s-numbers, eigenvalues and the trace theorem in Banach spaces

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Abstract. Any continuous linear operator \( T \) in a complex Banach space \( X \) with summable approximation numbers is of trace class. We prove that the trace formula

\[
\text{tr}(T) = \sum \lambda_i(T)
\]

holds for any such \( T \in \mathcal{L}(X) \). Here \( \lambda_i(T) \) denote the eigenvalues of \( T \) which are absolutely summable. This answers affirmatively a problem of A. S. Markus and V. I. Macarv and generalizes Lidskii's theorem in Hilbert spaces. We show further that the eigenvalues of \( T \) belong to a Lorentz sequence space \( \ell_{p,q} \) if the approximation numbers of \( T \) do so. In general, a similar result is not true for other types of sequence spaces and also fails for other s-numbers (in the sense of A. Pietsch) the approximation numbers. The Lorentz space result contains for \( q = \infty \) (\( q > p \)) a generalization of the weak (strong) form of Weyl's inequality in Hilbert spaces to Banach spaces, which yields results on the eigenvalue distribution of operators in \( L_p \). We show next that for the special class of \( \mathcal{S} \)-operators more can be said: their s-numbers have the same asymptotic order as their eigenvalues, provided the s-numbers are larger than the Bernstein numbers. Moreover, some questions related to the isomorphism numbers are considered.

1. Introduction. A Pietsch introduced in [13] the concept of s-numbers \( s_n(T) \) which are associated to any continuous linear operator \( T \) between Banach spaces \( X \) and \( Y \), \( T \in \mathcal{L}(X, Y) \). Examples are the approximation numbers

\[
s_n(T) = \inf \{ \| T - T_n \| : T_n \in \mathcal{L}(X, Y), \text{rank} T_n < n \}, \quad n \in \mathbb{N};
\]

the Gelfand numbers

\[
y_n(T) = \inf \{ \| T \|_Z : Z \subset X, \text{codim} Z < n \};
\]

the Kolmogorov numbers

\[
\delta_n(T) = \inf \{ \sup \{ \inf \{ \| Tz - y \| : y \in Y \} : \| z \| = 1 \} : \text{dim} Z < n \}
\]

and the isomorphism numbers

\[
i_n(T) = \sup \{ \| T^* \|^{-1} \| S \|^{-1} : \text{dim} Z > n, E \in \mathcal{L}(Z, X), S \in \mathcal{L}(Y, Z) \}
\]

with \( T R = \text{Id}_Y \).
In Hilbert spaces, all \( s \)-number sequences coincide with the singular numbers of the operator. In Banach spaces, the approximation numbers are the largest, the isomorphism numbers the smallest \( s \)-numbers. A \( s \)-number sequence is \( * \)-injective, if for each isometric imbedding \( I : Y \to L_{n}(X) \) one has \( s_{n}(T) = s_{n}(I(T)) \). \( s_{n} \) is additive, if for all \( s_{x}, T \in S(X, Y) \) and \( m, n \in \mathbb{N} \)

\[
 s_{n+m}(T+s_{m}(T)) \leq s_{n}(s_{m}(T)) + s_{m}(T).
\]

The previously defined sequences are additive with the exception of the isomorphism numbers. The Gelland numbers are injective. Let \( 0 < p < \infty, 0 < r < \infty \) or \( p = r = \infty \) and \( s_{n} \) be a \( s \)-number sequence. Define for Banach spaces \( X \) and \( Y \)

\[
 S^{(s)}_{p,r}(X, Y) = \{ T \in \mathcal{L}(X, Y) : \| s_{n}(T) \| \in \ell_{p,r} \}.
\]

Here \( \ell_{p,r} \) is the Lorentz sequence space of all \( (\alpha_{n}) \in \ell_{p} \) for which

\[
\left( \sum_{n \in \mathbb{N}} \alpha_{n}^{r} n^{r(p-1)} \right)^{1/r} < \infty, \quad r < \infty,
\]

\(
\sup_{n \in \mathbb{N}} \alpha_{n}^{r} n^{p} < \infty, \quad r = \infty,
\)

\( a_{n}^{*} \) being the monotone non-increasing non-negative rearrangement of \( (\alpha_{n}) \). Let \( \mathcal{S}^{(s)}_{p,r}(X, Y) = \{ s_{n}(T) \} \). If \( s_{n} \) is additive, \( (S^{(s)}_{p,r}, \mathcal{S}^{(s)}_{p,r}) \) is a complete quasi-normed operator ideal. The special case \( p = r \), denoted by \( (S^{(s)}_{p}, \mathcal{S}^{(s)}_{p}) \), extends the classes \( S_{p}(H) \) in Hilbert spaces \( H \).

