(1+2^j(a+2^k)K^{-1})^\lambda \leqslant (1+2^jK^{-1})^\lambda(1+a+2^k)^\lambda \leqslant A\,(1+2^j)^\lambda(1+a)^\lambda(1+2^k)^\lambda$$
$$\leqslant A\,(1+2^j)^\lambda(1+a)^\lambda(1+|y|)^\lambda.$$

Applying this to (6) and using the fact that $\phi_{jk} = \eta_k\psi_j$ is supported in N_k yields

$$(7)\qquad |f*\Phi_t(x)| \leqslant A \sum_j \sum_k (1+2^j)^\lambda(1+a)^\lambda \int_{N_k} (1+|y|)^\lambda |\psi_j(y)|\,dy$$
$$\leqslant 2A(1+a)^\lambda \sum_j (1+2^j)^\lambda \int (1+|y|)^\lambda |\psi_j(y)|\,dy.$$

The last inequality used the fact that no point of $\boldsymbol{R}^n$ is in more than two of the N_k. Since

$$(1+|y|^2)^{\frac{1}{2}N}|\psi_j(y)| = |\widehat{(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j}(y)| \leqslant \|(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j\|_1,$$

then

$$\int (1+|y|)^\lambda |\psi_j(y)|\,dy \leqslant A \int (1+|y|^2)^{\frac{1}{2}(N-n-1)}|\psi_j(y)|\,dy$$
$$\leqslant A\,\|(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j\|_1 \int (1+|y|^2)^{-\frac{1}{2}n-\frac{1}{2}}\,dy = A\,\|(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j\|_1.$$

Now (7) can be used to give

$$(8)\qquad |f*\Phi_t(x)| \leqslant A\,(1+a)^\lambda \sum_j (1+2^j)^\lambda \|(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j\|_1.$$

To estimate $\|(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j\|_1$ recall that $\hat{\psi}_j(z) = \eta_j(z)\hat{\Phi}(z)/\hat{\varphi}(Kz2^{-j})$. Thus

$$(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j(z) = (I-\Delta)^{\frac{1}{2}N}\{\eta_j(z)\hat{\Phi}(z)/\hat{\varphi}(Kz2^{-j})\}$$
$$= \sum_{|\gamma_1+\gamma_2+\gamma_3|\leqslant N} c_{\gamma_1\gamma_2\gamma_3} D^{\gamma_1}\eta_j(z)\,D^{\gamma_2}\left[\frac{1}{\hat{\varphi}(Kz2^{-j})}\right] D^{\gamma_3}\hat{\Phi}(z).$$

Using (4) and the fact that

$$D^{\gamma_2}\left[\frac{1}{\hat{\varphi}(Kz2^{-j})}\right] \leqslant (K2^{-j})^{|\gamma_2|}\sup_{|w|\leqslant K}\left|D^{\gamma_2}\left(\frac{1}{\hat{\varphi}}\right)(w)\right| = A\,(K2^{-j})^{|\gamma_2|}$$

gives the following estimate

$$(9)\qquad |(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j(z)| \leqslant \sum_{|\gamma_1+\gamma_2+\gamma_3|\leqslant N} A\cdot A2^{-j|\gamma_1|}A\,(K2^{-j})^{|\gamma_2|}|D^{\gamma_3}\hat{\Phi}(z)|$$
$$\leqslant A \sum_{|\gamma_1+\gamma_2|\leqslant N} 2^{-j|\gamma_1|}|D^{\gamma_2}\hat{\Phi}(z)| \leqslant A \sum_{|\gamma|\leqslant N}|D^\gamma\hat{\Phi}(z)|.$$

Remember that the support of $D^{\gamma_1}\eta_j$ is contained in N_j. Therefore integrating both sides of (9) over that set gives

$$\|(I-\Delta)^{\frac{1}{2}N}\hat{\psi}_j\|_1 \leqslant A \sum_{|\gamma|\leqslant N}\int_{N_j}|D^\gamma\hat{\Phi}(z)|\,dz.$$

Putting this into (8) completes the basic assertion of the theorem:

$$|f*\Phi_t(x)| \leqslant A\,(1+a)^\lambda \sum_{|\gamma|\leqslant N}\sum_j (1+2^j)^\lambda \int_{N_j}|D^\gamma\hat{\Phi}(z)|\,dz$$
$$\leqslant A\,(1+a)^\lambda 4 \sum_{|\gamma|\leqslant N}\sum_j \int_{N_j}(1+|z|)^\lambda |D^\gamma\hat{\Phi}(z)|\,dz$$
$$\leqslant 2A\,(1+a)^\lambda \sum_{|\gamma|\leqslant N}\|(1+|z|)^\lambda D^\gamma\hat{\Phi}(z)\|_1 = A\|\Phi\|(1+a)^\lambda.$$

Note that in the above computation we again used the fact that the sets N_j do not overlap very much. The other norm mentioned in the theorem can be obtained by using the uniform boundedness of the Fourier transform of an L^1 function:

$$|(1+|z|^2)^{\frac{1}{2}N}D^\gamma\hat{\Phi}(z)| = |\widehat{(I-\Delta)^{\frac{1}{2}N}x^\gamma\Phi}(z)| \leqslant \|(I-\Delta)^{\frac{1}{2}N}x^\gamma\Phi(z)\|_1$$
$$\leqslant A\sum_{\substack{|\gamma|\leqslant N\\|\beta|\leqslant N}}\|x^\gamma D^\gamma\Phi\|_1 \leqslant A\sum_{|\beta|\leqslant N}\|(1+|x|)^N D^\gamma\Phi\|_1.$$

Therefore

$$\|\Phi\| = \sum_{|\gamma|\leqslant N} \|(1+|z|)^{\lambda} D^{\gamma}\hat{\Phi}(z)\|_1 \leqslant \sum_{|\gamma|\leqslant N} \|(1+|z|)^{N} D^{\gamma}\hat{\Phi}(z)\|_{\infty} \|(1+|z|)^{-n-1}\|_1$$

$$\leqslant \sum_{|\gamma|\leqslant N} A \sum_{|\gamma|\leqslant N} \|(1+|z|)^{N} D^{\gamma}\Phi\|_1 A = A|||\Phi|||.$$

This completes the proof of the theorem.

It is clear that if (1) is true for some truncation height h then it must be true for any other height k. It will, however, be necessary for us to examine the effect of this change on equation (1). The following lemma is a simple consequence of Theorem 1:

LEMMA 2. *Let φ and N be as in Theorem 1. If*

$$\sup_{(x,t)\in\Gamma_{\alpha}^{h}(x_0)} |(f*\varphi_t)(x)| \geqslant B(1+\alpha)^{\lambda} \quad \textit{for all } \alpha > 0,$$

hen for every C^{∞} function Φ such that $\|\Phi\|$ is finite,

$$\text{(10)} \qquad \sup_{(x,t)\in\Gamma_{\alpha}^{k}(x_0)} |(f*\Phi_t)(x)| \leqslant AB(1+k)^{N}\|\Phi\|(1+\alpha)^{\lambda} \quad \textit{for all } \alpha > 0,$$

where A is a constant depending only on h, φ, n, λ, and $\|\Phi\|$ is the norm in Theorem 1.

Proof. Apply Theorem 1 to $\Psi = \Phi_{k(Kh)^{-1}}$. Then

$$\text{(11)} \qquad \sup_{(x,t)\in\Gamma_{\alpha}^{k}(x_0)} |f*\Phi_t(x)| \leqslant A\|\Psi\|(1+\alpha)^{\lambda}.$$

But

$$\|\Psi\| = \sum_{|t|\leqslant N} \int (1+|y|)^{\lambda} \left| D^{\gamma}\left\{\hat{\Phi}\left(k(Kh)^{-1}y\right)\right\}\right| dy$$

$$\leqslant \sum_{|\gamma|\leqslant N} \left(k(Kh)^{-1}\right)^{-n+|\gamma|} \int (1+|Khk^{-1}z|)^{\lambda} |D^{\gamma}\hat{\Phi}(z)|\,dz.$$

If $k \leqslant Kh$, then $\Gamma_{\alpha}^{k}(x_0) \subset \Gamma_{\alpha}^{Kh}(x_0)$ so we can use Theorem 1. This gives

$$\text{(12)} \qquad \sup_{(x,t)\in\Gamma_{\alpha}^{k}(x_0)} |(f*\Phi_t)(x)| \leqslant AB\|\Phi\|(1+\alpha)^{\lambda}.$$

On the other hand if $k \geqslant Kh$, then

$$\|\Psi\| \leqslant \sum_{|\gamma|\leqslant N} \left(k(Kh)^{-1}\right)^{|\gamma|} \int (1+|z|)^{\lambda} |D^{\gamma}\hat{\Phi}(z)|\,dz \leqslant ABk^{N}\|\Phi\|.$$

Therefore putting this into (11) and using (12) completes the proof of the lemma.

The notion of tempered nontangential boundedness is equivalent to the finiteness of a certain maximal function; namely,

$$\mathscr{M}_{\lambda}(f)(x_0) \equiv \sup_{(x,t)\in\Omega_h} \left(\frac{t}{t+|x-x_0|}\right)^{\lambda} |f*\varphi_t(x)|$$

where $\Omega_h = \{(x,t)\colon x\in\boldsymbol{R}^n,\ 0<t<h\}$. $\mathscr{M}_{\lambda}(f)(x_0)$ is equivalent to the smallest B such that

$$\sup_{\Gamma_{\alpha}^{h}(x_0)} |f*\varphi_t(x)| \leqslant B(1+\alpha)^{\lambda}.$$

Theorem 1 and Lemma 2 show that using a different φ and h produces little change in $\mathscr{M}_{\lambda}(f)(x_0)$. In fact,

$$\mathscr{M}_{\lambda,\Phi,k}(f)(x_0) \leqslant A(1+k)^{N}\|\Phi\|\mathscr{M}_{\lambda}(f)(x_0)$$

where $\mathscr{M}_{\lambda,\Phi,k}$ denotes the maximal function obtained by using Φ and k instead of φ and h.

If we assume a certain type of nontangential convergence, then we can change the mollifier φ and still preserve the convergence.

DEFINITION. We will say that f is *tempered nontangentially convergent at* x_0 if there exists an increasing function B_h such that B_h approaches zero as $h\to 0$ and

$$\text{(13)} \qquad \sup_{\Gamma_{\alpha}^{h}(x_0)} |(f*\varphi_{t_1})(x_1) - (f*\varphi_{t_2})(x_2)| \leqslant B_h(1+\alpha)^{\lambda} \quad \text{for all } \alpha > 0.$$

THEOREM 2. *Let φ and N be as in Theorem 1. If f is tempered nontangentially convergent at x_0, then for any smooth Φ,*

$$\sup_{\Gamma_{\alpha}^{h}(x_0)} |(f*\Phi_{t_1})(x_1) - (f*\Phi_{t_2})(x_2)| \leqslant AB_{Kh}\|\Phi\|(1+\alpha)^{\lambda};$$

A is a constant independent of f, x_0, B_h, Φ, α.

If $f_{\Phi}(x_0)$ denotes the nontangential limit of $f*\Phi_t(x)$ and if φ and Φ both have mean value one, then

$$\text{(14)} \qquad f_{\varphi}(x_0) = f_{\Phi}(x_0).$$

Proof. The verification is essentially a repetition of the proof of Theorem 1, and we adopt the notation of that proof. Using (5) and the inequality (13) we get

$$|(f*\Phi_{t_1})(x_1) - (f*\Phi_{t_2})(x_2)|$$

$$= \left|\sum_{j}\sum_{k} \int \{f*\Phi_{Kt_1 2^{-j}}(x_1-yt_1) - f*\Phi_{Kt_2 2^{-j}}(x_2-yt_2)\}\phi_{jk}(y)\,dy\right|$$

$$\leqslant \sum_{j}\sum_{k} AB_{Kh}\left(1+2^{j}K^{-1}(\alpha+2^{k})\right)^{\lambda} \int |\phi_{jk}(y)|\,dy.$$

The expression on the right-hand side is the same as (6). Thus the proof of Theorem 1 says that it is dominated by $AB_{Kh}\|\Phi\|(1+a)^\lambda$.

