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s-numbers, eigenvalues and the trace theorem in Banach spaces

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Abstract. Any continuous linear operator T in a complex Banach space X with summable approximation numbers is of trace class. We prove that the trace formula

$$\operatorname{tr}(T) = \sum_i \lambda_i(T)$$

holds for any such $T \in S_1^a(X)$. Here $\lambda_i(T)$ denote the eigenvalues of T which are absolutely summable. This answers affirmatively a problem of A. S. Markus and V. I. Macaev and generalizes Lidskij's theorem in Hilbert spaces. We show further that the eigenvalues of T belong to a Lorentz sequence space $l_{p,q}$, if the approximation numbers of T do so. In general, a similar result is not true for other types of sequence spaces and also false for other s-numbers (in the sense of A. Pietsch) the approximation numbers. The Lorentz space result contains for $q = \infty$ $[q = p]$ a generalization of the weak [strong] form of Weyl's inequality in Hilbert spaces to Banach spaces, which yields results on the eigenvalue distribution of operators in L_r. We show next that for the special class of $\mathscr{H}$-operators more can be said: their s-numbers have the same asymptotic order as their eigenvalues, provided the s-numbers are larger than the Bernstein numbers. Moreover, some questions related to the isomorphism numbers are considered.

1. Introduction. A Pietsch introduced in [13] the concept of s-numbers $s_n(T)$ which are associated to any continuous linear operator T between Banach spaces X and Y, $T \in \mathscr{L}(X, Y)$. Examples are the approximation numbers

$$a_n(T) = \inf\{\|T - T_n\|:\ T_n \in \mathscr{L}(X, Y),\ \operatorname{rank} T_n < n\}, \qquad n \in \boldsymbol{N};$$

the Gelfand numbers

$$\gamma_n(T) = \inf\{\|T\|_Z:\ Z \subset X,\ \operatorname{codim} Z < n\};$$

the Kolmogorov numbers

$$\delta_n(T) = \inf\bigl(\sup\{\inf(\|Tx - y\|:\ y \in Z \subset Y);\ \|x\| = 1\};\ \dim Z < n\bigr)$$

and the isomorphism numbers

$$i_n(T) = \sup\{\|R\|^{-1}\|S\|^{-1}:\ \dim Z \geqslant n,\ R \in \mathscr{L}(Z, X),\ S \in \mathscr{L}(Y, Z) \text{ with } STR = \mathrm{Id}_Z\}.$$

In Hilbert spaces, all s-number sequences coincide with the singular numbers of the operator. In Banach spaces, the approximation numbers are the largest, the isomorphism numbers the smallest s-numbers. A s-number sequence is *injective*, if for each isometric imbedding $I\colon Y\to L_\infty(\mu)$ one has $s_n(T) = s_n(IT)$. s_n is *additive*, if for all $S, T \in \mathscr{L}(X, Y)$ and $m, n \in \boldsymbol{N}$

$$s_{m+n-1}(S+T) \leqslant s_m(S)+s_n(T).$$

The previously defined sequences are additive with the exception of the isomorphism numbers. The Gelfand numbers are injective.

Let $0<p<\infty$, $0<r\leqslant\infty$ or $p=r=\infty$ and s_n be a s-number sequence. Define for Banach spaces X and Y the classes

$$S^s_{p,r}(X, Y) = \{T\in\mathscr{L}(X, Y)\colon (s_n(T))\in l_{p,r}\}.$$

Here $l_{p,r}$ is the Lorentz sequence space of all $(x_n)\in l_\infty$ for which

$$\|(x_n)\|_{p,r} = \begin{cases} \Big(\sum\limits_{n\in\mathbf{N}} x_n^{*r} n^{r/p-1}\Big)^{1/r}, & r<\infty,\\ \sup\{x_n^* n^{1/p}\colon n\in\boldsymbol{N}\}, & r=\infty,\end{cases}$$

x_n^* being the monotone non-increasing non-negative rearrangement of (x_n). Let $\sigma^s_p(T) = \|(x_n)\|_{p,r}$. If s_n is additive, $(S^s_{p,r}, \sigma^s_{p,r})$ is a complete quasi-normed operator ideal. The special case $p=r$, denoted by (S^s_p, σ^s_p), extends the classes $S_p(H)$ in Hilbert spaces H.

We denote the nuclear operators from X to Y by $\mathscr{N}_1(X, Y)$. Finite-dimensional l_p-spaces are written l_p^n, l_p-direct sums of Banach spaces (X_n) denoted by $l_p(X_n)$.

Given two sequences of non-negative numbers, (α_n) and (β_n), we write $\alpha_n\sim\beta_n$, if there are constants c_1, c_2 such that

$$c_1\beta_n\leqslant\alpha_n\leqslant c_2\beta_n\quad\text{for all } n\in\boldsymbol{N}.$$

We will also use the Landau symbols $O(\cdot)$ and $o(\cdot)$.

Since we are mainly interested in the trace formula and relations between s-numbers and eigenvalues, we shall always assume all Banach spaces to be complex. The eigenvalues of a compact operator $T\in\mathscr{K}(X)$ are denoted by $(\lambda_n(T))_{n\in\mathbf{N}}$ and assumed to be ordered in non-increasing absolute value and counted according to their multiplicity.

In the next section, we consider questions of the type: If the s-numbers of a compact operator $T\in\mathscr{K}(X)$ belong to a sequence space like e.g. $l_{p,r}$, what can be said of its eigenvalues? These are generalizations of Weyl's inequality in Hilbert spaces. Section 3 contains the trace theorem for S^a_1-type operators, showing that the spectral trace coincides

with the matrix trace. The final section contains questions related to inequalities between single eigenvalues and single s-numbers and some examples concerning the isomorphism numbers.