We denote the nuclear operators from \( X \) to \( Y \) by \( \mathcal{K}(X, Y) \). Finite-dimensional \( l_{2} \)-spaces are written \( l_{2}^{*} \), \( l_{2} \)-direct sums of Banach spaces \( (X_{n}) \) denoted by \( l_{2}(X_{n}) \).

Given two sequences of non-negative numbers, \( (a_{n}) \) and \( (\beta_{n}) \), we write \( a_{n} \sim \beta_{n} \), if there are constants \( c_{1}, c_{2} \) such that

\[
c_{1} \beta_{n} \leq a_{n} \leq c_{2} \beta_{n}, \quad n \in \mathbb{N}.
\]

We will also use the Landau symbols \( O(\cdot) \) and \( o(\cdot) \).

Since we are mainly interested in the trace formula and relations between \( s \)-numbers and eigenvalues, we shall always assume all Banach spaces to be complex. The eigenvalues of a compact operator \( T \in \mathcal{K}(X) \) are denoted by \( (\lambda_{n}(T))_{n \in \mathbb{N}} \) and assumed to be ordered in non-increasing absolute value and counted according to their multiplicity.

In the next section, we consider questions of the type: If the \( s \)-numbers of a compact operator \( T \in \mathcal{K}(X) \) belong to a sequence space like e.g. \( \ell_{p,r} \), what can be said of its eigenvalues? These are generalizations of Weyl's inequality in Hilbert spaces. Section 3 contains the trace theorem for \( S_{1} \)-type operators, showing that the spectral trace coincides with the matrix trace. The final section contains questions related to inequalities between single eigenvalues and single \( s \)-numbers and some examples concerning the isomorphism numbers.

2. Generalized Weyl inequalities. The singular numbers \( s_{n}(T) \) of a compact operator \( T \) in a Hilbert space \( H \) are the eigenvalues of \( (T^{*}T)^{1/2} \). H. Weyl [16] proved the following inequality between the eigenvalues of \( T \) and \( (T^{*}T)^{1/2} \):

\[
\sum_{j=1}^{n} |\lambda_{j}(T)|^{p} \leq \sum_{j=1}^{n} s_{j}(T)^{p}
\]

for any compact \( T \), any \( 0 < p < \infty \) and any \( n \in \mathbb{N} \). In [7] we derive the following generalization of Weyl's inequality to Banach spaces which we will need in the following:

**Proposition.** Let \( s_{n} \) denote either the approximation- or the Gelfand- or the Kolmogorov-numbers. For any \( 0 < p < \infty \) there is \( c_{p} > 0 \) such that for any Banach space \( X \), any compact operator \( T \in \mathcal{K}(X) \) and any \( n \in \mathbb{N} \)

\[
\left( \sum_{j=1}^{n} |\lambda_{j}(T)|^{p} \right)^{1/p} \leq c_{p} \left( \sum_{j=1}^{n} s_{j}(T)^{p} \right)^{1/p}.
\]

We extend this result to Lorentz sequence spaces.

**Theorem 1.** Let \( 0 < r < p < \infty, 0 < q < \infty \) and \( s_{n} \) be as in the proposition. Then there is a constant \( c_{p,q} \) such that for all Banach spaces \( X \) and any operator \( T \in \mathcal{K}_{p,q}(X) \) we have

\[
\| (\lambda_{n}(T))_{n \in \mathbb{N}} \|_{p, q} \leq c_{p,q} \| (s_{n}(T))_{n \in \mathbb{N}} \|_{p, q} = c_{p,q} \cdot \mathcal{S}^{(s)}_{p,q}(T).
\]

The main part of the proof is contained in

**Lemma 1.** Let \( 0 < r < p < \infty, 0 < q < \infty \). Then there is a constant \( c = c(p, q) \) depending only on \( p \) and \( q \) such that for all monotone non-increasing sequences \( (\alpha_{n}) \) of positive numbers

\[
\| (\alpha_{n})_{n \in \mathbb{N}} \|_{p, q} \leq c \| (\alpha_{n})_{n \in \mathbb{N}} \|_{p, q}.
\]

**Proof.** The "average" operator

\[
A : (\xi_{n})_{n \in \mathbb{N}} \mapsto \left( \frac{1}{n} \sum_{j=1}^{n} \xi_{j} \right)_{n \in \mathbb{N}}
\]

has norm one when considered as a map \( A : L_{q} \to L_{q} \). But \( A \) also is an operator from \( l_{\infty} \) into \( l_{\infty} \), again with norm one. To see this, note that \( \theta < \varepsilon_{n} \).
\[ \leq y_n \text{ implies } x_n \leq y_n^* \text{ and } \]
\[ x_n := \sum_{j=1}^{n} \frac{s_j}{n} \leq y_n := \sum_{j=1}^{n} \frac{s_j^*}{n}, \quad x_n^* \leq y_n = y_n. \]

Hence
\[ \| A \|_n \| x_n \| = \sup_{n \in N} \| x_n^* \| \leq \sup \| x_n^* \| = \| x_n \|. \]

By the (generalized) Marcinkiewicz interpolation theorem, \( A \) also induces a continuous linear map \( A : l_{p \lor \varrho} \to l_{q \lor \varrho} \), see [1]. Note here \( r < p \).