Now we look at the nontangential limit. As before

(15) $$f*\Phi_t(x) = \sum_j \sum_k \int (f*\varphi_{Kt2^{-j}})(x-yt)\phi_{jk}(y)\,dy.$$

We have already seen that this series converges absolutely. Therefore taking the limit as $(x,t)\to(x_0,0)$ in $\Gamma_a^h(x_0)$, the right-hand side of (15) becomes

$$f_\varphi(x_0)\sum_j\sum_k\int\phi_{jk}(y)\,dy = f_\varphi(x_0)\sum_j\sum_k\int\eta_k(y)\psi_j(y)\,dy = f_\varphi(x_0)\sum_j\int\psi_j(y)\,dy.$$

But $\int\psi_j(y)\,dy = \hat\psi_j(0) = \eta_j(0)\hat\Phi(0)/\hat\varphi(0) = \eta_j(0)$. Since $\{\eta_j\}$ is a partition of unity the limit is in fact $f_\varphi(x_0)$.

§ 2. The Poisson kernel. Since much of this paper is devoted to generalizing theorems about harmonic functions it is natural to examine the problem of changing the mollifier to and from the Poisson kernel $\mathscr{P}$. Since the Poisson kernel and its derivatives might not decrease rapidly enough Theorem 1 cannot be applied directly. To change to the Poisson kernel we will have to assume boundedness over untruncated cones ($h=\infty$). This is not however a serious restriction because we will see in the next section that if f is bounded over truncated cones, then by subtracting a very good distribution g, $f-g$ will be bounded over untruncated cones (Theorem 9).

THEOREM 3. *Let φ and N be as in Theorem 1 of the first section. If*

$$\sup_{\Gamma_a(x_0)}|f*\varphi_t(x)|\le B(1+a)^\lambda \quad \textit{for all } a>0,$$

then

$$\sup_{\Gamma_a(x_0)}|f*\mathscr{P}_t(x)|\le AB(1+a)^\lambda \quad \textit{for all } a>0.$$

Proof. Let $\{\eta_j\}_{j=0}^\infty$ be a C^∞ partition of unity such that $\eta_j(x) = \eta(x2^{-j})$ for $j\geqslant 1$ where η is supported in $\{\frac12\leqslant|x|\leqslant 2\}$. Then

$$\mathscr{P}(x) = \sum_{j=0}^\infty \eta_j(x)\mathscr{P}(x) = \sum_{j=1}^\infty \Phi_j(x2^{-j})$$

where $\Phi_j(x) = \eta(x)\mathscr{P}(2^jx)$ is supported in $\{\frac12\leqslant|x|\leqslant 2\}$ for $j\geqslant 1$ and $\Phi_0(x) = \eta_0(x)\mathscr{P}(x)$. Therefore

$$\mathscr{P}_t(x) = \sum_{j=0}^\infty 2^{jn}(\Phi_j)_{2^jt}(x)$$

and

$$|f*\mathscr{P}_t(x)|\le\sum_{j=0}^\infty 2^{jn}|f*(\Phi_j)_{2^jt}(x)|.$$

If (x,t) is in $\Gamma_a(x_0)$ then so is $(x,2^jt)$. Hence applying Theorem 1 gives

(16) $$|f*\mathscr{P}_t(x)|\le\sum_{j=0}^\infty 2^{jn}AB|||\Phi_j|||(1+a)^\lambda.$$

Therefore if we can show that $|||\Phi_j|||\le A2^{-j(n+1)}$ we will be done. But

$$|||\Phi_j||| = \sum_{|\gamma|\le N}\int(1+|x|)^N|D^\gamma\Phi_j(x)|\,dx$$
$$\le A\sum_{|\gamma_1+\gamma_2|\le N}\int(1+|x|)^N|D^{\gamma_1}\eta(x)|2^{j|\gamma_2|}|D^{\gamma_2}\mathscr{P}(2^jx)|\,dx.$$

Since $|D^{\gamma_2}\mathscr{P}(2^jx)|\le A2^{-j(n+1+|\gamma_2|)}$, the right-hand side is dominated by

$$A\sum_{|\gamma_1+\gamma_2|\le N}2^{-j(n+1)}\int(1+|x|)^N|D^{\gamma_1}\eta(x)|\,dx = A2^{-j(n+1)}|||\eta||| = A2^{-j(n+1)}.$$

Using this in (1) completes the proof of the theorem.

If in the last theorem we were willing to restrict ourselves to the case $\lambda<1$, then it is possible to modify the proof of Theorem 1 so that we can change from φ to $\mathscr{P}$ without assuming boundedness over untruncated cones.

In order to change the mollifier from the Poisson kernel to a Schwartz function it is not necessary to restrict ourselves to boundedness over untruncated cones:

THEOREM 4. *If*

$$\sup_{\Gamma_a^h(x_0)}|f*\mathscr{P}_t(x)|\le B(1+a)^\lambda \quad \textit{for all } a>0,$$

then for any $\varphi\in\mathscr{S}$,

$$\sup_{\Gamma_a^h(x_0)}|f*\varphi_t(x)|\le AB(1+a)^\lambda \quad \textit{for all } a>0.$$

Proof. It will be sufficient to change from $\mathscr{P}$ to a function φ such that we can apply Theorem 1. Fix $N>n+\lambda$. Let $g(t)$ be a C^∞ function supported in $[1,2]$ such that

$$\int g(t)\,dt = 1,$$

(17) $$\int t^kg(t)\,dt = 0 \quad \text{for} \quad k=1,\ldots,M+N-1.$$

Consider $\varphi(x) = \int \mathscr{P}_t(x) g(t)\,dt$. Then $f*\varphi_t(x) = \int \big(f*\mathscr{P}_{ts}(x)\big) g(s)\,ds$. Thus

$$|f*\varphi_t(x)| \leqslant \int A(1+a)^\lambda |g(s)|\,ds = A(1+a)^\lambda \quad \text{if} \quad (x,t)\in\Gamma_a^h(x_0).$$

Clearly φ is a C^∞ function. Therefore in order to be able to apply Theorem 1 we need only show that $\|\varphi\|$ is finite. Since

$$\hat{\varphi}(z) = \int e^{-t|z|} g(t)\,dt,$$

it can be shown by induction that

$$D^\gamma \hat{\varphi}(z) = \sum_{j=0}^{|\gamma|-1} \frac{1}{|z|^j} p_j\left(\frac{z_1}{|z|},\ldots,\frac{z_n}{|z|}\right) \int e^{-t|z|} t^{|\gamma|-j} g(t)\,dt \tag{18}$$

where p_j is a polynomial.

Consider

$$F(z) = \int_0^\infty e^{-t|z|} t^k g(t)\,dt \quad \text{for} \quad k>0.$$

Integrating by parts M times gives

$$F(z) = |z|^M \int_0^\infty e^{-t|z|} G(t)\,dt$$

where $G(t)$ is obtained by integrating $t^k g(t)$ M times. It can be shown that

$$G(t) = \int_0^t s^k g(s) p(s)\,ds$$

where p is a polynomial of degree $M-1$. Therefore because of (17) $G(t)$ has compact support if $k+(M-1)\leqslant M+N-1$ (that is, if $k\leqslant N$). In particular $G\in L^1(0,\infty)$, and so

$$|F(z)| \leqslant |z|^M \int_0^\infty |G(t)|\,dt = A|z|^M.$$

Using this estimate in (18) gives

$$|D^\gamma\hat{\varphi}(z)| \leqslant \sum_{j=0}^{|\gamma|-1} \frac{1}{|z|^j} A\cdot A|z|^M \leqslant A|z|^{M-|\gamma|+1} \quad \text{for} \quad |z|\leqslant 1.$$

Therefore if M is large enough $D^\gamma\hat{\varphi}$ will be continuous for all $|\gamma|\leqslant N$.

Also $D^\gamma\hat{\varphi}$ is rapidly decreasing because by (18)

$$|D^\gamma\hat{\varphi}(z)| \leqslant \sum_{j=0}^{|\gamma|-1} 1\cdot A\cdot e^{-|z|} \int t^{|\gamma|-j}|g(t)|\,dt = Ae^{-|z|} \quad \text{for} \quad |z|\geqslant 1.$$

Thus $\|\varphi\|$ is finite and we can apply Theorem 1.

The proof of the last result can also be used to show that we can preserve tempered nontangential convergence when changing from $\mathscr{P}$ to φ.

THEOREM 5. *If f is tempered nontangentially convergent at x_0 with mollifier $\mathscr{P}$, then f is tempered nontangentially convergent at x_0 with any mollifier $\varphi\in\mathscr{S}$. The new bound is the same as in Theorem 2.*

§ 3. Multiplication and restriction. In this section we examine the operation of multiplying f by a smooth ψ. The results of this section are all consequences of Theorems 1 and 2.

THEOREM 6. *Let N be an integer $> n+\lambda$. Suppose that ψ is a C^N function such that*

$$|D^\gamma\psi(x)| \leqslant B(1+|x|)^M \quad \textit{for all} \quad |\gamma|\leqslant N.$$

If

$$\sup_{\Gamma_a^h(x_0)} |f*\varphi_t(x)| \leqslant A(1+a)^\lambda,$$

then

$$\sup_{\Gamma_a^h(x_0)} |(\psi f)*\Phi_t(x)| \leqslant AB\|\Phi\|(1+a)^{\lambda+M},$$

where $\|\Phi\| = \sum_{|\gamma|\leqslant N}\int (1+|y|)^{N+M}|D^\gamma\Phi(y)|\,dy$ *and A is a constant independent of f, x_0, a, ψ.*

Proof. Consider a fixed $(x,t)\in\Gamma_a^h(x_0)$. Notice that

$$(\psi f)*\Phi_t(x) = \int_{R^n} \psi(x-y) f(x-y)\Phi_t(y)\,dy = \int f(x-y)\phi_t(y)\,dy$$

where $\phi(y) = \Phi(y)\psi(x-ty)$. By Theorem 1

$$|(\psi f)*\Phi_t(x)| \leqslant A\|\phi\|_*(1+a)^\lambda \tag{19}$$

where $\|\phi\|_*$ is the second norm in Theorem 1. Although ϕ depends on x and t we will be able to get a bound for $\|\phi\|_*$ that is independent of these variables. Note that

$$\|\phi\|_* \leqslant A \sum_{|\gamma_1+\gamma_2|\leqslant N} \int (1+|y|)^N |D^{\gamma_1}\Phi(y)|\,t^{|\gamma_2|}\,|D^{\gamma_2}\psi(x-ty)|\,dy. \tag{20}$$

But

$$D^{\gamma_2}\psi(x-ty) \leqslant B(1+|x-ty|)^M \leqslant B(1+|x-x_0|)^M(1+t|y|)^M \leqslant AB(1+a)^M(1+|y|)^M.$$

Therefore (20) simplifies to $\|\phi\|_* \leqslant AB(1+a)^M\|\Phi\|$. Putting this into (19) completes the proof of the theorem.