2. Generalized Weyl inequalities. The singular numbers $s_n(T)$ of a compact operator T in a Hilbert space H are the eigenvalues of $(T^*T)^{1/2}$. H. Weyl [16] proved the following inequality between the eigenvalues of T and $(T^*T)^{1/2}$:

$$\sum_{j=1}^n|\lambda_j(T)|^p\leqslant\sum_{j=1}^n s_j(T)^p$$

for any compact T, any $0<p<\infty$ and any $n\in\boldsymbol{N}$. In [7] we derive the following generalization of Weyl's inequality to Banach spaces which we will need in the following:

Proposition. *Let s_n denote either the approximation- or the Gelfand- or the Kolmogorov-numbers. For any $0<p<\infty$ there is $c_p>0$ such that for any Banach space X, any compact operator $T\in\mathscr{K}(X)$ and any $n\in\boldsymbol{N}$*

$$\Big(\sum_{j=1}^n|\lambda_i(T)|^p\Big)^{1/p}\leqslant c_p\Big(\sum_{j=1}^n s_j(T)^p\Big)^{1/p}.$$

We extend this result to Lorentz sequence spaces.

Theorem 1. *Let $0<p<\infty$, $0<q\leqslant\infty$ and s_n be as in the proposition. Then there is a constant $c_{p,q}$ such that for all Banach spaces X and any operator $T\in S^s_{p,q}(X)$ we have*

$$\|(\lambda_n(T))\|_{p,q}\leqslant c_{p,q}\|(s_n(T))\|_{p,q} = c_{p,q}\cdot\sigma^s_{p,q}(T).$$

The main part of the proof is contained in

Lemma 1. *Let $0<r<p<\infty$ and $0<q\leqslant\infty$. Then there is a constant $c=c(p/r, q/r)$ depending only on p/r and q/r, such that for all monotone non-increasing sequences (a_n) of positive numbers*

$$\|(a_n)\|_{p,q}\leqslant\Big(\Big\|\Big(\Big(\sum_{j=1}^n a_j^r\Big)/n\Big)_n\Big\|_{p/r,q/r}\Big)^{1/r}\leqslant c\|(a_n)\|_{p,q}.$$

Proof. The "average" operator

$$A\colon(\xi_n)_n\mapsto\Big(\Big(\sum_{j=1}^n\xi_j\Big)/n\Big)_n$$

has norm one when considered as a map $A\colon l_\infty\to l_\infty$. But A also is an operator from l_1 into $l_{1,\infty}$, again with norm one. To see this, note that $0<x_n$

$\leqslant y_n$ implies $x_n^* \leqslant y_n^*$ and

$$x_n := \Big|\sum_{j=1}^{n} \xi_j/n\Big| \leqslant y_n := \sum_{j=1}^{n} \xi_j^*/n, \qquad x_n^* \leqslant y_n^* = y_n.$$

Hence

$$\|A(\xi_n)_n\|_{1,\infty} = \sup_{n\in N} nx_n^* \leqslant \sup_{n\in N} ny_n^* \leqslant \sum_{n\in N} \xi_n^* = \|(\xi_n)_n\|_1.$$

By the (generalized) Marcinkiewicz interpolation theorem, A also induces a continuous linear map $A\colon l_{p/r,q/r} \to l_{p/r,q/r}$, see [1]. Note here $r<p$. Writing this as an inequality for sequences, with $\|A\| = c(p/r, q/r)$, and applying it to $\xi_n = a_n^r$, we get the right inequality in Lemma 1. The left one is trivial.

Proof of Theorem 1. Choose r with $0<r<p$. The proposition implies in view of the ordering of the eigenvalues for any $T \in S_{p,q}^s(X)$

$$|\lambda_n(T)|\, n^{1/r} \leqslant \Big(\sum_{j=1}^{n} |\lambda_j(T)|^r\Big)^{1/r} \leqslant c_r \Big(\sum_{j=1}^{n} s_j(T)^r\Big)^{1/r},$$

(2.1) $$|\lambda_n(T)| \leqslant c_r \Big(\Big(\sum_{j=1}^{n} s_j(T)^r\Big)/n\Big)^{1/r}$$

so that by Lemma 1

$$\begin{aligned}\|(\lambda_n(T))_n\|_{p,q} &\leqslant c_r \Big\| \Big(\Big(\sum_{j=1}^{n} s_j(T)^r\Big)/n\Big)_n^{1/r}\Big\|_{p,q}\\ &\leqslant c_r \Big(\Big\|\Big(\Big(\sum_{j=1}^{n} s_j(T)^r\Big)/n\Big)_n\Big\|_{p/r,q/r}\Big)^{1/r}\\ &\leqslant c_{p,q,r} \|(s_n(T))_n\|_{p,q} = c_{p,q,r}\cdot \sigma_{p,q}^s(T).\end{aligned}$$

For $p=q$, Lemma 1 is just Hardy's inequality and the statement of Theorem 1 is just the proposition. A case of special interest is also $q=\infty$:

Corollary 1. *Let s_n be as before, $0<a, p<\infty$ and $T \in \mathscr{K}(X)$.*

(a) *If $s_n(T) = O(n^{-a})$, also $|\lambda_n(T)| = O(n^{-a})$.*

(b) *If $s_n(T) = o(n^{-a})$, also $|\lambda_n(T)| = o(n^{-a})$.*

(c) *If $s_n(T) = O\big(n^{-1/p}(\ln n)^a\big)$, also $|\lambda_n(T)| = O\big(n^{-1/p}(\ln n)^a\big)$.*

(d) *The eigenvalues $\lambda_n(T)$ are rapidly decreasing if the $s_n(T)$ are.*

Proof. (a) is the case $q=\infty$ of Theorem 1. Choose $r<1/a$. A direct $\varepsilon-\delta$-argument together with (2.1) yields (b); (2.1) for $r<p$ also implies (c) if one uses that for $\lambda<1$, $\mu>0$

$$\sum_{j=1}^{n} j^{-\lambda}(\ln j)^\mu \sim n^{1-\lambda}(\ln n)^\mu.$$

Statement (d) follows from Weyl's inequality, applied to all $p>0$.