Writing this as an inequality for sequences, with \( \| A \| = o(p/r, q/r) \), and applying it to \( x_n = \xi_n \), we get the right inequality in Lemma 1. The left one is trivial.

**Proof of Theorem 1.** Choose \( r > 0 < \varrho < p \). The proposition implies in view of the ordering of the eigenvalues for any \( T \in S_{p,q}(X) \)
\[ |\lambda_n(T)^{n^{1/2}}| |x_n^{n^{1/2}}| \leq o \left( \sum_{j=1}^{n} s_j(T)^{n^{1/2}} \right)^{1/\varrho} \]
(2.1)
\[ |\lambda_n(T)^{n^{1/2}}| \leq o \left( \sum_{j=1}^{n} s_j(T)^{n^{1/2}} \right)^{1/\varrho} \]
so that by Lemma 1
\[ \| \lambda_n(T) \|_{p,q,n} \leq o \left( \left( \sum_{j=1}^{n} s_j(T)^{n^{1/2}} \right) -n^{1/2} \right)^{1/\varrho} \]
\[ \leq o \left( \left( \sum_{j=1}^{n} s_j(T)^{n^{1/2}} \right) -n^{1/2} \right)^{1/\varrho} \]
\[ \leq o_{p,q,n} \left( \left( \sum_{j=1}^{n} s_j(T)^{n^{1/2}} \right) -n^{1/2} \right)^{1/\varrho} \].

For \( p = q = \infty \), Lemma 1 is just Hardy's inequality and the statement of Theorem 1 is just the proposition. A case of special interest is also \( q = \infty \).

**Corollary 1.** Let \( s_n \) be as before, \( 0 < a < p < \infty \) and \( T \in \mathcal{F}(X) \).

(a) If \( s_n(T) = O(n^{-a}) \), also \( |\lambda_n(T)| = O(n^{-a}) \).

(b) If \( s_n(T) = O(n^{-\varrho}) \), also \( |\lambda_n(T)| = O(n^{-\varrho}) \).

(c) If \( s_n(T) = O(n^{-10 \log(n) n}) \), also \( |\lambda_n(T)| = O(n^{-10 \log(n) n}) \).

(d) The eigenvalues \( \lambda_n(T) \) are rapidly decreasing if the \( s_n(T) \) are.

**Proof.** (a) is the case \( q = \infty \) of Theorem 1. Choose \( r < 1/n \). A direct \( \epsilon = \delta \)-argument together with (2.1) yields (b); (2.1) for \( r < p \) also implies (c) if one uses that for \( \lambda < 1, \mu > 0 \)
\[ \sum_{j=1}^{n} j^{-1} \log(n)^{\mu} \sim n^{-1} \log(n)^{\mu}. \]

Statement (d) follows from Weyl's inequality, applied to all \( p > 0 \).

**Example 1.** There are Banach sequence spaces for which results analogous to those of Theorem 1 and Corollary 1 are false. Let \( R > 1 \) and \( l(R) = \{ x_n \in l_n : \sup x_n^* n \leq R < \infty \} \). Then there are operators \( T \in \mathcal{F}(l) \) such that the approximation numbers \( a_n(T) \) belong to \( l(R) \) whereas the eigenvalues \( \lambda_j(T) \) do not. For simplicity, let \( R = 2 \). Denote by \( A_n \) the Littlewood-matrices of order \( 2^n \),
\[ A_n = (1), \quad A_{n+1} = \begin{bmatrix} A_n & A_{n-1} \\ A_n & -A_{n-1} \end{bmatrix} \]
\[ n > 0, \]
and define \( T : l_1 \to l_1 \) as a direct sum of multiples of Littlewood matrices:
\[ T = \bigoplus_{n=0}^\infty A_n : l_1(2^n) \to l_1(2^n). \]

Since \( A_{n+1} = 2^n I_n \) the eigenvalues of \( A_n \) are \( \{ \pm 2^n \} \), both with multiplicity \( 2^{n-1} \). Hence
\[ |\lambda_n(T)| = 2^n n^{1/2} \quad \text{for} \quad 0 < n < 2^{n-1}, \quad n \in N. \]