THEOREM 7. *Let N be an integer $> n+\lambda$. Suppose that ψ is a C^{N+1} function with*

$$|D^\gamma \psi(x)| \leqslant B(1+|x|)^M \qquad \text{for all } |\gamma| \leqslant N+1.$$

If f is tempered nontangentially convergent (TNTC) *at x_0 (with bound $(1+\alpha)^\lambda$), then f is also* TNTC *at x_0 (with bound $(1+\alpha)^{\lambda+M+1}$). If the first limit is denoted by "$f(x_0)$", then the second limit is $\psi(x_0)$"$f(x_0)$".*

Proof. As in the previous proof we write

$$(\psi f)*\Phi_{t_1}(x_1)-(\psi f)*\Phi_{t_2}(x_2) = \int \{f(x_1-y)-f(x_2-y)\}\psi(x_1-y)\Phi_t(y)\,dy +$$
$$+\int f(x_2-y)\{\psi(x_1-y)-\psi(x_2-y)\}\Phi_t(y)\,dy$$
$$= I_1+I_2.$$

Let $(x_1, t_1), (x_2, t_2) \in \Gamma^h_\alpha(x_0)$. We handle the first integral by writing $\phi(y) = \Phi(y)\psi(x_1-ty)$ and using Theorem 2. To take care of the second integral we let $\phi(y) = \Phi(y)\{\psi(x_1-ty)-\psi(x_2-ty)\}$. Now since f must be TNTB at x_0 we can use Theorem 1.

$$\|\phi\| = \sum_{|\gamma_1+\gamma_2|\leqslant N} \int (1+|y|)^N |D^{\gamma_1}\Phi(y)|\, t^{|\gamma_2|}\, |D^{\gamma_2}\psi(x_1-ty)-D^{\gamma_2}\psi(x_2-ty)|\,dy$$
$$\leqslant |x_1-x_2|AB \sum_{|\gamma|\leqslant N}\int (1+|y|)^{N+M}|D^\gamma\Phi(y)|\,dy\,(1+\alpha)^M$$
$$= AB|x_1-x_2|\,\|\Phi\|(1+\alpha)^M.$$

Since $|x_1-x_2| \leqslant at \leqslant ah$, this shows that ψf is TNTC at x_0.

Suppose that $\int \Phi(x)\,dx = 1$. We write

$$\psi f*\Phi_t(x) = \psi(x_0)\int f(y)\Phi_t(x-y)\,dy + \int f(y)\{\psi(x_0)-\psi(x-y)\}\Phi_t(y)\,dy.$$

The second integral is handled like I_2 above. So it approaches zero. By assumption the first integral approaches $\psi(x_0)$ "$f(x_0)$".

LEMMA 3. *If $f \equiv 0$ in a neighborhood of x_0, then f is* TNTC *to zero at x_0.*

Proof. Take a mollifier φ with compact support. Outside a neighborhood of x_0 in Ω^* we have $f*\varphi_t(x) = 0$. This and the fact that $f \leqslant At^{-M}$ for some M can be combined to prove the assertion of the theorem.

THEOREM 8. *Let ψ be C^∞ and $\psi(x_0) \neq 0$. If ψf is* TNTB *(or* TNTC*) at x_0, then so is f.*

Proof. Multiply ψf by $1/\psi(x)$ in a small neighborhood of x_0. Then use Lemma 3.

THEOREM 9. *If f is* TNTB *at every point of some bounded set E, then there exist f_1 and g such that $f = f_1+g$ and $g \equiv 0$ in a neighborhood of E and f_1 is* TNTB *over untruncated cones at every point of E.*

Remark. This simple decomposition will be important when we want to use harmonic functions in Chapter IV. g is very good and causes no problems. f_1 is bounded over untruncated cones so we can use Theorem 3 to change to the Poisson kernel.

Proof. Since $f \in \mathscr{S}'$, it can be written as the derivative of a slowly increasing continuous function. Let $f = D^\gamma h$, where h is slowly increasing. Let χ be a C^∞ function with compact support that is identically one in a neighborhood of E. Let $f_1 = D^\gamma(\chi h)$; then clearly $g = f - f_1 = 0$ in a neighborhood of E. Also f_1 is TNTB over untruncated cones because

$$|f*\varphi_t(x)| = \frac{1}{t^{|\gamma|}}\,|(\chi h)*(D^\gamma\varphi)_t(x)| \leqslant \frac{\|\chi h\|\,\|D^\gamma\varphi\|_1}{t^{|\gamma|}} \leqslant \frac{A}{t^{|\gamma|}}.$$

II. A COUNTEREXAMPLE

It was shown in the last section that the notion of tempered nontangential boundedness does not depend on the mollifier φ. Such is not the case for nontangential boundedness. In fact even if we require nontangential convergence it is not always possible to change the mollifier and retain nontangential boundedness. This section will be devoted to constructing an example to illustrate that the set of points of nontangential boundedness depends on the mollifier that is used.

THEOREM 1. *There exists a tempered distribution f and mollifiers P and Q in $\mathscr{S}_*$ such that $f*P_t(x)$ is nontangentially convergent almost everywhere but $f*Q_t(x)$ is nontangentially bounded almost nowhere.*

It should be noted that f can be made to have compact support. In this case the conclusion will be that $f*Q_t(x)$ is nontangentially bounded almost nowhere in the interior of the support of f.

This construction works for any dimension n, but for simplicity we will assume that $n = 1$. No new ideas are needed for the general case.

Consider $\phi \in C^\infty(\boldsymbol{R})$. We require that $\hat{\phi}$ is supported in $[2, 3]$ and $\int\hat{\phi}(z)\,dz = 1$. Let $P \in \mathscr{S}$ be such that $\hat{P}(z) = 1$ for all $|z| \leqslant 4$. We will let χ denote the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$. Now define $\hat{\varphi} = \chi*\hat{\phi}$. Note that $\hat{\varphi}$ is supported in $[1, 4]$. Therefore $\hat{P}\hat{\varphi} = \hat{\varphi}$ and so

$$P*\varphi(x) = \varphi(x) = \frac{\sin(\pi x)}{\pi x}\,\varphi(x).$$

We now list the properties of P and φ that will be of interest to us.

(i) $\int P(x)\,dx = \hat{P}(0) = 1$,

(ii) $P \in \mathscr{S}$, $\varphi \in \mathscr{S}$,

(iii) $u(0, 1) = P*\varphi(0) = \phi(0) = \int\hat{\phi}(z)\,dz = 1$,

(iv) $u(j,1) = P*\varphi(j) = \dfrac{\sin(\pi j)}{\pi j}\hat{\varphi}(j) = 0$ for $j \in Z$, $j \neq 0$,

(v) $\int x^k \varphi(x)\,dx = D^k\hat{\varphi}(0) = 0$ for all $k \in Z$, $k \geqslant 0$.

For any Q in $\mathscr{S}_*$ define

$$u^Q(x,t) = \varphi * Q_t(x) = \int \varphi(x-y)Q_t(y)\,dy$$

and let $u(x,t) = u^P(x,t)$. This function will be the basic building block in the construction of f. Note that $u(0,1) = \varphi * P(0) = 1$. We will now show that u decreases rapidly at infinity.

LEMMA 1. *For all N,*

$$|u^Q(x,t)| \leqslant A\,\|Q\|(1+|x|+t)^{-N}$$

where $\|Q\| = \|Q\|_1 + \|z^N Q(z)\|_\infty + \|D^N Q\|_\infty + \|z^{N+1}D^N Q(z)\|_\infty$ *and A is a constant depending on N and φ but independent of Q, x, and t.*

Proof.

$$u^Q(x,t) = \varphi(x)\int_{|y|<\frac{1}{2}|x|} Q_t(y)\,dy + \int_{|y|<\frac{1}{2}|x|} Q_t(y)\{\varphi(x-y)-\varphi(x)\}\,dx + \int_{|y|>\frac{1}{2}|x|} Q_t(y)\varphi(x-y)\,dy.$$

The first integral is dominated by $|\varphi(x)|\,\|Q\|_1 \leqslant A\,\|Q\|_1|x|^{-N}$. The second term is majorized by

$$\{\sup_{|z-x|\leqslant\frac{1}{2}|x|} |D\varphi(z)|\}\int_{|y|<\frac{1}{2}|x|} |Q_t(y)|\,|y|\,dy \leqslant A\,\|z^{N+1}D\varphi(z)\|\,|x|^{-N}\|Q\|_1 = A\,\|Q\|_1|x|^{-N}.$$

The last integral is taken care of by using

$$|Q_t(y)| \leqslant t^{N-1}|y|^{-N}\|z^N Q(z)\|_\infty \leqslant A\,\|z^N Q(z)\|_\infty |x|^{-N} \quad \text{for} \quad t \leqslant 1.$$

Therefore putting these estimates together we get

(1) $$|u^Q(x,t)| \leqslant A\,\{\|Q\|_1 + \|z^N Q(z)\|_\infty\}|x|^{-N} \quad \text{for} \quad t \leqslant 1.$$

We will now obtain a similar estimate for large t.

(2) $$u^Q(x,t) = \int \varphi(y)\frac{1}{t}Q\left(\frac{x-y}{t}\right)dy = \int \varphi(y)\frac{1}{t}(D^N Q)\left(\frac{c}{t}\right)\frac{1}{N!}\left(\frac{y}{t}\right)^N dy$$

for some $c = c_{x,y}$ between x and $x-y$. Note that the first N terms of the Taylor series of Q about x/t make no contribution because all the moments of φ are zero.

For $|y| \geqslant \frac{1}{2}|x|$ we can use the estimate

$$\left|(D^N Q)\left(\frac{c}{t}\right)\right| \leqslant \|D^N Q\|_\infty,$$

but if $|y| \leqslant \frac{1}{2}|x|$, then

$$\left|(D^N Q)\left(\frac{c}{t}\right)\right| \leqslant \sup_{|z-x|\leqslant|y|}\left|(D^N Q)\left(\frac{z}{t}\right)\right| \leqslant A\left(\frac{t}{|x|}\right)^{N+1}\|z^{N+1}D^N Q(z)\|_\infty.$$

Therefore dividing the integral in (2) into two parts and using these estimates

(3) $$|u^Q(x,t)| \leqslant A\,\|D^N Q\|_\infty \frac{1}{t^{N+1}}\int_{|y|\geqslant\frac{1}{2}|x|} |y^N\varphi(y)|\,dy + A\,\|z^{N+1}D^N Q\|_\infty |x|^{-(N+1)}\int_{|y|<\frac{1}{2}|x|} |y^N\varphi(y)|\,dy$$
$$\leqslant A\,\{\|D^N Q\| + \|z^{N+1}D^N Q(z)\|_\infty\}|x|^{-N-1} \quad \text{for} \quad \text{all } t \geqslant 1.$$

Observe that it also follows from (2) that

(4) $$|u^Q(x,t)| \leqslant At^{-N-1}\|D^N Q\|_\infty.$$

Estimates (1), (3), and (4), together with

$$|u^Q(x,t)| \leqslant \|\varphi\|\,\|Q_t\|_1 = A\,\|Q\|_1,$$

can be combined to complete the proof of the lemma.

Now we know that $u(x,t)$ decreases very rapidly at infinity. We will, however, need a better majorant. Define

$$p(r) = \frac{1}{\sup_{|x|+t\geqslant r} |u(x,t)|}.$$

This means that

(5) $$|u(x,t)| \leqslant \frac{1}{p(|x|+t)}.$$

Lemma 1 asserts that p is a rapidly increasing function.

We are now in a position to construct a function $U(x,t)$, which will turn out to be $f*P_t(x)$. In view of Theorem 1 of Section I $U(x,t)$ will be nontangentially bounded but the nontangential bound will increase very rapidly as a function of the aperture. The strategy therefore will be to make $U(x,t)$ big at certain carefully located points of the upper half plane.

$\{d_k\}$ and $\{M_k\}$ are positive sequences that will be defined shortly. $\{(jd_k, 1/M_k)\colon j \in Z,\ k = 1, 2, \ldots\}$ are the points where $U(x,t)$ will be big. They will be referred to as the lattice points. The set $\{(jd_k, 1/M_k)\colon j \in Z\}$ will be called the *k-th level* and will be denoted by L_k. Thus the distance between consecutive lattice points on the kth level will be d_k. The kth level is at a distance $1/M_k$ from the real line.

To each lattice point $(jd_k, 1/M_k)$ there corresponds a dilate of u:

$$M_k u\big((x-jd_k)M_k,\ tM_k\big).$$

The value of this function at the lattice point is M_k, the reciprocal of its distance to the real line. Define

$$U^Q(x, t) = \sum_{k=1}^{\infty}\sum_{j=-\infty}^{\infty} M_k u^Q\big((x-jd_k)M_k,\ tM_k\big)$$

and set $U(x, t) = U^P(x, t)$.

We will now find appropriate sequences M_k and d_k and then verify the relevant properties of U.

