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A characterization of multiplicative linear functionals in Banach algebras

by

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Abstract. A characterization is given of multiplicative linear functionals in complex Banach algebras without the linearity assumption.

1. Introduction. The following characterization of multiplicative linear functionals in complex Banach algebras, given independently by Gleason and Kahane–Żelazko, is well known (cf. [1], [3], [5]):

1.1. THEOREM. *Let A be a complex Banach algebra (not necessarily unital and commutative). Let $f\colon A \to C$ be a linear selection from the spectrum, i.e.*

$$f(x) \in \sigma(x) \quad \textit{for every } x \in A.$$

Then f is multiplicative.

In this note we weaken the assumptions of this theorem in the following way:

1.2. THEOREM. *Let A be a complex Banach algebra (not necessarily commutative and unital). Let $f\colon A \to C$ satisfy*

$$f(0) = 0, \tag{1.1}$$

$$f(x)-f(y) \in \sigma(x-y) \quad \textit{for every } x, y \in A. \tag{1.2}$$

Then f is multiplicative and linear.

We have found this theorem trying to weaken an assumption of the spectral mapping theorem of [6]. This application is contained in Section 4 of the present paper. Theorem 1.2 is proved in Section 2; Section 3 contains a proof of an auxiliary lemma.

2. Proof of the main theorem.

DEFINITION. Let X be a complex linear space. We say that a map $\varphi\colon X \to C$ is *complex* (*real*) linear (shortly *C-linear* or *R-linear*) if it is additive and homogeneous with respect to complex (real) scalars.

2.1. LEMMA. *Let A be a complex Banach algebra and let φ be an R-linear selection from the spectrum. Then φ is C-linear.*

Proof. We have

$$e^{ir}\varphi(e^{-ir}x) \in e^{ir}\sigma(e^{-ir}x) = \sigma(x)$$

for every $r \in R$, $x \in A$. Since φ is R-linear,

$$e^{ir}\varphi(e^{-ir}x) = \frac{\varphi(x)-i\varphi(ix)}{2} + e^{2ir}\frac{\varphi(x)+i\varphi(ix)}{2};$$

so for a fixed $x \in A$ the set

$$\Gamma_x = \{e^{ir}\varphi(e^{-ir}x):\ r \in R\}$$

is a circle contained in $\sigma(x)$ with centre $\frac{1}{2}(\varphi(x)-i\varphi(ix))$. It is easy to verify that the numbers $\mathrm{Re}\varphi(x)+i\mathrm{Im}(-i\varphi(ix))$ and $\mathrm{Re}(-i\varphi(ix))+$ $+i\mathrm{Im}\varphi(x)$ are in Γ_x and so in $\sigma(x)$. Thus the functionals

$$(2.1)\qquad \begin{aligned} \varphi_1(x) &= \mathrm{Re}\varphi(x)+i\mathrm{Im}(-i\varphi(ix)) = \mathrm{Re}\varphi(x)-i\mathrm{Re}\varphi(ix),\\ \varphi_2(x) &= \mathrm{Re}(-i\varphi(ix))+i\mathrm{Im}\varphi(x) = \mathrm{Im}\varphi(ix)+i\mathrm{Im}\varphi(x)\end{aligned}$$

are selections from the spectrum. Moreover, from

$$\varphi_k(ix) = i\varphi_k(x), \qquad k = 1, 2,$$

they are C-linear. By the Gleason–Kahane–Żelazko theorem they are multiplicative on A. We shall show that $\varphi_1 = \varphi_2$. Otherwise we would have, for a certain $a \in A$,

$$(2.2)\qquad \varphi_1(a) = 1, \quad \varphi_2(a) = 0.$$

Denote $h(z) = e^{i\pi iz}-1$. The function h operates in every Banach algebra, (not necessarily unital). By formulas (2.1) it follows that

$$\begin{aligned}\varphi(h(a)) &= \mathrm{Re}\varphi(h(a))+i\mathrm{Im}\varphi(h(a)) = \mathrm{Re}\varphi_1(h(a))+i\mathrm{Im}\varphi_2(h(a))\\ &= \mathrm{Re}\,h(\varphi_1(a))+i\mathrm{Im}\,h(\varphi_2(a)) = \mathrm{Re}\,h(1)+i\mathrm{Im}\,h(0)\\ &= -1 \in \sigma(h(a)) = h(\sigma(a)) \subset h(C) = C\setminus\{-1\},\end{aligned}$$