Example 1. There are Banach sequence spaces for which results analogous to those of Theorem 1 and Corollary 1 are false. Let $R>1$ and $l(R) = \{(\xi_n) \in l_\infty\colon \sup\{\xi_n^* R^n\colon n \in N\} < \infty\}$. Then there are operators $T \in \mathscr{K}(l_1)$ such that the approximation numbers $a_j(T)$ belong to $l(R)$ whereas the eigenvalues $\lambda_j(T)$ do not. For simplicity, let $R=2$. Denote by A_{2^n} the Littlewood-matrices of order 2^n,

$$A_{2^0} = (1), \qquad A_{2^{n+1}} = \begin{bmatrix} A_{2^n} & A_{2^n}\\ A_{2^n} & -A_{2^n}\end{bmatrix}, \qquad n \geqslant 0,$$

and define $T\colon l_1 \to l_1$ as a direct sum of multiples of Littlewood matrices:

$$T = \bigoplus_{n=0}^{\infty} 2^{-2^{n+1}} A_{2^n}\colon l_1(l_1^{2^n}) \to l_1(l_1^{2^n}).$$

Since $A_{2^n}^2 = 2^n\,\mathrm{Id}$, the eigenvalues of A_{2^n} are $\{\pm 2^{n/2}\}$, both with multiplicity 2^{n-1}. Hence

$$|\lambda_{2^n+i}(T)| = 2^{n/2}/2^{2^{n+1}} \quad \text{for} \quad 0 \leqslant i \leqslant 2^n-1,\ n \in N.$$

Therefore

$$\sup_{n,i} |\lambda_{2^n+i}(T)|\, 2^{2^n+i} = \sup_{n,i} 2^{n/2-1} = \infty, \qquad (\lambda_j(T))_j \notin l(2).$$

We show that nevertheless $(a_j(T))_j \in l(2)$. Let $T_{2^n+i}\colon l_1 \to l_1$ be the operator defined by the same first (2^n-1+i) rows as T and otherwise zero. Since $\operatorname{rank} T_{2^n+i} < 2^n+i$, we conclude

$$a_{2^n+i}(T) \leqslant \|T - T_n\|_1 = (2^n - i)/2^{2^{n+1}}.$$

Note here that the matrix elements in $T-T_n$ are either zero or $\pm 1/2^{2^{j+1}}$. This implies

$$\sup_{n,i} a_{2^n+i}(T)\, 2^{2^n+i} \leqslant \sup_{n,i} (2^n - i)\, 2^{2^n+i}/2^{2^{n+1}}.$$

It is easy to see that the last supremum is attained for $i = 2^n-1$, therefore

$$\sup_{n,i} a_{2^n+i}(T)\, 2^{2^n+i} \leqslant 1/2, \qquad (a_j(T))_j \in l(2).$$

However, a weaker positive result is available:

Lemma 2. *Let $R>1$ and s_n as above. Assume $T \in \mathscr{L}(X)$ is compact with $(s_n(T)) \in l(R)$. Then for any $1<S<\sqrt{R}$, $(\lambda_n(T)) \in l(S)$.*

Proof. By a perturbation argument, we may assume all eigenvalues to have multiplicity one. Let X_n be the space spanned by the first n eigenvectors associated to $\lambda_1(T), \ldots, \lambda_n(T)$. Since $\dim X_n = n$, there are operators $A_n\colon X_n \to l_2^n$ with $\|A_n\|\,\|A_n^{-1}\| \leqslant \sqrt{n}$, cf. F. John [6]. Let $S_n = A_n T A_n^{-1}\colon l_2^n \to l_2^n$. Then $\lambda_j(S_n) = \lambda_j(T)$ for $j = 1, \ldots, n$. By H. Weyl [16], for the Hilbert space operator S_n,

$$|\lambda_1(S_n) \ldots \lambda_n(S_n)| \leqslant s_1(S_n) \ldots s_n(S_n).$$

Hence

$$|\lambda_n(T)| = |\lambda_n(S_n)| \leqslant |\lambda_1(S_n) \ldots \lambda_n(S_n)|^{1/n} \leqslant \big(s_1(S_n) \ldots s_n(S_n)\big)^{1/n} \leqslant \|A_n\| \, \|A_n^{-1}\| \big(s_1(T) \ldots s_n(T)\big)^{1/n}.$$

Since $(s_n(T)) \in l(R)$, for some constant $c > 0$

$$|\lambda_n(T)| = c\sqrt{n}\Big(\prod_{i=1}^{n} R^{-i}\Big)^{1/n} = c\sqrt{n/R}\,\sqrt{R}^{-n}.$$

This implies for any $S < \sqrt{R}$

$$|\lambda_n(T)| \leqslant dS^{-n}, \qquad (\lambda_n(T)) \in l(S).$$

We now give some applications of the previous results to operators in L_p-spaces.

PROPOSITION 1. *Let $\Omega \subset \boldsymbol{R}^N$ be a bounded domain and $T: L_p(\Omega) \to L_p(\Omega)$ an operator with image contained in a Sobolev space $W_p^k(\Omega)$. Here $k \in \boldsymbol{R}^+$ and $1 \leqslant p \leqslant \infty$. Then $|\lambda_n(T)| = O(n^{-k/N})$. If the image of the unit ball of $L_p(\Omega)$ is relatively compact in $W_p^k(\Omega)$, one even has $|\lambda_n(T)| = o(n^{-k/N})$.*

Proof. The map $T: L_p(\Omega) \to W_p^k(\Omega)$ has a closed graph and hence is continuous with respect to the Sobolev norm. Hence

$$a_{2n-1}(T: L_p \to L_p) \leqslant a_n(T: L_p \to W_p^k) \cdot a_n(\mathrm{Id}: W_p^k \to L_p).$$

But the approximation numbers $a_n(\mathrm{Id})$ of the Sobolev imbedding tend to zero of order $O(n^{-k/N})$, see e.g. R. S. Ismagilov [5], which gives $a_n(T) = O(n^{-k/N})$ and $|\lambda_n(T)| = O(n^{-k/N})$ by Corollary 1. If $T(\text{ball } L_p)$ is relatively compact in W_p^k, $a_n(T: L_p \to W_p^k)$ tends to zero for $n \to \infty$, hence $a_n(T) = o(n^{-k/N})$. Apply again Corollary 1.