Let m_k be a sequence of positive integers increasing to infinity. Define $M_k = (p(m_k))^{\frac{1}{4}}$. By taking subsequences we can ensure that m_k satisfies the following conditions

(i) $M_k^{\frac{1}{4}} \geqslant \sum_{i=1}^{k-1} M_i$,

(ii) $\displaystyle\sum_{i=k+1}^{\infty} \frac{1}{M_i} \leqslant M_k^{-2}$,

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{m_k} < \tfrac{1}{8}$,

(iv) $M_k \geqslant 1$, $m_k^4 \leqslant M_k$.

Condition (i) guarantees that the first $k-1$ levels do not interfere with the kth, and condition (ii) prevents interference between later levels and the kth. (iii) is needed to show that the set of points of nontangential boundednees has positive measure and the inequalities in (iv) are purely for convenience.

Define $d_k = m_k^4/M_k$, and note that the second inequality in (iv) says that $d_k \leqslant 1$.

In order to show that the series defining $U^Q(x, t)$ is convergent everywhere we use the estimate

$$|u^Q((x-jd_k)M_k, tM_k)| \leqslant A(1+|x-jd_k|M_k+tM_k)^{-3}. \tag{6}$$

The sum of the right-hand side over j can be dominated by an integral, as in the integral test:

$$\begin{aligned}(1+|x-jd_k|M_k)^{-3} &\leqslant 2\sum_{=0}^{\infty}(1+jd_kM_k+tM_k)^{-3} \\ &\leqslant 2(1+tM_k)^{-3}+2\int_0^{\infty}(1+rd_kM_k+tM_k)^{-3}\,dr \\ &= 2(1+tM_k)^{-3}+\frac{1}{d_kM_k}(1+tM_k)^{-2} \leqslant A(tM_k)^{-2}.\end{aligned} \tag{7}$$

Putting this into the series for $U^Q(x, t)$ we get

$$|U^Q(x, t)| \leqslant A\sum_{k=1}^{\infty} M_k A(tM_k)^{-2} = At^{-2} < \infty. \tag{8}$$

Therefore the series for $U^Q(x, t)$ converges absolutely in the upper half plane.

In order to estimate $U^Q(x, t)$ at the lattice points of the kth level we must show that the contribution of the other levels is small. More precisely,

$$\Big|\sum_{i\neq k} M_i \sum_{j=-\infty}^{\infty} u^Q\Big((x-jd_i)M_i, \frac{M_i}{M_k}\Big)\Big| \leqslant AM_k^{\frac{1}{4}}. \tag{9}$$

Denote the left-hand side of (9) by S. Estimates (6) and (7) yield

$$\begin{aligned}|S| &\leqslant \sum_{i\neq k} M_i\Big\{2\Big(1+\frac{M_i}{M_k}\Big)^{-3}+\frac{1}{d_iM_i}\Big(1+\frac{M_i}{M_k}\Big)^{-2}\Big\} \\ &\leqslant A\sum_{i\neq k} M_i\Big(1+\frac{M_i}{M_k}\Big)^{-2}.\end{aligned}$$

If $i < k$, then we may ignore the denominator and use inequality (i). For $i > k$, eliminate the one from the denominator. Now condition (ii) handles the sum over these i. Thus

$$|S| \leqslant A\{M_k^{\frac{1}{4}}+M_k^2AM_k^{-2}\} = AM_k^{\frac{1}{4}}.$$

This proves (9).

Now we would like to show that $U^Q(jd_k, 1/M_k)$ is approximately equal to M_k. First we prove that

$$\Big|M_k\sum_{i\neq j} u\big((jd_k-id_k)M_k, M_k/M_k\big)\Big| \leqslant \tfrac{1}{4}M_k. \tag{10}$$

Note that the left-hand side of (10) equals $M_k\big|\sum_{i\neq 0} u(id_kM_k, 1)\big|$. Again use Lemma 1 and dominate the resulting series by an integral. Let S denote the left-hand side of (10). Then

$$\begin{aligned}|S| &\leqslant M_kA\sum_{i\neq 0}(2+|i|d_kM_k)^{-2} \leqslant 2AM_k\int_0^{\infty}(2+d_kM_kr)^{-2}\,dr \\ &= AM_k\frac{1}{d_kM_k} = \frac{A}{d_k} = \frac{AM_k}{m_k^4} \leqslant \frac{1}{4}M_k \quad \text{for all large } k.\end{aligned}$$

Since $u(0, 1) = 1$, the estimates (9) and (10) mean that $U(jd_k, 1/M_k)$ is roughly equal to M_k. In fact

$$|U(jd_k, 1/M_k)-M_k| \leqslant AM_k^{\frac{1}{4}}+\tfrac{1}{4}M_k \leqslant \tfrac{1}{2}M_k \quad \text{for large } k. \tag{11}$$

Next we define a set on which we hope U is nontangentially bounded. Suppose $e_k = m_k^2$ and define

$$E = \{x\colon \operatorname{dist}(x, L_k) > e_k/M_k \text{ for all } k\},$$

where $\operatorname{dist}(x, L_k)$ is the distance between x and the kth level, L_k. Let M be a big integer. Consider $I = [-M, M]$. We will show that $E \cap I$ contains most points of I. Hence E has positive measure.

$$E \cap I = I - \{x \in I:\ \exists k \text{ such that } \operatorname{dist}(x, L_k) \leqslant e_k/N_k\}$$
$$= I - \bigcup_k \{x \in I:\ \operatorname{dist}(x, L_k) \leqslant e_k/M_k\}.$$

The set $I \cap \{x:\ \operatorname{dist}(x, L_k) \leqslant e_k/M_k\}$ is contained in $2M/d_k + 2$ intervals of length $2e_k/M_k$. So its measure is less than or equal to

$$\left(\frac{2M}{d_k}+2\right)\frac{2e_k}{M_k} \leqslant \left(\frac{4M}{d_k}\right)\frac{2e_k}{M_k} = 4\frac{e_k}{d_k M_k}|I| = \frac{4}{m_k^2}|I|.$$

But the complement of $E \cap I$ in I is the union of these sets. Therefore

$$|I - E \cap I| \leqslant \sum_k \frac{4}{m_k^2}|I| \leqslant \frac{1}{2}|I|.$$

The second inequality follows from condition (iii). Hence $|E \cap I| \geqslant \frac{1}{2}|I|$.

We will now show that U is nontangentially bounded at every point of E for every aperture. Consider a specific point x_0 in E. Take any positive integer k.

The first claim is that there are only a finite number of lattice points in $\Gamma_{e_k}(x_0)$. Suppose that $(jd_i, 1/M_i)$ is one such lattice point. Then $|x_0 - jd_i| \leqslant e_k/M_i$. But since x_0 is in E, $|x_0 - jd_i| > e_i/M_i$. Since e_i is increasing these two inequalities imply that $i \leqslant k$. This proves the assertion that there are only a finite number of lattice points in $\Gamma_{e_k}(x_0)$. Therefore the sum of the dilates of u associated with these points is a function bounded on the upper half plane by some big constant, say B.

Consider a point $(x, t) \in \Gamma_{e_k}(x_0)$. To estimate U at this point only the lattice points outside $\Gamma_{e_k}(x_0)$ need be considered. Let $(jd_i, 1/M_i)$ be such a lattice point. Then

$$(12)\qquad M_i|x - jd_i| + M_i t \geqslant M_i|x - jd_i| + M_i\frac{|x - x_0|}{e_k}$$
$$\geqslant \frac{M_i}{e_k}(|x - jd_i| + |x - x_0|) \geqslant \frac{M_i}{e_k}|x_0 - jd_i| \geqslant \frac{M_i}{e_k}\frac{e_i}{M_i} = \frac{e_i}{e_k}$$

since $x \in \Gamma_{e_k}(x_0)$, $e_k \geqslant 1$, and $x_0 \in E$.

$\sum''\sum''$ will denote the sum over i and j such that $(jd_i, 1/M_i) \notin \Gamma_{e_k}(x_0)$. Using this notation $U(x, t)$ is dominated by

$$|U(x, t)| \leqslant B + \sum''\sum'' M_i\left|u\left(M_i(x - jd_i), M_i t\right)\right|.$$

It is at this point that the function p is needed. Estimates (5) and (12) yield

$$(13)\qquad |U(x, t)| \leqslant B + 2\sum_{i=1}^{\infty} M_i \sum_{j=0}^{\infty} \frac{1}{p(e_i/e_k + jd_i M_i)}.$$

The sum over j is dominated by the integral

$$(14)\qquad \frac{1}{p(e_i/e_k)} + \int_0^{\infty} \frac{dr}{p(e_i/e_k + d_k M_k r)}.$$

For large i, $p(e_i/e_k) = p(m_i^2/e_k) \geqslant p(m_i) = M_i^4$. Substituting this into (14) gives the larger expression

$$\frac{1}{M_i^3} + \frac{1}{d_i M_i}\int_0^{\infty}\frac{dr}{p(e_i/e_k + r)} \leqslant \frac{1}{M_i^3} + \frac{1}{d_i M_i}\frac{1}{p(e_i/e_k)^{2/3}}\int_0^{\infty}\frac{dr}{p(e_i/e_k + r)^{1/3}}$$
$$\leqslant \frac{1}{M_i^3} + \frac{1}{d_i M_i}\frac{1}{M_i^2}\int_0^{\infty}\frac{dr}{p(r)^{1/3}} \leqslant \frac{1}{M_i^3} + \frac{A}{M_i^2} \leqslant \frac{A}{M_i^2}.$$

Putting this into (13) completes the estimate of $U(x, t)$:

$$|U(x, t)| \leqslant B + 2\sum_{i=1}^{\infty} M_i \frac{A}{M_i^2} \leqslant B + A.$$

Since $U(x, t)$ is bounded by a constant independent of $(x, t) \in \Gamma_{e_k}(x_0)$ then U is nontangentially bounded at x_0.

We wish also to show that $U(x, t)$ is nontangentially convergent at x_0. Given $\varepsilon > 0$. Take I and J to be so large that $I > k$ and

$$(15)\qquad \sum_{i=I+1}^{\infty} M_i \sum_{j=J+1}^{\infty} \frac{1}{p(e_i/e_k + jd_i M_i)} < \tfrac{1}{4}\varepsilon.$$

This leaves a finite sum

$$U'(x, t) = \sum_{i=1}^{I}\sum_{j=1}^{J} M_i u\left(M_i(x - jd_i), M_i t\right).$$

Since $U'(x, t)$ is continuous in the closed upper half plane we can take h so small that if (x_1, t_1) and (x_2, t_2) are both in $\Gamma_{e_k}^h(x_0)$, then

$$(16)\qquad |U'(x_1, t_1) - U'(x_2, t_2)| < \tfrac{1}{2}\varepsilon.$$

Therefore (15) and (16) give nontangential convergence: for all $\varepsilon > 0$ there exists $h > 0$ such that if $(x_1, t_1), (x_2, t_2) \in \Gamma_{e_k}^h(x_0)$, then

$$|U(x_1, t_1) - U(x_2, t_2)| < \tfrac{1}{2}\varepsilon + 2\cdot\tfrac{1}{4}\varepsilon = \varepsilon.$$

It is of interest to show that U does not satisfy the hypothesis of the change of approximate identity theorem. In other words, the nontangential bound of U increases rapidly as a function of the aperture.

Let $a = \frac{1}{2}d_k M_k = \frac{1}{2}m_k^4$. Consider $\Gamma_a(x_0)$, where x_0 is arbitrary. There exists a jd_k that is at a distance no greater than $\frac{1}{2}d_k$ from x_0. So $|x_0 - jd_k| \leqslant \frac{1}{2}d_k = a/M_k$. Hence $(jd_k, 1/M_k)$ is in $\Gamma_a^1(x_0)$. Therefore, using (11),

$$\sup_{\Gamma_a(x_0)} |U(x,t)| \geqslant |U(jd_k, 1/M_k)| \geqslant \tfrac{1}{2}M_k = \tfrac{1}{2}(p(m_k))^{\frac{1}{4}} = \tfrac{1}{2}[p(\sqrt[4]{2a})]^{\frac{1}{4}}. \tag{17}$$

Since $p(a)$ increases faster than any polynomial, then so does the expression on the right-hand side of (17).