which is impossible. Having $\varphi_1 = \varphi_2$, we see by formulas (2.1) that

$$\mathrm{Re}\varphi(x) = \mathrm{Re}(-i\varphi(ix)) \quad\text{and}\quad \mathrm{Im}\varphi(x) = \mathrm{Im}(-i\varphi(ix)),$$

which implies $\varphi(x) = -i\varphi(ix)$ and φ is C-linear. ∎

In the proof of the theorem we use an extension of Rademacher's theorem on differentiation of Lipschitz functions; Definition 2.2 and Theorem 2.3 were given by P. Mankiewicz [4].

Let Q be the Hilbert cube, i.e.

$$Q = \prod_{i=1}^{\infty}[-2^{-i}, 2^{-i}]$$

and let μ denote the natural product measure on Q.

2.2. Definition. A subset Z of a separable Fréchet space X is a *zero set* if for every affine continuous mapping $j: Q \to X$ with linearly dense image we have

$$\mu(j^{-1}(Z)) = 0.$$

We say that a mapping in a Fréchet space has a *real differential at a point* if it has a Gateaux differential with respect to real scalars which, in addition, is continuous.

2.3. Theorem. *If $f: X \to C$ is a Lipschitz mapping defined in a separable Fréchet space, then it has real differentials except for some zero set.*

In the proof of Theorem 1.2 we also need the following criterion for a Lipschitz function to be holomorphic:

2.4. Lemma. *Let X be a separable, complex Fréchet space, and let $f: U \to C$ be a locally Lipschitz mapping defined in its open subset. Assume that it has a C-linear differential at every point except for some zero set. Then f is holomorphic on U.*

The proof of this lemma is postponed till Section 3.

Proof of Theorem 1.2. (i) First we assume that A is separable. Suppose that f has a real differential at a point $a \in A$. Then

$$\frac{f(a+rx)-f(a)}{r} \in \frac{\sigma(a+rx-a)}{r} = \sigma(x) \qquad r \in R,\ r \neq 0,\ x \in A$$

and

$$(\mathrm{D}f)_a(x) = \lim_{r\to 0}\frac{f(a+rx)-f(a)}{r} \in \sigma(x).$$

Thus the differential is an R-linear selection from the spectrum and by Lemma 2.1 it is C-linear. On the other hand,

$$|f(x)-f(y)| \in |\sigma(x-y)| \leqslant \|x-y\|,$$

i.e. f is Lipschitz. By Theorem 2.3, f has real differentials except some zero set (here we use the separability assumption) and we have just proved that all these differentials are C-linear. Thus f fulfils the conditions of Lemma 2.4, which in turn implies that f is holomorphic in A, i.e. entire.

For $a, b \in A$ the function $f_{a,b}: C \to C$ defined by the formula

$$f_{a,b}(z) = f(az+b)$$

is Lipschitz and entire, hence it is affine, i.e.

$$f_{a,b}(z) = z[f_{a,b}(1)-f_{a,b}(0)]+f_{a,b}(0), \qquad z \in C,$$

$$(2.3)\qquad f(az+b) = z[f(a+b)-f(b)]+f(b).$$

Putting $b = 0$ in (2.3), we obtain

$$f(az) = zf(a), \tag{2.4}$$

because $f(0) = 0$ by assumption (1.1).

Now put in (2.3) $a = \frac{1}{2}(c-d)$, $b = d$ and $z = 2$. Then we have

$$f(c) = f(2a+b) = 2[f(a+b)-f(b)]+f(b) = 2\left[f\left(\tfrac{1}{2}(c+d)\right)-f(d)\right]+f(d)$$

and

$$f\left(\tfrac{1}{2}(c+d)\right) = \tfrac{1}{2}f(c)+\tfrac{1}{2}f(d). \tag{2.5}$$

Relations (2.4) and (2.5) mean that f is a $\boldsymbol{C}$-linear selection from the spectrum; hence, by the Gleason–Kahane–Żelazko theorem, f is multiplicative and linear.