PROPOSITION 2. *Let $K: [0,1]^2 \to \boldsymbol{C}$ be a measurable kernel, which is $(n-1)$ times continuously differentiable with respect to the second variable y and absolutely continuous $(n-1)$-st derivative. Assume*

$$K_j f(x) = \int_0^1 \Big(\frac{\partial}{\partial y}\Big)^j K(x, y) f(y)\, dy$$

defines continuous linear operators $K_j: L_p \to L_p$ for $j = 0, \ldots, n$. Let further $0 < q \leqslant \infty$ and $K_j \in S_q^a(L_p)$, for $q = \infty$ no extra condition. Then the eigenvalues of $K = K_0$ in L_p obey

$$\Big(\sum_{k \in \boldsymbol{N}} k^{nq} |\lambda_k(K)|^q\Big)^{1/q} < \infty,$$

which for $q < \infty$ means especially $|\lambda_k(K)| = o(k^{-(n+1/q)})$ and for $q = \infty$ should read $|\lambda_k(K)| = O(k^{-n})$.

This generalizes a result in I. Z. Gohberg–M.G. Krein [3], p. 120 for $p = q = 2$.

Proof. We indicate only the case $n = 1$. Partial integration yields

$$\int_0^1 K(x, y) f(y)\, dy = \Big[K(x, y) \int_0^t f(t)\, dt\Big]_{y=0}^1 - \int_0^1 \Big(\frac{\partial}{\partial y} K(x, y)\Big)\Big(\int_0^1 f(t)\, dt\Big)\, dy.$$

Hence $K_0 = L - K_1 J_1$, where $J_1: L_p \to L_p$ is the integration operator $J_1 f(x) = \int_0^x f(t)\, dt$ and $L: L_p \to L_p$ is the rank one operator $Lf = \big(\int_0^t f(t)\, dt\big) \times K(\cdot, 1)$. Since L does not influence the asymptotic order of the approximation numbers, we have

$$a_{2j-1}(K_0) \sim a_{2j-1}(K_1 J_1) \leqslant a_j(K_1) \cdot a_j(J_1) \leqslant c a_j(K_1)/j,$$

where $a_j(J_1) \leqslant c/j$ by the previous proposition. Since by assumption $K_1 \in S_q^a(L_p)$, we get

$$\Big(\sum_{j \in \boldsymbol{N}} \big(j a_{2j-1}(K_0)\big)^q\Big)^{1/q} < \infty,$$

i.e. $(a_j(K_0)) \in l_{r,q}$ with $1/r = 1 + 1/q$. The result now follows directly from Theorem 1.

3. The trace formula for S_1-type operators. V. B. Lidskij [10] proved for operators of trace class in Hilbert spaces $T = \mathscr{N}_1(H) = S_1(H)$ the trace formula

$$\operatorname{tr}(T) = \sum_{j \in \boldsymbol{N}}{}' \lambda_i(T). \tag{3.1}$$

A. Grothendieck [4] showed that the notion of "trace" can be defined for nuclear operators in Banach spaces with the approximation property. However, formula (3.1) no longer makes sense for these operators, since the eigenvalues of nuclear operators in Banach spaces are only square summable in general. On the other side, A.S. Markus and V. I. Macaev [12] proved (3.1) for any operator T of type $S_p^a(X)$ with $p < 1$ in general Banach spaces, without requiring the approximation property. We show in the following that (3.1) remains valid for any T in the larger class $S_1^a(X)$ and thus answer a problem of A. S. Markus and V. I. Macaev affirmatively.

Let X be a Banach space and $T \in \mathscr{F}(X)$ a finite rank operator

$$T = \sum_{i \in \boldsymbol{N}} x_i' \otimes y_i, \qquad x_i' \in X', \ y_i \in X. \tag{3.2}$$

Then $x_i'(y_i)$ is independent of the special representation of T and called the *trace of* T, denoted $\operatorname{tr}(T)$. One has $\operatorname{tr}(T) = \sum_{j\in\mathbf{N}} \lambda_j(T)$. Any such $T \in \mathscr{F}(X)$ has a representation (3.2) with

$$\sum_{i\in\mathbf{N}} \|x_i'\|\,\|y_i\| \leqslant 8\,\sigma_1^a(T),$$

cf. A. Pietsch [14]. Hence the linear functional trace can be uniquely and continuously extended to $\overline{(\mathscr{F}(X), \sigma_1^a)} = S_1^a(X)$, again denoted by tr.

LEMMA 3. *Let (λ_i) and (γ_i) be two sequences of positive, monotone non-increasing numbers with $\prod_{i=1}^{n} \lambda_i \leqslant \prod_{i=1}^{n} \gamma_i$ for all $n \in \mathbf{N}$. Then for all $r \in \mathbf{R}^+$ and any $n \in \mathbf{N}$*

$$\prod_{j=1}^{n} (1+r\lambda_j) \leqslant \prod_{j=1}^{n} (1+r\gamma_j).$$

This was shown by H. Weyl [16], cf. also [3].

THEOREM 2 (Trace formula). *Let X be a Banach space and $T \in S_1^a(X)$. Then the previously defined trace coincides with the sum of the eigenvalues,*

$$\operatorname{tr}(T) = \sum_{n\in\mathbf{N}} \lambda_n(T).$$

Proof. The sum exists by Theorem 1. We will use the fundamental idea of A. S. Markus and V. I. Macaev [12]. However, the central estimate of the characteristic polynomials defined below proceeds differently. First, there are operators $T_n \in \mathscr{F}(X)$ of finite rank with $\sigma_1^a(T-T_n) \to 0$, since $\mathscr{F}(X)$ is σ_1^a-dense. For these T_n, the characteristic polynomials

$$D_n(\lambda) = \prod_j \big(1-\lambda\lambda_j(T_n)\big) = \det(I-\lambda T_n)$$

are well-defined, since only finitely many eigenvalues $\lambda_j(T_n)$ are non-zero. Let $r<1$ and n_0 such that $\sigma_1^a(T-T_n) \leqslant 1$ for $n \geqslant n_0$. By (2.1) there is c_r such that for all $S \in \mathscr{K}(X)$

$$|\lambda_j(S)| \leqslant c_r\Big(\sum_{i=1}^{j} \alpha_i(S)^r/j\Big)^{1/r} =: \beta_j(S). \tag{3.3}$$