We have yet to show that $U(x,t)$ is actually of the form $(f*P_t)(x)$ for some tempered distribution f. Let ψ be any Schwartz function and use the notation $\tilde{\psi}(x) = \psi(-x)$. A simple change of variables shows that

$$\int u\big(M_k(x-jd_k), tM_k\big)\psi(x)\,dx = \int u^{\tilde{\psi}}\big(M_k(-x-jd_k), M_k\big)P_t(x)\,dx.$$

Since the sums defining U and $U^{\tilde{\psi}}$ are absolutely convergent we get from this

$$\int U(x,t)\psi(x)\,dx = \int U^{\tilde{\psi}}(-x,1)P_t(x)\,dx.$$

Since $U^{\tilde{\psi}}(\cdot,1)$ is continuous the limit of the right-hand side as $t\to 0$ is $U^{\tilde{\psi}}(0,1)$. Use this to define a functional on $\mathscr{S}$,

$$f(\psi) = \lim_{t\to 0}\int U(x,t)\psi(x)\,dx = U^{\tilde{\psi}}(0,1).$$

(8) gave an estimate of $U^{\hat{\psi}}(x,t)$ but if we had used Lemma 1 instead of (6) we would have been able to isolate the dependence on ψ. Doing this yields

$$|(U^{\psi}(0,1)| \leqslant A\,\|\psi\|_3$$

where by $\|\psi\|_3$ we mean the norm in Lemma 1 with $N = 3$. This shows that f is a tempered distribution. If we write $\varphi(y) = Q_t(y-x)$, then

$$(f*Q_t)(x) = f(\tilde{\varphi}) = U^{\varphi}(0,1) = U^Q(x,t).$$

Now we approach the problem of finding a new mollifier Q such that $f*Q_t$ is not nontangentially bounded. Q will be formed as a sum of translates of P. Define $q(x) = [p(x^{\frac{1}{4}})]^{\frac{1}{4}}$ and

$$Q(x) = \sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}\,P(x-i).$$

Note first of all that since P is a Schwartz function and q increases faster than any polynomial then Q must also be a Schwartz function.

u^Q and U^Q can be written in terms of u and U:

$$u^Q(x,t) = (\varphi*Q_t)(x) = \sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}(\varphi*P_t)(x-it) = \sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}u(x-it,t),$$

$$U^Q(x,t) = \sum_{k=1}^{\infty}\sum_{j=-\infty}^{\infty}M_k\sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}u\big(M_k(x-jd_k-it), M_kt\big).$$

We will be interested in estimating $U^Q(x,t)$ at points of the form $(J/M_{k_0}, 1/M_{k_0})$ where J is an integer. These points have the property that for any fixed k_0, any cone of aperture one contains one of these points. So we will try to show that U^Q is big at $(J/M_{k_0}, 1/M_{k_0})$. At this point,

$$U^Q\left(\frac{J}{M_{k_0}}, \frac{1}{M_{k_0}}\right) = \sum_{k=1}^{\infty}\sum_{j=-\infty}^{\infty}M_k\sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}u\left(M_k\left(\frac{J}{M_{k_0}} - jd_k - \frac{i}{M_{k_0}}\right), \frac{M_k}{M_{k_0}}\right).$$

But inequality (9) says that the terms for $k \neq k_0$ are comparatively small:

$$\sum_{k\neq k_0}^{\infty}M_k\sum_{j=-\infty}^{\infty}u\left(M_k\left(\frac{J}{M_{k_0}} - \frac{i}{M_{k_0}} - jd_k\right), \frac{M_k}{M_{k_0}}\right) \leqslant AM_{k_0}^{\frac{1}{4}}.$$

So the contribution of these terms is less than

$$\sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}AM_{k_0}^{\frac{1}{4}} = AM_{k_0}^{\frac{1}{4}}.$$

Therefore

$$U^Q\left(\frac{J}{M_{k_0}}, \frac{1}{M_{k_0}}\right) = \sum_{j=-\infty}^{\infty}M_{k_0}\sum_{i=-\infty}^{\infty}\frac{1}{q(|i|)}u(J - jd_{k_0}M_{k_0} - i, 1) + O(M_{k_0}^{\frac{1}{4}}). \tag{18}$$

But recall that $d_{k_0}M_{k_0} = m_k^4$ is an integer and $u(j,1) = 0$ whenever j is a nonzero integer. Thus we see that the terms in (18) are zero unless $i = J - jd_{k_0}M_{k_0}$. Hence (18) equals

$$U^Q\left(\frac{J}{M_{k_0}}, \frac{1}{M_{k_0}}\right) = \sum_{j=-\infty}^{\infty}M_{k_0}\frac{1}{q(|J-jd_{k_0}M_{k_0}|)} + O(M_{k_0}^{\frac{1}{4}}). \tag{19}$$

Pick a j such that $|J - jd_{k_0}M_{k_0}|$ is the smallest possible. Then $|J - jd_{k_0}M_{k_0}| \leqslant d_{k_0}M_{k_0}$. Thus

$$q(|J - jd_{k_0}M_{k_0}|) \leqslant q(d_{k_0}M_{k_0}) = q(m_{k_0}^4) = [p(m_{k_0})]^{\frac{1}{4}} = M_{k_0}^{\frac{1}{4}}.$$

Since all the terms in (19) are positive,

$$\sum_{j=-\infty}^{\infty} M_{k_0} \frac{1}{q(|J-jd_{k_0}M_{k_0}|)} \geqslant M_{k_0}\cdot \frac{1}{M_{k_0}^{\frac{1}{2}}} = M_{k_0}^{-\frac{1}{2}}.$$

Therefore

$$U^Q\left(\frac{J}{M_{k_0}}, \frac{1}{M_{k_0}}\right) \geqslant M_{k_0}^{\frac{1}{2}} - O(M_{k_0}^{\frac{1}{2}}). \tag{20}$$

Take any cone $\Gamma_1^1(x)$. Consider any fixed k_0. There must be a point of the form $(J/M_{k_0}, 1/M_{k_0})$ in $\Gamma_1^1(x_0)$. Therefore

$$\sup_{\Gamma_1^1(x_0)} |U^Q(x,t)| \geqslant M_{k_0}^{\frac{1}{2}} - O(M_{k_0}^{\frac{1}{2}}).$$

But since this is true for every k_0 and since $M_{k_0}\to\infty$ then

$$\sup_{\Gamma_1^1(x_0)} |U^Q(x,t)| = \infty.$$

This is true for every x_0. Furthermore, by a point of density argument (see Stein [5], p. 201), for every aperture (no matter how small)

$$\sup_{\Gamma_a^1(x_0)} |U^Q(x,t)| = \infty \quad \text{for almost every } x_0.$$

So $U^Q(x,t) = f*Q_t(x)$ is not nontangentially bounded at almost every point x_0.

This completes the proof of Theorem 1.

The phenomenon expressed in Theorem 1 is not a result of some property of P. In fact the theorem can be strengthened to give the following:

THEOREM 2. *Given any $P\in\mathcal{S}$ such that $\int P(x)\,dx \neq 0$, there exists a tempered distribution f and a $Q\in\mathcal{S}$ such that $f*P_t(x)$ is nontangentially convergent almost everywhere but $f*Q_t(x)$ is nontangentially bounded almost nowhere.*

Proof. The proof can be reduced to the problem of finding a correct $\varphi\in\mathcal{S}$ with which to build the basic peak $u(x,t) = \varphi*P_t(x)$. The rest of the construction in Theorem 1 will then proceed as before.

Clearly we may assume that $\hat{P}(0) = \int P(x)\,dx = 1$. Since $\hat{P}$ is smooth there exists an ε such that if $|x|<3$ then $|\hat{P}(\varepsilon x)| > \frac{1}{2}$. Let $\hat{\phi}$ be a Schwartz function supported in $[1, 2]$ such that $\int\hat{\phi}(z)\,dz = 1$. If χ denotes the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$, define

$$\hat{\varphi}(x) = \frac{\chi*\hat{\phi}(x)}{\hat{P}(\varepsilon x)}.$$

Since $|\hat{P}(\varepsilon x)| > \frac{1}{2}$ when x is in the support of $\chi*\hat{\phi}$ then $\hat{\varphi}$ is a Schwartz function. Now $\hat{\varphi}(x)\hat{P}_\varepsilon(x) = \chi*\hat{\phi}(x)$. So

$$\varphi*P_\varepsilon(x) = \frac{\sin(\pi x)}{(\pi x)}\,\phi(x).$$

Let $R = P_\varepsilon$. Then it is evident that

$$\varphi*R(j) = \begin{cases} 1 & \text{if} \quad j = 0,\\ 0 & \text{if} \quad j\in Z,\ j\neq 0.\end{cases}$$

Also $\int x^k\varphi(x)\,dx = 0$ since $\hat{\varphi}$ is identically zero near the origin. Therefore continuing in the construction of Theorem 1 we arrive at the conclusion that $f*R_t(x)$ is nontangentially convergent almost everywhere but $f*Q_t(x)$ is nontangentially bounded almost nowhere. Since $R = P_\varepsilon$, by changing the aperture of the cones we see that $f*P_t(x)$ is nontangentially convergent almost everywhere.

III. DIFFERENTIABILITY AT A POINT

We now turn to the problem of describing the smoothness of a tempered distribution f at a point x_0. Again we begin by forming a function $u(x,t) = f*\varphi_t(x)$ in the upper half space Ω and examining it on cones with vertex x_0. As in the Littlewood-Paley theory the smoothness can be measured by integrals involving various derivatives of u. These integrals will be denoted by $G(f)(x_0)$ and they can be combined to form norms $N(f)(x_0)$.

These norms will be used to define certain Banach spaces $\Lambda_\gamma^{p,\lambda}(x_0)$. The parameter λ gives the order of growth of u and its derivatives as we examine it over cones of larger and larger aperture with vertex x_0. Very often we are not interested in this order of growth and so we will consider instead the spaces $\Lambda_\gamma^p(x_0) = \bigcup_{\lambda>0}\Lambda_\gamma^{p,\lambda}(x_0)$. Roughly speaking, if $f\in\Lambda_\gamma^p(x_0)$, then f is a distribution that has smoothness of order γ at the point x_0. The p here denotes the type of mean value being used.

The results given here are analogous to the usual Sobolev spaces $L_k^p(\mathbf{R}^n)$. The essential difference however is that rather than measuring an average global differentiability, the spaces $\Lambda_\gamma^p(x_0)$ will measure differentiability at a single point x_0.

The proofs of the results stated in this part can be found in [3].

§ 1. The spaces $\Lambda_\gamma^{p,\lambda}(x_0)$. Let $\Omega = \{(x,t)\colon x\in\mathbf{R}^n, 0<t<1\}$. We will also write Z_0 for the nonnegative integers and Z^n for the corresponding

n-fold Cartesian product. If $\beta = (\beta_1, \ldots, \beta_n) \in Z_0^n$, then $|\beta| = \sum_{i=1}^{n} \beta_i$, $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$ and $\left(\frac{\partial}{\partial x}\right)^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$.

Now we define the G functions by

$$(1)\quad G_{\gamma,\beta}^{p,\lambda}(f)(x_0) = \left[\iint_{\Omega} \left(\frac{t}{t+|x-x_0|}\right)^{p\lambda} \left\{t^{|\beta|-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|\right\}^p \frac{dx\,dt}{t^{n+1}}\right]^{1/p}$$

where $u(x,t) = f*\varphi_t(x)$ and $1 \leqslant p \leqslant \infty$, $\lambda > 0$, $\beta \in Z_0^n$. For $p = \infty$ the expression in (1) should be interpreted as

$$G_{\gamma,\beta}^{\infty,\lambda}(f)(x_0) = \sup_{(x,t)\in\Omega} \left(\frac{t}{t+|x-x_0|}\right)^{\lambda} t^{|\beta|-\gamma}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)\right|.$$

As we have stated earlier, γ will give the order of differentiability of f at x_0.