(ii) Now we consider the general case. Let $a_1, a_2 \in A$. The function f of Theorem 1.2 restricted to subalgebra $[a_1, a_2]$ of A generated by a_1 and a_2 satisfies conditions (1.1), (1.2). As the subalgebra $[a_1, a_2]$ is separable and $f|_{[a_1,a_2]}$ is multiplicative and linear by step (i), f is multiplicative linear in the whole of A, because a_1, a_2 were chosen arbitrarily. ■

3. Proof of Lemma 2.4. We begin this section with the lemma relating a partial derivative of a Lipschitz function in the sense of distribution theory to the limit of the difference quotient. The lemma seems to belong to the so called mathematical folklore, but as we cannot point to reference, we shall prove it for completeness.

3.1. Lemma. *Let $F\colon U \to \boldsymbol{R}$ be a locally Lipschitz function defined in an open subset of $\boldsymbol{C}$. Let*

$$g(x, y) := \lim_{h\to 0} \frac{f(x+h, y)-f(x, y)}{h}$$

(this limit exists a.e. by Rademacher's theorem). Then the partial derivative $\partial/\partial x$ of the distribution represented by f is the distribution represented by the function g.

Proof. If $p \in L^1_{\text{loc}}(U, dx\,dy)$, we denote by $[p]$ the distribution represented by p. In order to prove that $\frac{\partial}{\partial x}[f] = [g]$ it suffices to show that for every smooth function φ with compact support contained in U (i.e. $\varphi \in C_c^\infty(U)$) we have

$$\iint g(x, y)\varphi(x, y)\,dx\,dy = -\iint f(x, y)\,\frac{\partial}{\partial x}\varphi(x, y)\,dx\,dy.$$

For the proof of this equality put

$$(\Delta_h p)(x, y) = \frac{p(x+h, y)-p(x, y)}{h}, \quad h \neq 0,$$

for any function p on U. $\Delta_h p$ is well defined on some subdomain of U depending on h. For a given $\varphi \in C_c^\infty(U)$ and sufficiently small $\varepsilon > 0$ all functions $\Delta_h f$, $\Delta_h \varphi$; $|h| < \varepsilon$ are defined on some compact set K satisfying in addition

$$\operatorname*{Int}_{K\subset U} K \supset \operatorname{supp} \Delta_h \varphi, \quad |h| < \varepsilon.$$

(We omit the proof of the existence of K.) We have

$$\iint (\Delta_h f)(x, y)\varphi(x, y)\,dx\,dy = -\iint f(x, y)(\Delta_{-h}\varphi)(x, y)\,dx\,dy. \tag{3.1}$$

The functions f and φ are locally Lipschitz; hence the families

$$(\Delta_h f)(x, y)\varphi(x, y), \quad (\Delta_{-h}\varphi)(x, y)f(x, y), \quad |h| < \varepsilon$$

are uniformly bounded on the compact K for sufficiently small ε (we omit the details of the proof). By the Lebesgue theorem we may pass to the limit ($h \to 0$) in integrals (3.1) thus obtaining

$$\iint g(x, y)\varphi(x, y)\,dx\,dy = -\iint f(x, y)\,\frac{\partial}{\partial x}\varphi(x, y)$$

(because $\Delta_h f \to g$ pointwise a.e. by the Rademacher theorem). ■

Using this lemma, we are now able to prove Lemma 2.4 in the special case of $X = \boldsymbol{C}$.

3.2. Lemma. *If $f\colon U \to \boldsymbol{C}$ is a locally Lipschitz function defined on an open subset of $\boldsymbol{C}$ and it has a $\boldsymbol{C}$-linear differential a.e. with respect to the Lebesgue measure, then it is homolorphic in U.*

Proof. It is easy to see that any function $f\colon U \to \boldsymbol{C}$ has a $\boldsymbol{C}$-linear differential at a point if and only if it has an $\boldsymbol{R}$-linear differential at that point and partial derivatives fulfil the Cauchy–Riemann equations.