Lemma 1 implies for $r<p=q=1$ that the $(\beta_j(S))$ are summable for any $S \in S_1^a(X)$ with

$$\sum_{j\in\mathbf{N}} \beta_j(S) \leqslant d \sum_{j\in\mathbf{N}} \alpha_j(S), \tag{3.4}$$

d independent of S. Let $|\lambda| = r$. Weyl's Lemma 3 yields

$$|D_n(\lambda)| = \prod_j \big(1+r\,|\lambda_j(T_n)|\big) \leqslant \prod_j \big(1+r\beta_j(T_n)\big),$$

$$\ln|D_n(\lambda)| \leqslant \sum_j \ln\big(1+r\beta_j(T_n)\big)$$

$$\leqslant \sum_j r\beta_j(T_n) \leqslant dr \sum_j \alpha_j(T_n) \leqslant dr\big(\sigma_1^a(T)+1\big).$$

By Montel's theorem, this implies as in [12] that the sequence of polynomials D_n converges compactly to an entire function D. We will show now that D is a function of order one, i.e.

$$\max_{|\lambda|=r} \ln|D(\lambda)| = o(r) \quad \text{for} \quad r\to\infty.$$

Let $\varepsilon>0$. By (3.4) there are $r_0 \in \mathbf{R}^+$ and $n_0 \in \mathbf{N}$ such that for all $r \geqslant r_0$ and $n \geqslant n_0$

$$\sum_{j\in\mathbf{N}} \beta_j(T-T_n) < \varepsilon, \qquad \sum_{j>[\sqrt{r}/2]} \beta_j(T) < \varepsilon, \qquad \ln\big(1+c_r(\|T\|+1)r\big)/\sqrt{r} < \varepsilon. \tag{3.5}$$

Using that the approximation numbers are monotone non-increasing and additive, we get

$$\beta_{2j}(T_n) \leqslant c_r\Big(\sum_{k=1}^{2j} \alpha_k(T_n)^r/2j\Big)^{1/r} \leqslant c_r\Big(\sum_{i=1}^{j} \alpha_{2i-1}(T_n)^r/j\Big)^{1/r}$$

$$\leqslant c_r\Big(\Big(\sum_{i=1}^{j} \alpha_i(T)^r+\alpha_i(T-T_n)^r\Big)/j\Big)^{1/r} \leqslant d_r\big(\beta_j(T)+\beta_j(T-T_n)\big).$$

Hence, taking sums

$$\sum_{k>[\sqrt{r}]} \beta_k(T_n) \leqslant 2 \sum_{k>[\sqrt{r}/2]} \beta_{2k}(T_n) \leqslant 2d_r \sum_{k>[\sqrt{r}/2]} \big(\beta_k(T)+\beta_k(T-T_n)\big) \leqslant 4d_r\,\varepsilon.$$

This enables a better estimate for $|D_n(\lambda)|$, $|\lambda| = r$. Using (3.5) we get

$$(1/r)\ln|D_n(\lambda)| \leqslant (1/r)\ln\prod_{j=1}^{[\sqrt{r}]} \big(1+r\beta_j(T_n)\big) + (1/r)\ln\prod_{j>[\sqrt{r}]} \big(1+r\beta_j(T_n)\big)$$

$$\leqslant (1/r)\ln\big(1+rc_r(\|T\|+1)\big)^{[\sqrt{r}]} + (1/r)\sum_{j>[\sqrt{r}]} \ln\big(1+r\beta_j(T_n)\big)$$

$$\leqslant (1/\sqrt{r})\ln\big(1+rc_r(\|T\|+1)\big) + \sum_{j>[\sqrt{r}]} \beta_j(T_n) \leqslant (1+4d_r)\,\varepsilon.$$

Since this holds for all $n \geqslant n_0$ and $r \geqslant r_0$, we conclude for $r \geqslant r_0$

$$\ln|D(\lambda)| \leqslant (1+4d_r)\,\varepsilon r,$$

i.e. $D(\lambda)$ is an entire function of order one. The last part in the proof is now the same as in [12]: By a theorem of Hurwitz the zeros of $D(\lambda)$ coincide with the limits of the zeros of the polynomials $D_n(\lambda)$, i.e. with $\left(\lim_{n\to\infty}\lambda_j(T_n)\right)^{-1}$, which is equal to $\lambda_j(T)^{-1}$ for an appropriate order of the $\lambda_j(T_n)$. Since $\sum_{j\in\boldsymbol{N}}|\lambda_j(T)| < \infty$ by Weyl's generalized inequality, one gets for the order one function $D(\lambda)$

$$D(\lambda) = \prod_{j\in\boldsymbol{N}}\left(1-\lambda\lambda_j(T)\right)$$

and

$$-\sum_{j\in\boldsymbol{N}}\lambda_j(T) = D'(0)/D(0) = \lim_{n\to\infty} D_n'(0)/D_n(0) = \lim\left(-\operatorname{tr}(T_n)\right) = -\operatorname{tr}(T).$$

This proves Theorem 2.

COROLLARY 2. *Let $S, T \in S_1^a(X)$. Then we have the eigenvalue equality*

$$\sum_{j\in\boldsymbol{N}}\lambda_j(S+T) = \sum_{j\in\boldsymbol{N}}\lambda_j(S)+\sum_{j\in\boldsymbol{N}}\lambda_j(T).$$

PROPOSITION ([12]). *Assume the Banach space X has an unconditional basis $\{e_j\}$ with biorthogonal coefficient functionals $\{f_j\}$. Let $T\in\mathcal{N}_1(X)$ be a nuclear operator. Then $\sum_{j\in\boldsymbol{N}} f_j(Te_j)$ converges absolutely and the sum is equal to the trace of T.*

In view of $S_1^a(X)\subset\mathcal{N}_1(X)$, this result and Theorem 2 imply

COROLLARY 3. *Let X be a Banach space with an unconditional basis $\{e_j\}$ with biorthogonal coefficient functionals $\{f_j\}$. Assume T is in $S_1^a(X)$. Then the "matrix trace" is equal to the "spectral trace",*

$$\sum_{j\in\boldsymbol{N}} f_j(Te_j) = \sum_{j\in\boldsymbol{N}}\lambda_j(T).$$

4. s-numbers and eigenvalues. In Section 2 we considered estimates for sequence space norms of all eigenvalues of a compact operator against all s-numbers. Sometimes it may be more interesting to have an estimate for single eigenvalues against single s-numbers. In [9] it was shown that for any compact operator $T\in\mathcal{K}(X)$ in a Banach space X and $j\in\boldsymbol{N}$

$$|\lambda_j(T)| = \lim_{m\to\infty} s_j(T^m)^{1/m},$$

where s_j is an arbitrary s-number sequence. But as the Littlewood matrices show easily, an estimate of the type

$$|\lambda_j(T)| \leqslant c a_j(T),$$

c independent of $j\in\boldsymbol{N}$, is false in general. This is true, however, for a certain class of operators in Banach spaces which were studied by A. S. Markus [11]:

A compact operator in a Banach space $T\in\mathcal{K}(X)$ is called *$\mathcal{H}$-operator*, if the spectrum of T is real and if the resolvent of T fulfils

$$\|(\lambda I - T)^{-1}\| \leqslant c/|\operatorname{Im}\lambda|, \quad \lambda\in\boldsymbol{C}\sim\boldsymbol{R}.$$

Any selfadjoint operator in a Hilbert space is an $\mathcal{H}$-operator with $c=1$. For selfadjoint operators, $|\lambda_n(T)| = s_n(T)$. The following result generalizes this. For the definition of the Bernstein- and Mitjagin-numbers we refer to [13].

PROPOSITION 3. *Let s_n be a s-number sequence which is larger than or equal to the Bernstein- or Mitjagin-numbers. Then for any compact $\mathcal{H}$-operator T with constant c in a Banach space X*

$$(2c)^{-1}s_n(T) \leqslant |\lambda_n(T)| \leqslant 2(c+1)s_n(T). \tag{4.1}$$

Especially, $a_n(T)\leqslant 4c(c+1)s_n(T)$: all s-numbers considered have the same asymptotic order as the eigenvalues.

Proof. A. S. Markus [11] showed this for the Kolmogorov and approximation numbers. Hence the left inequality is clear in view of $s_n(T)\leqslant a_n(T)$; instead of $(2\sqrt{2}c)^{-1}$ as in [11], $(2c)^{-1}$ is sufficient. If $T\in\mathcal{K}(X)$ is a $\mathcal{H}$-operator with constant c, the same is true for $T'\in\mathcal{K}(X')$. By A. Pietsch [13] the Mitjagin and Bernstein-numbers are dual, $v_n(T) = u_n(T')$. Therefore it is enough to prove the right inequality for the Bernstein-numbers u_n. We will use similar ideas as in [11].

Assume without loss of generality $\lambda_n(T)\neq 0$ and let $k\in\boldsymbol{N}$ be minimal with $|\lambda_{n+k}(T)| < |\lambda_n(T)|$. Let P_n be the spectral projection of T relative to $\Lambda = \{\lambda_1(T), \ldots, \lambda_{n+k-1}(T)\}$. Then $\dim P_n(X)\geqslant n$ and the restriction $T_n\colon P_n(X)\to P_n(X)$ of T to $P_n(X)$ is an isomorphism. Choose $r>0$ with

$$|\lambda_n(T)|^{-1} < r < |\lambda_{n+k}(T)|^{-1}.$$

The spectrum of $T_n^{-1}\in\mathcal{L}(P_n(X))$ is equal to $\{\lambda_1(T)^{-1}, \ldots, \lambda_{n+k-1}(T)^{-1}\}$ and contained in the interior of the circle $\Gamma = \{\lambda\in\boldsymbol{C}\colon |\lambda| = r\}$. Hence the identity on $P_n(X)$ has a representation by the Dunford integral

$$I_n = (2\pi i)^{-1}\int_\Gamma (\lambda I_n - T_n^{-1})^{-1}\,d\lambda,$$

with $(\lambda I_n - T_n^{-1})^{-1}\in\mathcal{L}(P_n(X))$, cf. [2], chap. 7. Since

$$I_n = (2\pi i)^{-1}\int_\Gamma I_n/\lambda\,d\lambda,$$

we get by subtraction

$$0 = -T_n^{-1}/(2\pi i) \int_\Gamma (1/\lambda)(\lambda I_n - T_n^{-1})^{-1} d\lambda.$$

We multiply this by $r^2 T_n$ and take the sum with

$$T_n^{-1} = (2\pi i)^{-1} \int_\Gamma \lambda(\lambda I_n - T_n^{-1})^{-1} d\lambda,$$

which yields

$$T_n^{-1} = (2\pi i)^{-1} \int_\Gamma \frac{\lambda^2 - r^2}{\lambda} (\lambda I_n - T_n^{-1})^{-1} d\lambda. \tag{4.2}$$

The equality

$$(\lambda I_n - T_n^{-1})^{-1} = \lambda^{-1} I_n - \lambda^{-2}(\lambda^{-1} I_n - T_n)^{-1}$$

implies that T_n^{-1} is a $\mathscr{H}$-operator with constant $(c+1)$. Hence (4.2) enables the estimate

$$\|T_n^{-1}\| \leqslant (c+1) r \max_{|\lambda|=r} \left| \frac{\lambda^2 - r^2}{\lambda(\operatorname{Im}\lambda)} \right| = 2(c+1)r.$$

By [13], the isomorphism numbers are the smallest and the Bernstein numbers the smallest injective s-numbers. Factoring the identity on $P_n(X)$ as $I_n = T_n^{-1} T_n I_n$, we conclude by the definition of the isomorphism numbers

$$\begin{aligned}\big(2(c+1)r\big)^{-1} \leqslant \|T_n^{-1}\|^{-1} \leqslant i_n\big(T_n\colon P_n(X) \to P_n(X)\big) \leqslant u_n\big(T_n\colon P_n(X) \to P_n(X)\big) \\ \leqslant u_n(T\colon X \to X),\end{aligned}$$

using the injectivity of the u_n's. Since $r > |\lambda_n(T)|^{-1}$ was arbitrary, this shows $|\lambda_n(T)| \leqslant 2(c+1) u_n(T)$.

Weyl's inequality in Banach spaces

$$\sum_{j\in\boldsymbol{N}} |\lambda_j(T)|^p \leqslant c_p \sum_{j\in\boldsymbol{N}} s_j(T)^p,\ 0 < p < \infty$$

was formulated in Section 2 for the Gelfand, Kolmogorov or approximation numbers. We want to show now that in general this inequality does not hold for other s-number sequences. To do so, we need a fact on the isomorphism numbers which were considered in the previous proof.