If $G_{\gamma,\beta}^{p,\lambda}(f)(x_0) < \infty$, then we control $\left(\frac{\partial}{\partial x}\right)^\beta u(x,t)$. To control all the derivatives of a specific order we consider

$$\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) = \left[\sum_{|\beta|=k} \{G_{\gamma,\beta}^{p,\lambda}(f)(x_0)\}^p\right]^{1/p}$$

and

$$\mathscr{G}_{\gamma,k}^{\infty,\lambda}(f)(x_0) = \sup_{(x,t)\in\Omega} G_{\gamma,\beta}^{\infty,\lambda}(f)(x_0).$$

In the case $p = 2$, $\gamma = 0$, the expression $\mathscr{G}(f)(x_0)$ becomes the Littlewood–Paley function $g_{2\lambda/n}^*(f)(x_0)$:

$$\mathscr{G}_{0,1}^{2,\lambda}(f)(x_0) = \left[\iint_{\Omega}\left(\frac{t}{t+|x-x_0|}\right)^{2\lambda} |\nabla u(x,t)|^2 t^{1-n}\,dx\,dt\right]^{\frac{1}{2}} = g_{2\lambda/n}^*(f)(x_0)$$

where

$$|\nabla u|^2 = \sum_{i=1}^{n}\left|\left(\frac{\partial}{\partial x_i}\right)u\right|^2.$$

This case will be important when we study the connection between the $\mathscr{G}$ functions and tempered nontangential boundedness.

Our first result is that the finiteness of $G(f)(x_0)$ is independent of our choice of mollifier $\varphi \in \mathscr{S}$, $\int \varphi(x)\,dx \neq 0$. Since we are concerned with the effect of a change of mollifier, we will consider f and x_0 to be fixed and we will emphasize the dependence of G on φ. Consequently we will denote $G_{\gamma,\beta}^{p,\lambda}(f)(x_0)$ by $G_{\gamma,\beta}^{p,\lambda}(\varphi)$.

THEOREM 1. *Suppose that N is an integer $> n+\lambda$, $\lambda > 0$, and $1 \leqslant p \leqslant \infty$. Let φ be a C^∞ function such that $\hat{\varphi}(0) \neq 0$ and $\hat{\varphi}$ is of class C^N in a neighborhood of the origin. Then for every C^∞ function Φ,*

$$G_{\gamma,\beta}^{p,\lambda}(\Phi) \leqslant A\,\|\Phi\|\,G_{\gamma,\beta}^{p,\lambda}(\varphi)$$

where

$$\|\Phi\| \leqslant \sum_{|\delta|\leqslant N} \int (1+|z|)^{\lambda+|\beta|-\gamma}\left|\left(\frac{\partial}{\partial t}\right)^\delta \hat{\Phi}(z)\right| dz$$

and A is a constant independent of Φ, f, and x_0.

Since the G functions contain only derivatives of high order, certain harmless lower order terms must be added to form a norm. These lower order terms are defined as follows:

$$R_\beta^{p,\lambda}(f)(x_0) = \left[\int_{R^n} (1+|x-x_0|)^{-p\lambda}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right|^p dx\right]^{1/p}$$

and

$$R_\beta^{\infty,\lambda}(f)(x_0) = \sup_{x\in R^n} (1+|x-x_0|)^{-\lambda}\left|\left(\frac{\partial}{\partial x}\right)^\beta u(x,1)\right|,$$

where as usual $u(x,t) = f*\varphi_t(x)$ and $1 \leqslant p \leqslant \infty$, $\lambda > 0$, $\beta \in Z_0^n$.

These terms are harmless because if f is in $\mathscr{S}'$, then $u(x,1) = f*\varphi(x)$ is a tempered C^∞ function. Therefore $R_\beta^{p,\lambda}(f)(x_0)$ is finite for all λ sufficiently large.

The norms are then defined by

$$N_{\gamma,k}^{p,\lambda}(f)(x_0) = \mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) + \sum_{|\beta|<k} R_\beta^{p,\lambda}(f)(x_0).$$

As in Theorem 1 we can show that $N_{\gamma,k}^{p,\lambda}(f)(x_0)$ is essentially independent of the mollifier φ and that changing the mollifier gives an equivalent norm:

THEOREM 2. *Let N, λ, p and φ be as in Theorem 1. Then for every C^∞ function Φ,*

$$N_{\gamma,k}^{p,\lambda}(\Phi) \leqslant A\,\|\Phi\|\,N_{\gamma,k}^{p,\lambda}(\varphi),$$

where

$$\|\Phi\| = \sum_{|\delta|\leqslant N} \int (1+|z|)^{k+\lambda+\max(0,-\gamma_0)}\left|\left(\frac{\partial}{\partial z}\right)^\delta \hat{\Phi}(z)\right| dz$$

and A is a constant independent of Φ, f, x_0, and γ_0 is any real $< \gamma$.

In the norm $N_{\gamma,k}^{p,\lambda}(f)(x_0)$ the most critical information comes from the derivatives of $u(x,t)$ of order k (from the $\mathscr{G}$ function). It turns out though that for any $k > \gamma + n/p$ we get an equivalent norm.

THEOREM 3. *For any $k > \gamma + n/p$*

$$A_1 N_{\gamma,k+1}^{p,\lambda}(f)(x_0) \leqslant N_{\gamma,k}^{p,\lambda}(f)(x_0) \leqslant A_2 N_{\gamma,k+1}^{p,\lambda}(f)(x_0).$$

Furthermore, for any $k > \gamma$, $N^{p,\lambda}_{\gamma,k+1}(f)(x_0) \leqslant A_3 N^{p,\lambda}_{\gamma,k}(f)(x_0)$ and $N^{p,\lambda+2}_{\gamma,k}(f)(x_0) \leqslant A_4 N^{p,\lambda}_{\gamma,k+1}(f)(x_0)$. The A's here are constants independent of f, x_0, and φ.

Now that we know that the norms are independent of $\varphi \in \mathscr{S}$, $\int \varphi(x)\,dx \neq 0$, and $k > \gamma + n/p$ we are in a position to define the spaces $\Lambda^{p,\lambda}_\gamma(x_0)$:

$$\Lambda^{p,\lambda}_\gamma(x_0) = \Big\{f \in \mathscr{S}' \colon N^{p,\lambda}_{\gamma,k}(f)(x_0) < \infty \text{ for some } k > \gamma + n/p \text{ and } \varphi \in \mathscr{S},\ \int \varphi(x)\,dx \neq 0\Big\}.$$

From now on we will assume that k is the smallest integer greater than $\gamma + n/p$ and will write simply $N^{p,\lambda}_\gamma$.

THEOREM 4. *Let $\lambda > 0$, $1 \leqslant p \leqslant \infty$, $\gamma \in \boldsymbol{R}$.*

(a) *$\Lambda^{p,\lambda}_\gamma(x_0)$ is a Banach space with norm $N^{p,\lambda}_\gamma(f)(x_0)$.*

(b) *Using a different $\varphi \in \mathscr{S}$, $\int \varphi(x)\,dx \neq 0$, and $k > \gamma + n/p$ gives an equivalent norm for $\Lambda^{p,\lambda}_\gamma(x_0)$.*

For $1 \leqslant p < \infty$, the C^∞ functions with compact support are dense in $\Lambda^{p,\lambda}_\gamma(x_0)$.

The λ gives the order of growth as we examine $u(x, t) = f * \varphi_t(x)$ over cones of larger and larger aperture. Often, particularly in Chapter IV, this parameter will be unimportant. In these cases we consider the spaces $\Lambda^p_\gamma(x_0) = \bigcup_{\lambda > 0} \Lambda^{p,\lambda}_\gamma(x_0)$. It is important to note that because of the second part of Theorem 3, for the spaces $\Lambda^p_\gamma(x_0)$, we can use any $k > \gamma$ (and not just $k > \gamma + n/p$). Thus

$$\Lambda^p_\gamma(x_0) = \{f \in \mathscr{S}' \colon N^{p,\lambda}_{\gamma,k}(f)(x_0) < \infty \text{ for some } \lambda,\ k > \gamma\}$$
$$= \{f \in \mathscr{S}' \colon \mathscr{G}^{p,\lambda}_{\gamma,k}(f)(x_0) < \infty \text{ for some } \lambda,\ k > \gamma\}.$$

For each $1 \leqslant p \leqslant \infty$, every tempered distribution is in some $\Lambda^p_\gamma(x_0)$:

THEOREM 5. $\bigcup_{\gamma \in \boldsymbol{R}} \Lambda^p_\gamma(x_0) = \mathscr{S}'$.

§ 2. The Poisson kernel. Just as in the case of tempered nontangential boundedness the problem of changing to and from the Poisson kernel must be dealt with separately.

In this case the G functions are obtained by integrating over $\Omega^* = \{(x, t) \colon x \in \boldsymbol{R}^n,\ t > 0\}$ rather than over Ω. The resulting integrals are then denoted by $G^{p,\lambda *}_{\gamma,\beta}(f)(x_0)$.

THEOREM 6. *Let $\varphi \in \mathscr{S}$ and $\mathscr{P}$ be the Poisson kernel. Also let $1 \leqslant p \leqslant \infty$.*

(i) *If $\int \varphi(x)\,dx \neq 0$, then $G^{p,\lambda *}_{\gamma,\beta}(\mathscr{P}) \leqslant A G^{p,\lambda *}_{\gamma,\beta}(\varphi)$ for $|\beta| > \gamma + n/p - 1$. In addition, for any $k > \gamma$ if $\mathscr{G}^{p,\lambda}_{\gamma,k}(\varphi)$ is finite for some λ, then $G^{p,\mu *}_{\gamma,k}(\mathscr{P}) < \infty$ for $\mu \in \boldsymbol{R}$ large enough.*

(ii) *For all $\varphi \in \mathscr{S}$, $G^{p,\lambda}_{\gamma,\beta}(\varphi) \leqslant A G^{p,\lambda}_{\gamma,\beta}(\mathscr{P})$.*

The constants A here depend on φ and $\mathscr{P}$ but not on f or x_0.

In the applications of the next chapter we will change the mollifier to the Poisson kernel in order to take advantage of certain results about harmonic functions. To do this however we must be able to control $f * \varphi_t(x)$ over all of Ω^* rather than just Ω. The following lemma will be used to accomplish this.

LEMMA 1. *Let E be a bounded set and let W be a bounded open set containing E. If $f \in \Lambda^p_\gamma(x_0)$ for every x_0 in E, then there exists a distribution g with compact support such that $f = g$ in W and $g \in \Lambda^{p*}_\gamma(x_0)$, where the asterisk means that integration is over Ω^* rather than Ω.*

We will write $f = g + (f - g)$. $f - g$ is harmless because it is identically zero in a neighborhood of every x_0 in W. Thus $f - g \in \Lambda^p_\gamma(x_0)$ for all $\gamma \in \boldsymbol{R}$, $x_0 \in W$. We state this separately.

LEMMA 2. *If g is identically zero in a neighborhood of E, then $g \in \Lambda^p_\gamma(x_0)$ for every $\gamma \in \boldsymbol{R}$, $x_0 \in E$.*

§ 3. Operators on $\Lambda^{p,\lambda}_\gamma(x_0)$. A C^∞ function $a(\xi)$ on $\boldsymbol{R}^n - \{0\}$ will be called a *multiplier of order m* if for all $\beta \in Z^n_0$

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\beta a(\xi)\right| \leqslant A(1 + |\xi|)^{m - |\beta|}.$$

Let f be a tempered distribution with compact support such that

$$\int \hat{f}(\xi)(1 + |\xi|)^m\,d\xi < \infty.$$

The singular integral operator associated with a is defined by

$$(Tf)(x) = \int e^{ix\xi} a(\xi)\hat{f}(\xi)\,d\xi.$$

THEOREM 7. *If f is a C^∞ function with compact support and T is a singular integral operator of order m, then there exists a constant A such that*

$$N^{p,\lambda}_{\gamma-m}(Tf)(x_0) \leqslant A N^{p,\lambda}_\gamma(f)(x_0).$$

Therefore T can be extended to a bounded linear operator from $\Lambda^{p,\lambda}_\gamma(x_0)$ to $\Lambda^{p,\lambda}_{\gamma-m}(x_0)$, for $1 \leqslant p < \infty$.

Certain special cases of this theorem deserve to be stated separately. Let f be a C^∞ function with compact support.