Denote $u := \operatorname{Re} f$, $v := \operatorname{Im} f$, $u, v\colon U \to \boldsymbol{R}$. By the assumptions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{a.e.} \tag{3.2}$$

The functions u and v are locally Lipschitz; hence it follows from the preceding lemma that

$$\Delta[u] = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}[u]\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}[u]\right) = \frac{\partial}{\partial x}\left(\left[\frac{\partial u}{\partial x}\right]\right) + \frac{\partial}{\partial y}\left(\left[\frac{\partial u}{\partial y}\right]\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial y}\left[-\frac{\partial v}{\partial x}\right] = \frac{\partial}{\partial x}\frac{\partial}{\partial y}[v] - \frac{\partial}{\partial y}\frac{\partial}{\partial x}[v] = 0.$$

The function u fulfils the conditions of Weyl's lemma (Herve [2], Corollary p. 3), i.e. it is locally L^1 and its distributional Laplacian vanishes; hence by this lemma u is equal a.e. to a smooth harmonic function. As u is continuous, it is smooth and harmonic itself. In a similar way v has these properties, and so, by the Cauchy–Riemann equations (3.2), $f = u+iv$ is holomorphic. ∎

For the proof of Lemma 2.4 we also need a property of zero sets formulated in the following lemma (for convenience we write these zero sets as a difference $U\setminus W$):

3.3. LEMMA. *Let $W \subset U \subset X$ where X is a complex, separable Fréchet space and let $U\setminus W$ be a zero set.*

Then for every b in X there is a dense subset of U such that for its every element a the set

$$\{z \in C:\ a+zb \in U\setminus W\} \tag{3.3}$$

has plane Lebesgue measure zero.

Proof. Let b be an arbitrary point of X and u an arbitrary point of U. It is sufficient to show that u lies in the closure of the set of those a's which fulfil condition (3.3).

Let $(c_i)_{i=1}^{\infty}$ be a bounded sequence which is linearly independent and linearly dense in X with respect to real scalars and satisfies

$$c_1 = b, \quad c_2 = ib.$$

Put

$$j_K\big((x_i)_{i=1}^{\infty}\big) = u+Kx_1c_1+Kx_2c_2+\sum_{i=3}^{\infty} x_ic_i$$

for every $(x_i)_{i=1}^{\infty} \in Q$ and $K > 0$.

The mapping j_K is well defined on the Hilbert cube Q, affine and continuous, and its image is linearly dense in X (with respect to real scalars).

Hence $\mu\big(j_K^{-1}(U\setminus W)\big) = 0$ (where μ is the natural product measure on Q) because $U\setminus W$ is a zero set (cf. Def. 2.2).

Let $q = (x_i)_{i=1}^{\infty} \in Q$. Denote $q' = (x_3, x_4, \ldots)$ and write $q = (x_1, x_2, q')$. Similarly, we write $Q = [-\frac{1}{2}, \frac{1}{2}]\times[-\frac{1}{4}, \frac{1}{4}]\times Q'$ and we denote by μ' the natural product measure on Q'.

Notice that, for $q' \in Q'$, $j_K\big((0, 0, q')\big)$ does not depend on K, and so we write $j(q')$ instead of $j_K(0, 0, q')$. By the Fubini theorem

$$0 = \mu\big(j_K^{-1}(U\setminus W)\big) = \int_{Q'} (dq')\,|\{(x_1, x_2) \in [-\tfrac{1}{2}, \tfrac{1}{2}]\times[-\tfrac{1}{4}, \tfrac{1}{4}]:\ u+Kx_1b+Kx_2ib+j(q') \in U\setminus W\}|.$$

Hence the set

$$A_K = \{q':\ |\{(x_1, x_2) \in [-\tfrac{1}{2}, \tfrac{1}{2}]\times[-\tfrac{1}{4}, \tfrac{1}{4}]:\ u+Kx_1b+Kx_2ib+j(q') \in U\setminus W\}| = 0\}$$

has measure 1 in Q'. Put $A = \bigcap_{K=1}^{\infty} A_K$. Observe that for every q' in A

$$\{z \in C:\ u+zb+j(q') \in U\setminus W\} \text{ has plane Lebesgue measure zero.} \tag{3.4}$$

On the other hand, the set A is of full measure in Q'; hence there is a sequence $q'_n \to 0$, and the sequence $a_n := u+j(q'_n)$ converges to u. It follows from (3.4) that a_n's have property (3.3). ∎