Lemma 4. *For* $1 \leqslant n \leqslant m$, $(1/4)(m/n) \leqslant i_n(\mathrm{Id}\colon l_\infty^m \to l_1^m) \leqslant m/n$.

Proof. (a) Let $\dim X_n \geqslant n$ and take a factorization of the identity on X_n,

$$X_n \xrightarrow{P} l_\infty^m \xrightarrow{\mathrm{Id}} l_1^m \xrightarrow{Q} X_n.$$

Then $\|P\|^{-1}\|Q\|^{-1} \leqslant \|PQ\colon l_1^m \to l_\infty^m\|^{-1}$. The norm of PQ is the maximum of the absolute values of the elements in the matrix representation, hence larger than or equal to $1/m$ times the trace,

$$\|P\|^{-1}\|Q\|^{-1} \leqslant |m/\mathrm{tr}(\mathrm{Id}PQ)|.$$

$\mathrm{Id}PQ\colon l_1^m \to l_1^m$ is a projection of rank $\geqslant n$. Therefore

$$\mathrm{tr}(\mathrm{Id}PQ) = \sum_{i=1}^m \lambda_i(\mathrm{Id}PQ) \geqslant n.$$

Taking the supremum over all factorizations, we get

$$i_n(\mathrm{Id}\colon l_\infty^m \to l_1^m) \leqslant m/n.$$

(b) Let $m = 2^M$, $n = 2^N$, $M, N \in \boldsymbol{N}$. Let $\tilde{A}$ be the $(n\times m)$-matrix of the first n rows of the Littlewood matrices A_{2^M} of order $m = 2^M$ and $Q\colon l_1^m \to l_\infty^n$ the operator defined by $1/\sqrt{m}\tilde{A}$. Let $P\colon l_\infty^n \to l_\infty^m$ be defined by $1/\sqrt{m}\tilde{A}^t$. Then we have a factorization of the identity on l_∞^n as

$$l_\infty^n \xrightarrow{P} l_\infty^m \xrightarrow{\mathrm{Id}} l_1^m \xrightarrow{Q} l_\infty^n,$$

implying

$$i_n(\mathrm{Id}\colon l_\infty^m \to l_\infty^m) \geqslant \|P\|^{-1}\|Q\|^{-1} = m/n.$$

The left inequality of Lemma 4 is now an easy consequence.

Example 2. We prove that there is no constant c_p such that for all $T \in S_p^i(l_1)$ one has

$$\sum_{j\in\boldsymbol{N}} |\lambda_j(T)|^p \leqslant c_p \sum_{j\in\boldsymbol{N}} i_j(T)^p, \qquad p < 2. \tag{4.3}$$

Let $n = 2^N$ and $T_n\colon l_1^n \to l_1^n$ the operator defined by the Littlewood matrices A_{2^N} (or by imbedding, $T_n\colon l_1 \to l_1$). Both eigenvalues $\pm\sqrt{n}$ are of order $n/2$. Hence

$$\sum_{j=1}^n |\lambda_j(T_n)|^p = n^{1+p/2}. \tag{4.4}$$

On the other hand,

$$\sum_{j=1}^n i_j(T_n\colon l_1^n \to l_1^n)^p \leqslant \|T_n\colon l_1^n \to l_\infty^n\|^p \sum_{j=1}^n i_j(\mathrm{Id}\colon l_\infty^n \to l_1^n)^p.$$

But $\|T_n\colon l_1^n \to l_\infty^n\| = 1$. Hence by Lemma 4

$$\sum_{j=1}^n i_j(T_n)^p \leqslant n^p \sum_{j=1}^n j^{-p} \sim \begin{cases} n, & p < 1, \\ n\ln n, & p = 1, \\ n^p, & p > 1. \end{cases}$$

For large n, this and (4.4) contradict (4.3). We remark that it is not known whether the classes S_p^i are operator ideals.

Example 3. By A. Pietsch [13] one has for any invertible operator T in $\mathscr{L}(X, Y)$ with $\dim X = \dim Y = m$

$$i_n(T)\cdot a_{m-n+1}(T^{-1}) \leqslant 1. \tag{4.5}$$

In general, however, there may be a strict inequality "<", as will follow from Lemma 4. This answers a question in [13] negatively and seems a bit surprising, since for other corresponding pairs of s-numbers like Bernstein and Gelfand numbers there is an equality of type (4.5). Let $m = 2^M$, $n = m/2$. As is well known,

$$a_{n+1}(\mathrm{Id}\colon l_1^m \to l_\infty^m) \leqslant (\sqrt{n}+1)^{-1},$$

cf. B. S. Kasin [8]. This and Lemma 4 imply for $T = \mathrm{Id}\colon l_\infty^m \to l_1^m$ and $n = m/2 > 1$ that

$$i_n(T)\cdot a_{m-n+1}(T^{-1}) \leqslant 2/(\sqrt{n}+1) < 1.$$

This shows also that the estimate $u_n(T) \leqslant n^{1/2} i_n(T)$ in [13] is of optimal order, since

$$u_n(T) = \gamma_{n+1}(T^{-1})^{-1} = a_{n+1}(T^{-1})^{-1} \geqslant \sqrt{n} \geqslant \sqrt{n}/2\ \ i_n(T).$$

Any $T \in S_1^a(X, Y)$ is nuclear. On the other hand, one can ask whether the nuclear operators belong to some p-summability classes of s-numbers and whether $\sigma_p^s(T) \leqslant c\cdot \nu_1(T)$ holds. This is false in general for $p = 1$, as the identity $T = \mathrm{Id}\colon l_\infty^m \to l_1^m$ shows:

$$m\ln m \sim \sigma_1^i(T) \leqslant \sigma_1^s(T) \not\leqslant c\cdot \nu_1(T) = c\cdot m.$$

A weaker positive result is true for $p > 2$:

Proposition 4. *For all Banach spaces X and Y, $\mathscr{N}_1(X, Y) \subset S_{2,\infty}^i(X, Y)$; i.e. the isomorphism numbers i_n of nuclear operators are of order $n^{-1/2}$.*