COROLLARY 1. *If $\left|\left(\frac{\partial}{\partial \xi}\right)^\beta m(\xi)\right| \leqslant A|\xi|^{-|\beta|}$ for all $\beta \in Z^n_0$ and $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$, then T extends to a bounded linear operator from $\Lambda^{p,\lambda}_\gamma(x_0)$ to itself.*

COROLLARY 2. *$Tf = \left(\frac{\partial}{\partial x}\right)^{\beta} f$ is bounded from $\Lambda_{\gamma}^{p,\lambda}(x_0)$ to $\Lambda_{\gamma-|\beta|}^{p,\lambda}(x_0)$. Also, $Tf = (I-\Delta)^{\alpha/2} f$ is a Banach space isomorphism from $\Lambda_{\gamma}^{p,\lambda}(x_0)$ to $\Lambda_{\gamma-\alpha}^{p,\beta}(x_0)$, for $1 \leqslant p < \infty$.*

THEOREM 8. *$f \in \Lambda_{\gamma}^{p,\cdot}(x_0)$ if and only if $\left(\frac{\partial}{\partial x_i}\right) f \in \Lambda_{\gamma-1}^{p,\lambda}(x_0)$ for all $i = 1, 2, \ldots, n$.*

IV. DIFFERENTIABILITY ON A SET OF POSITIVE MEASURE

The chief purpose of this final chapter is to prove analogs of the various theorems concerning nontangential boundedness of harmonic functions. Our results will be applicable to functions of the form $u(x, t) = f*\varphi_t(x)$ where $f \in \mathscr{S}'$ and $\varphi \in \mathscr{S}$. Since we want the freedom to change the mollifier φ we will be concerned with tempered nontangential boundedness rather than nontangential boundedness.

We will prove analogs of the following theorems.

THEOREM 1. *Let $u(x, t)$ be harmonic in Ω^*. If u is nontangentially bounded at almost every point of a set $E \subset \boldsymbol{R}^n$, then u is nontangentially convergent at almost every point of E.*

For u harmonic in Ω^* the area integral of Lusin is defined by

$$S_{\alpha}(u)(x_0) = \Bigg(\iint_{\Gamma_{\alpha}^1(x_0)} \{t|\nabla u|\}^2 \frac{dx\,dt}{t^{n+1}} \Bigg)^{\frac{1}{2}}$$

where

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial t}\right|^2 + \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2.$$

THEOREM 2. *Let u be harmonic in Ω^*. Then u has a nontangential limit at almost every point of E iff $S_{\alpha}(u)(x_0)$ is finite for almost every point of E.*

Let $t = x_0$. The harmonic functions $u_0, u_1, \ldots, u_n$ in Ω^* are conjugate harmonic if

$$\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \quad \text{for all } 0 \leqslant i,\ j \leqslant n.$$

THEOREM 3. *Suppose that $u_0, \ldots, u_n$ are conjugate harmonic functions in Ω^*. If one of these functions has a nontangential limit at almost every point of $E \subset \boldsymbol{R}^n$, then all of them have nontangential limits at almost every point of E.*

In addition to proving analogs of these results we will also look at the spaces $\Lambda_{\gamma}^2(E)$. Define these spaces by

$$\Lambda_{\gamma}^2(E) = \{f \in \mathscr{S}' \colon f \in \Lambda_{\gamma}^2(x_0) \text{ for almost every } x_0 \text{ in } E\}$$
$$= \{f \in \mathscr{S}' \colon \text{for a.e. } x_0 \text{ in } E\ \exists\, \lambda > 0 \text{ and } k > \gamma \text{ such that } \mathscr{G}_{\gamma,k}^{2,\lambda}(f)(x_0) < \infty\}.$$

It will turn out that for any nonnegative integer k, f will be in $\Lambda_k^2(E)$ if and only if all its derivatives of order k are tempered nontangentially convergent at almost every point of E. Thus these two notions of differentiability are equivalent.

This is closely related to the idea of differentiation in the harmonic sense. For f in $L^1(\boldsymbol{R}^n)$, form the harmonic function $u(x, t) = f*\mathscr{P}_t(x)$. f is said to have a *harmonic derivative at* x_0 if u and all its first order derivatives $\partial u/\partial x_i$, $i = 1, \ldots, n$ are nontangentially convergent at x_0. A similar definition is used for higher order derivatives. See Stein [4], [5].

§ 1. Harmonic functions. In the next section we will need certain variants of standard results about harmonic functions. In particular we will want to keep track of the dependence of various constants on the aperture of the cone being considered. The proofs of this section have been adapted from Stein [5].

LEMMA 1. *Let $u(x, t)$ be harmonic in Ω^*. If u is nontangentially bounded at x_0 and for some $\lambda > 0$*

$$\iint_{\Omega^*} \left(\frac{t}{t+|x-x_0|}\right)^{2\lambda} \{t|\nabla u(x, t)|\}^p \frac{dx\,dt}{t^{n+1}} < \infty, \tag{16}$$

then u is tempered nontangentially bounded at x_0.

Proof. Consider two cones $\Gamma_{2\alpha}^1(x_0)$ and $\Gamma_{\alpha}^{\frac{1}{2}}(x_0)$ with $\alpha \geqslant 1$. There exists a constant c such that for every $(x, t) \in \Gamma_{\alpha}^{\frac{1}{2}}(x_0)$

$$B = \{(z, s) \colon |(x, t)-(z, s)| \leqslant ct\} \subset \Gamma_{2\alpha}^1(x_0),$$

where c does not depend on α ($c = \frac{1}{4}$ works). Using the mean value theorem for harmonic functions it follows that

$$|(\nabla u)(x, t)| \leqslant \Bigg(\frac{A}{|B|} \int_B |(\nabla u)(z, s)|^2 dz\,ds\Bigg)^{\frac{1}{2}}$$

since $|B| = A(ct)^{n+1}$ and s and t are comparable, then

$$t|(\nabla u)(x, t)| \leqslant A\Bigg(\int_B \{s|\nabla u(z, s)|\}^2 \frac{dz\,ds}{s^{n+1}}\Bigg)^{\frac{1}{2}}.$$
$$\leqslant A\Bigg(\iint_{\Gamma_{2\alpha}^1} \{s|\nabla u(z, s)|\}^2 \frac{dz\,ds}{s^{n+1}}\Bigg)^{\frac{1}{2}}.$$

Since $\left(\frac{s}{s+|z-x_0|}\right)(1+2\alpha)\geqslant 1$ for $(z,s)\in\Gamma^1_{2\alpha}(x_0)$ we can use (16) to show that

(17)
$$\sup_{(x,t)\in\Gamma^{1/2}_{\alpha}(x_0)} t|(\nabla u)(x,t)| \leqslant A(2\alpha)^\lambda \left(\iint_{\Gamma^1_{2\alpha}} \left(\frac{s}{s+|z-x_0|}\right)^{2\lambda}\{s|\nabla u(z,s)|\}^2\frac{dz\,ds}{s^{n+1}}\right)^{1/2}$$
$$\leqslant A\alpha^\lambda.$$

Consider $(x,t)\in\Gamma^1_\alpha(x_0)$. Since

(18) $$u(x,t)=u(x_0,t)+\int_L(\nabla_x u)(z,t)\,dz$$

where L is the line from x_0 to x, we can use (17) to show that

$$|u(x,t)|\leqslant|u(x_0,t)|+|x-x_0|\cdot\frac{A\alpha^\lambda}{t}.$$

But since u is nontangentially bounded then $|u(x_0,t)|\leqslant A$ where A is a constant independent of t. Also $\frac{|x-x_0|}{t}\leqslant\alpha$. Hence

$$|u(x,t)|\leqslant A+A\alpha^{\lambda+1}\quad \text{for all } (x,t)\in\Gamma^{1/2}_\alpha(x_0) \text{ and } \alpha\geqslant 1.$$

Therefore $u(x,t)$ is TNTB at x_0.

Lemma 2. *Let $u(x,t)$ be harmonic in Ω^*. If $u(x,t)$ is nontangentially convergent at x_0 and for some $\lambda>0$*

(19) $$\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{2\lambda}\{t|\nabla u(x,t)|\}^2\frac{dx\,dt}{t^{n+1}}<\infty,$$

then $u(x,t)$ is tempered nontangentially convergent at x_0.

Proof. Let $I(h)$ denote the integral in (19) except integrated over $\Omega_h=\{(x,t)\colon x\in \boldsymbol{R}^n,\ 0<t<h\}$ rather than Ω^*. Note that $I(h)\to 0$ as $h\to 0$. Also let $B_h=\sup_{0<t_1,t_2<h}|u(x_0,t_1)-u(x_0,t_2)|$. Then $B_h\to 0$ as $h\to 0$. As in (18) we have

$$u(x_1,t_1)-u(x_2,t_2)=u(x_0,t_1)-u(x_0,t_2)+\int_{L_1}(\nabla_x u)(z,t_1)\,dz-\int_{L_2}(\nabla_x u)(z,t_2)\,dz$$

where L_1 and L_2 are the lines from x_0 to x_1 and from x_0 to x_2. As in the proof of Lemma 1

$$t|\nabla_x u(z,t)|\leqslant A(1+2\alpha)^\lambda I(h).$$

Therefore if $(x_1,t_1),(x_2,t_2)\in\Gamma^h_\alpha(x_0)$, then

$$|u(x_1,t_1)-u(x_2,t_2)|\leqslant B_h+2\frac{|x-x_0|}{t}A(1+\alpha)^\lambda I(h)$$
$$\leqslant A(1+\alpha)^{\lambda+1}\big(B_h+I(h)\big).$$

Since $B_h+I(h)\to 0$ the proof is complete.

Let $g^*_\lambda(u)(x_0)$ be the Littlewood-Paley function defined by

$$g^*_\lambda(u)(x_0)=\left(\iint_\Omega\left(\frac{t}{t+|x-x_0|}\right)^{2\lambda}\{t|\nabla u(x,t)|\}^2\frac{dx\,dt}{t^{n+1}}\right)^{1/2}$$

where $u(x,t)$ is harmonic in Ω.

Theorem 4. *If for almost every x_0 in E there exists $\lambda>0$ such that $g^*_\lambda(u)(x_0)<\infty$, then $u(x,t)$ is TNTC for almost every x_0 in E.*

Proof. By assumption the area integral of Lusin

$$S(u)(x_0)=\left(\iint_{\Gamma^1_1(x_0)}\{t|\nabla u(x,t)|\}^2\frac{dx\,dt}{t^{n+1}}\right)^{1/2}$$

is finite for almost every x_0 in E. But by Theorem 2 this means that u is nontangentially convergent at almost every point of E. Therefore this theorem follows from Lemma 2.

Now we will prove the converse to Theorem 4.

Lemma 3 (Stein [5], p. 206). *Let E_0 be a compact subset of $\boldsymbol{R}^n$ and let $\mathscr{R}_\alpha=\bigcup_{x_0\in E_0}\Gamma^1_\alpha(x_0)$. There exists a sequence of subregions $\mathscr{R}_{\alpha,\varepsilon}$ with smooth boundary such that*

(i) $\overline{\mathscr{R}}_{\alpha,\varepsilon}\subset\mathscr{R}_\alpha$ *and* $\mathscr{R}_{\alpha,\varepsilon_1}\subset\mathscr{R}_{\alpha,\varepsilon_2}$ *if* $\varepsilon_2<\varepsilon_1$,

(ii) $\mathscr{R}_{\alpha,\varepsilon}\to\mathscr{R}_\alpha$,

(iii) $\int_{\partial\mathscr{R}_{\alpha,\varepsilon}}d\tau_{\alpha,\varepsilon}\leqslant A(1+\alpha)^n$

where $d\tau_{\alpha,\varepsilon}$ is an element of surface area on the boundary $\partial\mathscr{R}_{\alpha,\varepsilon}$.