Proof of Lemma 2.4. Let W denote the set of those $u \in U$ for which $(\mathrm{D}f)_u$ exists and is C-linear. We apply Lemma 3.3 to the set W and the set U (by the assumption of Lemma 2.4 $U\setminus W$ is a zero set). Let a, b satisfy condition (3.3). It means that the function $f_{a,b}$: $\{z:\ a+bz \in U\} \to C$, where $f_{a,b}(z) := f(a+zb)$, has a C-linear differential a.e. In addition, $f_{a,b}$ is locally Lipschitz (because f is such); hence by Lemma 3.2 it is holomorphic. In order to complete the proof it is sufficient to show that for every $a \in U$, $b \in X$, there exists an $\varepsilon > 0$ such that $f_{a,b}$ is holomorphic on $\{z:\ |z| < \varepsilon\}$. By the preceding lemma there exists a sequence $a_n \to a$ such that a_n, b satisfy condition (3.3) and, by the above consideration, $f_{a_n,b}$ is homolorphic on $\{z:\ a_n+zb \in U\}$.

Since f is locally Lipschitz, it is bounded on a neighbourhood of a; hence there exist an $\varepsilon > 0$ and an integer N such that

$$a_n+zb \in U \quad \text{for} \quad |z| < \varepsilon,\ n > N$$

and $f_{a_n,b}$ are uniformly bounded on $|z| < \varepsilon$.

As $f_{a_n,b}$ converges pointwise to $f_{a,b}$, the latter is holomorphic. ∎

4. An application. As we have mentioned in the introduction, the main result of the paper may be applied to the strengthening of the spectral mapping theorem of [6] (Thm 3.3). For the convenience of the reader we recall this theorem here, but we formulate it in the newer terms of [8]. First we recall some necessary definitions. They are all taken from [8].

Let A be a complex unital Banach algebra. The family of all non-void subsets of A consisting of pairwise commuting elements will be designated by $c(A)$. The elements of $c(A)$ will be denoted by $x_{\mathfrak{A}}$, $x_{\mathfrak{B}}$ etc. ($x_{\mathfrak{A}} = \{x_a\}_{a\in\mathfrak{A}}$).

Suppose that to each family $x_{\mathfrak{A}} \in c(A)$ there corresponds a non-void compact subset of $C^{\mathfrak{A}}$ ($C^{\mathfrak{A}} = \prod_{a\in\mathfrak{A}} C_a$, where $C_a = C$)

$$x_{\mathfrak{A}} \to \tilde{\sigma}(x_{\mathfrak{A}}) \subset C^{\mathfrak{A}}.$$

We recall some axioms of [8] for the map $\tilde{\sigma}$,

$$\tilde{\sigma}(x_{\mathfrak{A}}) \subset \prod_{a\in\mathfrak{A}} \sigma(x_a) \tag{I}$$

where $x_{\mathfrak{A}} = \{x_a\}_{a\in\mathfrak{A}} \in c(A)$, and $\sigma(x_a)$ is the usual spectrum of an element $x_a \in A$.

(III) The spectral mapping property of $\tilde{\sigma}$:

$$\tilde{\sigma}\big(p_{\mathfrak{B}}(x_{\mathfrak{A}})\big) = p_{\mathfrak{B}}\tilde{\sigma}(x_{\mathfrak{A}}),$$

here $x_{\mathfrak{A}} \in c(A)$ and $p_{\mathfrak{B}}$ is a system of complex polynomials in indeterminate t_a, or a polynomial map.

DEFINITION. A map $\tilde{\sigma}$ is called a *subspectrum on A* if axioms (I) and (III) are satisfied.

The following two axioms are consequences of (III).

Suppose first that $\mathfrak{B} \subset \mathfrak{A}$ and put $p_{\mathfrak{B}}(t_a) = t_\beta$ for all $\beta \in \mathfrak{B}$. Axiom (III) then implies

(IV) $$\tilde{\sigma}(x_{\mathfrak{B}}) = \pi\tilde{\sigma}(x_{\mathfrak{A}}),$$

where $x_{\mathfrak{A}} \in c(A)$, $\mathfrak{B}$ is a non-void subset of $\mathfrak{A}$, and π is the projection of $C^{\mathfrak{A}}$ onto $C^{\mathfrak{B}}$ given by $\pi(z_{\mathfrak{A}}) = z_{\mathfrak{B}}$ $\big(z_{\mathfrak{A}} = (z_a)_{a\in\mathfrak{A}} \in C^{\mathfrak{A}}\big)$.