Proof. The factorization diagram for nuclear operators [14] and the properties of the s-numbers imply that it is enough to consider diagonal operators $D_\sigma\colon l_\infty \to l_1$, $(x_n) \mapsto (\sigma_n x_n)$, $\sigma \in l_1$ and show

$$\sup_{n\in N} n^{1/2}\cdot i_n(D_\sigma) \leqslant \|\sigma\|_1 = \nu_1(D_\sigma).$$

We will assume without loss of generality that the σ_i are positive, non-increasing. Let $\mu_i = \sqrt{\sigma_i}$. Then $\mu \in l_2$ and

$$i_n(D_\sigma) \leqslant i_n(D_\mu\colon l_\infty \to l_2)\cdot \|D_\mu\colon l_2 \to l_1\| = \|\sigma\|_1^{1/2}\cdot i_n(D_\mu\colon l_\infty \to l_2).$$

Let D_μ^m be the "restriction" of D_μ to l_∞^m. Then

$$i_n(D_\mu) = \sup\{i_n(D_\mu^m)\colon m \in N\}$$

since

$$|i_n(D_\mu) - i_n(D_\mu^m)| \leqslant \|D_\mu - D_\mu^m\colon l_\infty \to l_2\| \to 0 \quad \text{for} \quad m \to \infty.$$

Using the relation between the isomorphism and approximation numbers considered in Example 3 and the results of M. Z. Solomjak and V. M. Tichomirow [15], one gets

$$i_n(D_\mu^m) \leqslant a_{m-n+1}\big((D_\mu^m)^{-1}\colon l_2^m \to l_\infty^m\big)^{-1}$$
$$= \min_{m-n+1\leqslant k\leqslant m} \sqrt{\Big(\sum_{i=m-k+1}^{m} \mu_i^2\Big)\Big/(k-m+n)} \leqslant (\|\sigma\|_1/n)^{1/2}.$$

This yields

$$i_n(D_\sigma\colon l_\infty \to l_1) \leqslant \|\sigma\|_1^{1/2}\cdot i_n(D_\mu) \leqslant \|\sigma\|_1/\sqrt{n}.$$

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ERRATA

Page, line	For	Read
112^{12}	$\hat{\otimes}$	$\check{\otimes}$
116_{15}	$(G\hat{\otimes}G)$, $(E\hat{\otimes}E)$	$(G\check{\otimes}G)$, $(F\check{\otimes}F)$
120^{24}	Let $\bar{N} = \{\mathrm{co}(N)$:	Let $\hat{N} = \{\mathrm{co}(N)$:
120^{25}	Then $\bar{N}$	Then $\hat{N}$
120_2	$(x_{n_j})_{n=1}^{\infty}$	$(x_{n_j})_{j=1}^{\infty}$
122^2	$\sum_{n=1}^{\infty} t_j y_{n_j}$	$\sum_{j=1}^{\infty} t_j y_{n_j}$
123^3	unit-vectors	unit-vectors basis
125^5	$\{\lvert f_i(z_n)\rvert =$	$\{\lvert f_i(z_n)\rvert$:
128^{10}	$\sup_{\lVert x\rVert<1}$	$\sup_{\lVert x\rVert_1<1}$
128_3	$z_j = \sum_{i=n_{j-1}+1}^{n_j} = a_i e_i$	$z_j = \sum_{i=n_{j-1}+1}^{n_j} a_i e_i$
141_2	$C = c\left(c(E(S))\cup D\right)$	$\tilde{C} = c\left(\tilde{c}(E(S))\cup D\right)$
141_1	$\tilde{c}(ES))$	$\tilde{c}(E(S))$
204^{13}	$(x-ty, \xi/t)(y)\}\rvert$	$(x-ty, \xi/t)\varphi(y)\}\rvert$
211_4	$a>0.$	$a>0.\ \Gamma_a^h(x_0) = \{(x,t)\colon \lvert x-x_0\rvert < at,\ t<h\}.$

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Differentiability of distributions at a single point

by

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Abstract. We develop tools needed to study the differentiability of distributions at a point of $\mathbf{R}^n$.

The purpose of this paper is to develop tools to study distributions at a point x_0. The integral expressions that we will be examining are closely related to the area integral of Lusin and the Littlewood–Paley g_λ^* functions. Rather than use the theory of harmonic functions we will consider analogous expressions that are independent of the mollifier that is used. The results of this paper are used to study tempered nontangential boundedness and convergence in [3].

Let φ be a Schwartz function with $\int \varphi(x)\,dx = 1$. Using this as a mollifier, form

$$u(x, t) = f*\varphi_t(x) \quad \text{where} \quad \varphi_t(x) = t^{-n}\varphi\left(\frac{x}{t}\right).$$

If f is a continuous function then $u(x, t)$ approaches $f(x)$ as $t\to 0$. It seems reasonable then that by examining $u(x, t)$ in the set $\Omega = \{(x, t)\colon x\in\mathbf{R}^n,\ 0<t<1\}$ we should be able to understand the behavior of f at a point, say x_0. We form a certain integral $\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0)$ of $u(x, t)$ over the set Ω. Of the various parameters involved the most important is $\gamma\in\mathbf{R}$. We will see that if $\mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0)<\infty$ then γ gives the order of differentiability of f at x_0.

If we add certain harmless terms to these $\mathscr{G}$ functions we can form norms $N_{\gamma,k}^{p,\lambda}$. Thus

$$N_{\gamma,k}^{p,\lambda}(f)(x_0) = \mathscr{G}_{\gamma,k}^{p,\lambda}(f)(x_0) + \text{“other terms”}.$$

In the second and third sections we show that both the $\mathscr{G}$ functions and the norm N are essentially independent of the mollifier φ. In addition, using a different k gives an equivalent norm.

With these norms we can define Banach spaces

$$\Lambda_\gamma^{p,\lambda}(x_0) = \{f\in\mathscr{S}'\colon N_{\gamma,k}^{p,\lambda}(f)(x_0)<\infty \text{ for some } k>\gamma+n/p\}.$$

Contrast these with the Sobolev spaces $L_k^p(\mathbf{R}^n)$ of functions that have