Lemma 4. *If $u(x,t)$ is harmonic, then*

$$\sup_{\Gamma^{1/2}_\alpha(x_0)} t|\nabla u(x,t)|\leqslant A\sup_{\Gamma^1_\alpha(x_0)}|u(x,t)|.$$

Proof. There exists a constant c independent of α such that if $(x,t)\in\Gamma^{1/2}_\alpha(x_0)$, then the ball B of radius αt about (x,t) is in $\Gamma^1_{2\alpha}(x_0)$. By the mean value theorem for harmonic functions

$$t|\nabla u(x,t)|\leqslant A\sup_B|u(x,t)|\leqslant A\sup_{\Gamma^1_{2\alpha}(x_0)}|u(x,t)|.$$

THEOREM 5. *If $u(x,t)$ is TNTB for almost every x_0 in E then for almost every x_0 in E there exists $\lambda > 0$ such that $g_\lambda^*(u)(x_0)$.*

Proof. By considering a compact subset $E_0 \subset E$ we can simplify the problem. Thus it suffices to prove the theorem under the following assumption. There exist A and λ such that for every x_0 in E_0

(20) $$\sup_{\Gamma_\alpha^1(x_0)} |u(x,t)| \leqslant A(1+\alpha)^\lambda \quad \text{for all } \alpha > 0.$$

We will now estimate $S_\alpha(u)(x_0)$ by considering the integral

(21) $$I = \int_{E_0} \{S_\alpha(u)(x_0)\}^2 dx_0 = \int_{E_0} \iint_{\Gamma_\alpha^1(x_0)} \{t|\nabla u(x,t)|\}^2 \frac{dx\,dt}{t^{n+1}}\, dx_0.$$

As before we will write $\mathscr{R}_\alpha = \bigcup_{x\in E_0} \Gamma_\alpha^1(x_0)$, and $\chi(x_0, x, t)$ will be the characteristic function of the set

$$\{(x_0, x, t):\ x_0 \in E_0,\ (x,t) \in \Gamma_\alpha^1(x_0)\}.$$

Using this notation we can change the order of integration in (21). So

$$I = \iint_{\mathscr{R}_\alpha} \Big\{\int_{E_0} \chi(x_0, x, t)\, dx_0\Big\} \{t|\nabla u(x,t)|\}^2 \frac{dx\,dt}{t^{n+1}}.$$

Since

$$\int_{E_0} \chi(x_0, x, t)\, dx_0 \leqslant \int_{|x_0 - x| \leqslant \alpha t} dx_0 = A\alpha^n t^n,$$

then

(22) $$I \leqslant A \iint_{\mathscr{R}_\alpha} t|\nabla u(x,t)|^2 dx\,dt.$$

We now examine the integrals

$$\iint_{\mathscr{R}_{\alpha,\varepsilon}} t|\nabla u(x,t)|^2 dx\,dt$$

where $\mathscr{R}_{\alpha,\varepsilon}$ are the subregions given in Lemma 3, and $\varepsilon > 0$. However since $\Delta t = 0$ and $\Delta(\frac{1}{2}u^2) = |\nabla u|^2$ this integral equals

$$\iint_{\mathscr{R}_{\alpha,\varepsilon}} t\Delta(\tfrac{1}{2}u^2) - (\Delta t)(\tfrac{1}{2}u^2)\, dx\,dt.$$

Therefore by Green's Theorem

(23) $$\iint_{\mathscr{R}_{\alpha,\varepsilon}} t|\nabla u(x,t)|^2 dx\,dt = \int_{\partial\mathscr{R}_{\alpha,\varepsilon}} t\left(\frac{\partial}{\partial n_{\alpha,\varepsilon}}\right)\left(\frac{1}{2}u^2\right) - \frac{1}{2}u^2\left(\frac{\partial E}{\partial n_{\alpha,\varepsilon}}\right) d\tau_{\alpha,\varepsilon}$$

where $n_{\alpha,\varepsilon}$ is the outward normal to $\mathscr{R}_{\alpha,\varepsilon}$ and $d\tau_{\alpha,\varepsilon}$ is an element of surface area.

By Lemma 4 we have $|\nabla u(x,t)| \leqslant \frac{A}{t}(1+2\alpha)^\lambda$. Consequently,

(24) $$\left|\frac{\partial}{\partial n_{\alpha,\varepsilon}}\left(\frac{1}{2}u^2\right)\right| \leqslant |u|\,|\nabla u| \leqslant A(1+\alpha)^\lambda \frac{A}{t}(1+2\alpha)^\lambda = \frac{A}{t}(1+\alpha)^{2\lambda}.$$

Also $\left|\frac{\partial\tau}{\partial n_{\alpha,\varepsilon}}\right| \leqslant 1$. Therefore $\left|\frac{1}{2}u^2 \frac{\partial t}{\partial n_{\alpha,\varepsilon}}\right| \leqslant A(1+\alpha)^{2\lambda}$. Therefore this and (24) can be used to simplify (23):

(25) $$\iint_{\mathscr{R}_{\alpha,\varepsilon}} t|\nabla u(x,t)|^2 dx\,dt \leqslant A(1+\alpha)^{2\lambda} \int_{\partial\mathscr{R}_{\alpha,\varepsilon}} d\tau_{\alpha,\varepsilon}.$$

By property (iii) of Lemma 3 we know that $\int_{\partial\mathscr{R}_{\alpha,\varepsilon}} d\tau_{\alpha,\varepsilon} \leqslant A(1+\alpha)^n$. Therefore we can put this into (25) and let $\varepsilon \to 0$.

This gives

$$\iint_{\mathscr{R}_\alpha} t|\nabla u(x,t)|^2 dx\,dt \leqslant A(1+\alpha)^{2\lambda+n}.$$

Therefore it follows from (22) that

(26) $$\int_{E_0} \{S_\alpha(u)(x_0)\}^2 dx_0 \leqslant A(1+\alpha)^{2\lambda+2n}.$$

Let $\lambda_1 > \lambda + n + \frac{1}{2}$. Then multiplying (26) by $(1+\alpha)^{-2\lambda_1}$ and integrating α from 0 to ∞ produces

$$\int_{E_0}\int_0^\infty \{S_\alpha(u)(x_0)\}^2 (1+\alpha)^{-2\lambda_1} d\alpha\, dx_0 \leqslant A \int_0^\infty (1+\alpha)^{2\lambda+2n-2\lambda_1} d\alpha = A.$$

So we have shown that for almost every x_0 in E_0

$$\int_0^\infty S_\alpha(u)(x_0)^2 (1+\alpha)^{-2\lambda_1} d\alpha < \infty.$$

But since $S_\alpha(u)(x_0)$ is increasing,

$$S_\alpha(u)(x_0)^2 \int_\alpha^\infty (1+\beta)^{-2\lambda_1} d\beta \leqslant \int_\alpha^\infty S_\beta(u)(x_0)^2 (1+\beta)^{-2\lambda_1} d\beta \leqslant A$$

and

$$S_\alpha(u)(x_0) \leqslant A(1+\alpha)^{2\lambda_1 - 1}.$$

Therefore

$$g^*_{\lambda_1}(u)(x_0) = \iint\limits_{\Omega} \left(\frac{t}{t+|x-x_0|}\right)^{2\lambda_1} \{t|\nabla u(x,t)|\}^2 \frac{dx\,dt}{t^{n+1}}$$

$$\leqslant S_1(u)(x_0)^2 + \sum_{j=1}^{\infty} \iint\limits_{\Gamma^1_{2^j}(x_0)-\Gamma^1_{2^{j-1}}(x_0)} \{t|\nabla u(x,t)|\}^2 \frac{dx\,dt}{t^{n+1}} \frac{1}{(1+2^{j-1})^{2\lambda_1}}$$

$$\leqslant S_1(u)(x_0)^2 + \sum_{j=1}^{\infty} S_{2^j}(u)(x_0)^2 2^{-2j\lambda_1} \leqslant A + A\sum_{j=1}^{\infty} 2^{j(2\lambda_1-1)} 2^{-j\lambda_1} = A\sum_{j=0}^{\infty} 2^{-j} = A.$$

This completes the proof of Theorem 5.

THEOREM 6. *Let $u(x, t)$ be harmonic in Ω. If u is* TNTB *at every point x_0 in E, then u is* TNTC *at almost every point of E.*

Proof. This follows from Theorem 5, Theorem 1, and Lemma 2.

§ 2. Applications to tempered nontangential boundedness. By combining the change of approximate identity theorems with the results of the last section about harmonic functions, we are able to give analogs of the theorems mentioned in the introduction to this chapter.

THEOREM 1′. *If f is* TNTB *at almost every point in E, then f is* TNTC *at almost every point in E.*

Proof. As in the restriction Theorem I.9, f can be written as $f = f_1+g$, where f_1 is TNTB over untruncated cones and $g \equiv 0$ in a neighborhood of E (It suffices to consider the case where E is bounded.). Clearly g is TNTC in E (Lemma I.3).

Now consider f_1. By Theorem I.3 we can change mollifiers from φ to the Poisson kernel. Since $U(x, t) = f_1 * \mathscr{P}_t(x)$ is harmonic we can apply Theorem 7 to conclude that $f_1 * \mathscr{P}_t(x)$ is TNTC almost everywhere in E. So by the change of approximate identity Theorem I.4 we conclude that $f_1 * \varphi_t(x)$ is TNTC almost everywhere in E.

THEOREM 2′. *f is* TNTC *at almost every point of E if and only if $f \in \Lambda^2_0(E)$.*

Proof. Suppose that f is TNTC at almost every point of E. Using Theorem I.9 we write $f = f_1+g$, where f_1 is bounded over untruncated cones and $g \equiv 0$ in a neighborhood of E. By Lemma III.2 $g \in \Lambda^2_0(x_0)$. Since f_1 is bounded over untruncated cones we can switch to the Poisson kernel (Theorem I.3). Now by the theory for harmonic functions some g^*_λ for $f_1 * \mathscr{P}_t(x)$ is finite (Theorem 5). Thus by changing back to our original mollifier we see that $f_1 \in \Lambda^2_0(E)$ (Theorem III.6).

Now suppose that $f \in \Lambda^2_0(E)$. This time we use the decomposition $f = f_1+g$ in Lemma III.1. Since $g \equiv 0$ in a neighborhood of E then g is TNTC in E. For f_1, we change mollifiers to take advantage of the theorem for harmonic functions (Theorem III.6, Theorem 4, Theorem I.2).

If $f \in L^2(\boldsymbol{R}^n)$ then finding a harmonic conjugate of $f * \mathscr{P}_t(x)$ is equivalent to applying a Riesz transform to f. Therefore our analog of Theorem 3 will be a statement about the effect of singular integral operators on tempered nontangential boundedness.

THEOREM 3′. *Let $m(z)$ be C^∞ in $\boldsymbol{R}^n \setminus \{0\}$ such that there exists a constant B with*

(i) $|m(z)| \leqslant B$,

(ii) $\left|\left(\frac{\partial}{\partial z}\right)^\alpha m(z)\right| \leqslant B|z|^{-|\alpha|}$ *for all* $\alpha \in Z^n_0$.

For every C^∞ function f with compact support define T_m by $\widehat{T_m f}(z) = m(z)\hat{f}(z)$. Then T_m extends to a bounded operator on $\Lambda^{2,\lambda}_0(x_0)$ for all $x_0 \in \boldsymbol{R}^n$, $\lambda > 0$. If f is tempered nontangentially convergent at almost every point of E, then $T_m f$ is tempered nontangentially convergent at almost every point of E.

Proof. By Theorem 2′ we need only show that if $f \in \Lambda^2_0(E)$, then $T_m f \in \Lambda^2_0(E)$. This statement however is exactly the assertion of the multiplier theorem (Chapter III, Corollary 2).

As in differentiation in the harmonic sense we can say that f is differentiable if its first order derivatives are tempered nontangentially convergent. This idea of differentiation is closely related to the spaces $\Lambda^2_\gamma(E)$.

THEOREM 7. *For any nonnegative integer k, $f \in \Lambda^2_k(E)$ if and only if all its derivatives of order k are* TNTC *a.e. E; that is, if and only if for every $|\beta| = k$, $\left(\frac{\partial}{\partial x}\right)^\beta f$ is* TNTC *a.e. E.*

Proof. This is an immediate consequence of Theorem 2′ and Theorem III.8.

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