Put $\mathfrak{B} := \mathfrak{A}$ and $p_a(t_a) = t_a + \lambda_a$, where $\lambda_a \in C$. Then

(V) $$\tilde{\sigma}(x_{\mathfrak{A}} + \lambda_{\mathfrak{A}}e) = \tilde{\sigma}(x_{\mathfrak{A}}) + \lambda_{\mathfrak{A}},$$

where $x_{\mathfrak{A}} \in c(A)$, $\lambda_{\mathfrak{A}} = (\lambda_a)_{a\in\mathfrak{A}} \in C^{\mathfrak{A}}$,

$$x_{\mathfrak{A}} + \lambda_{\mathfrak{A}}e = \{x_a + \lambda_a e\}_{a\in\mathfrak{A}},$$

e denotes the unit of A.

DEFINITION. A *spectroid* on A is a map $\tilde{\sigma}$ satisfying axioms (I), (IV) and (V).

4.1. THEOREM. *Let A be a complex Banach algebra with unit element e. Suppose that $\tilde{\sigma}$ is a spectroid on A. Then the following conditions are equivalent*

(ii) *For any three mutually commuting elements $x_1, x_2, x_3 \in A$,*

$$\tilde{\sigma}(x_1, x_2, x_3) \subset \sigma_{[x_1,x_2,x_3]}(x_1, x_2, x_3),$$

where $[x_1, x_2, x_3]$ is the smallest unital Banach subalgebra of A containing the elements x_1, x_2, x_3 and $\sigma_{[x_1,x_2,x_3]}(x_{\mathfrak{A}})$ denotes the usual joint spectrum of $x_{\mathfrak{A}} \in c([x_1, x_2, x_3])$.

(iv) *$\tilde{\sigma}$ is a subspectrum on A.*

It is natural (though not very important) to ask whether condition (ii) might be replaced by the following weaker one:

(ii)' For any two commuting elements $y_1, y_2 \in A$,

$$\tilde{\sigma}(y_1, y_2) \subset \sigma_{[y_1,y_2]}(y_1, y_2).$$

The answer to this question is "yes". We will show it by using the main theorem.

Remark. From the proof of Theorem 4.1 (cf. [6], § 3) it easily follows that each of the conditions (ii) and (iv) is equivalent to the following one:

(*) For each family $x \in c(A)$,

$$\tilde{\sigma}(x_{\mathfrak{A}}) \subset \sigma_{[x_{\mathfrak{A}}]}(x_{\mathfrak{A}}).$$

Proof. Only the implication (ii)' ⇒ (ii) needs proof. Let $\tilde{\sigma}$ satisfy (ii)' and let $(c_1, c_2, c_3) \in \tilde{\sigma}(x_1, x_2, x_3)$. Since $\tilde{\sigma}$ has the projection property (IV), there exists a map

$$c\colon A_0 \to C, \quad \text{where } A_0 = [x_1, x_2, x_3]$$

such that

$$c \in \tilde{\sigma}(A_0) \quad \big(A_0 \in c(A)\big)$$

$$c(x_i) = c_i, \quad i = 1, 2, 3.$$

Let $y_1, y_2 \in A_0$. Then

$$\big(c(y_1), c(y_2)\big) \in \tilde{\sigma}(y_1, y_2) \subset \sigma_{[y_1,y_2]}(y_1, y_2).$$

From this it follows that there exists a multiplicative linear functional φ in $[y_1, y_2]$ such that

$$c(y_i) = \varphi(y_i), \quad i = 1, 2.$$

Thus

(4.1) $$c(y_i) - c(y_2) = \varphi(y_1 - y_2) \in \sigma(y_1 - y_2) \quad \text{for any } y_1, y_2 \in A_0.$$

In addition

$$\big(c(0), c(0)\big) \in \sigma_{[0,0]}(0, 0) = \sigma_{Ce}(0, 0) = (0, 0),$$

which means that

(4.2) $$c(0) = 0.$$

Conditions (4.1) and (4.2) imply, by Theorem 1.2, that c is a multiplicative linear functional in A_0 and thus $(c_1, c_2, c_3) \in \sigma_{A_0}(x_1, x_2, x_3)$. ■

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