Therefore
\[ \text{sup}_{n,i} |\lambda_n(T)|^2 n^{1/2} \leq \text{sup}_{n,i} 2^n n^{1/2} = \infty, \quad \{ \lambda_i(T) \} \notin l(2). \]

We show that nevertheless \( s_n(T) \notin l(2) \). Let \( T_{n+i} : l_1 \to l_1 \) be the operator defined by the same first \( (2^n-1)+i \) rows as \( T \) and otherwise zero. Since \( \text{rank} T_{n+i} \leq 2^{n+i} \), we conclude
\[ a_n(T_{n+i}) = |T - T_n|_1 = (2^n-i)2^{n+i}. \]

Note here that the matrix elements in \( T - T_n \) are either zero or \( \pm 1/2^{n+i} \). This implies
\[ \text{sup}_{n,i} |\lambda_n(T)|^2 n^{1/2} \leq \text{sup}_{n,i} (2^n-i)2^{n+i} = \infty. \]

It is easy to see that the last supremum is attained for \( n = 2^n-1 \), therefore
\[ \text{sup}_{n,i} |\lambda_n(T)|^2 n^{1/2} \leq 1/2, \quad \{ \lambda_i(T) \} \notin l(2). \]

However, a weaker positive result is available:

**Lemma 2.** Let \( R > 1 \) and \( s_n \) as above. Assume \( T \in \mathcal{F}(X) \) is compact with \( s_n(T) \in l(R) \). Then for any \( 1 < s < R \sqrt{R}, |\lambda_n(T)| \in l(S) \).

**Proof.** By a perturbation argument, we may assume all eigenvalues to have multiplicity one. Let \( X_n \) be the space spanned by the first \( n \) eigenvectors associated to \( \lambda_1(T), \ldots, \lambda_n(T) \). Since \( \text{dim} X_n = n \), there are operators \( A_n : X_n \to X_n^* \) with \( A_n \| A_n \| \leq n \), cf. F. John [6]. Let \( \delta_n = A_n(T)A_n^{-1} ; 1 \leq n \leq R \). Then \( |\lambda_n(T)| = |\lambda_j(T)| \) for \( j = 1, \ldots, n \). By H. Weyl [16], for the Hilbert space operator \( \delta_n \)
\[ |\lambda_n(\delta_n) \ldots \lambda_n(\delta_n)| \leq n^2 \delta_n \ldots \delta_n = n^2 \delta_n. \]
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\[ |\lambda_n(T)| = |\lambda_n(S_n)| \leq |\lambda_1(S) \ldots \lambda_n(S)|^{1/n} \leq \|A_n\| \|A_n^{-1}\|^{1/n} \|\varepsilon_1(T) \ldots \varepsilon_n(T)\|^{1/n}. \]

Since \(\{\varepsilon_n(T)\} \in l(R)\), for some constant \(c > 0\)
\[ |\lambda_n(T)| = c \sqrt{n} \left( \sum_{i=1}^{n} R^{-1/2n} \right) = c \sqrt{n} |R| \sqrt{R - n}. \]

This implies for any \(S < \sqrt{R}\)
\[ |\lambda_n(T)| \leq dS^{-n}, \quad \{\varepsilon_n(T)\} \in l(S). \]

We now give some applications of the previous results to operators in \(L_q\)-spaces.

**Proposition 1.** Let \(Q \subseteq \mathbb{R}^d\) be a bounded domain and \(T: L_p(Q) \to L_q(Q)\) an operator with image contained in a Sobolev space \(W^s_p(Q)\). Here \(k \in \mathbb{R}^d\) and \(1 < p < \infty\). Then \(|\lambda_n(T)| = O(n^{-k/d})\). If the image of the unit ball of \(L_p(Q)\) is relatively compact in \(W^s_p(Q)\), one even has \(\{\varepsilon_n(T)\} \in l(S)\).

**Proof.** The map \(T: L_p(Q) \to W^s_p(Q)\) has a closed graph and hence is continuous with respect to the Sobolev norm. Hence
\[ a_{n-1}(T: L_p \to L_q) = \alpha_n(T: L_p \to W^s_p), \quad \alpha_n(T: W^s_p \to L_q). \]

But the approximation numbers \(\alpha_n(T)\) of the Sobolev embedding tend to zero of order \(O(n^{-k/d})\), see e.g. R. S. Ismagilov [5], which gives \(a_{n-1}(T) = O(n^{-k/d})\) and \(\lambda_n(T) = O(n^{-k/d})\).

**Proposition 2.** Let \(K[0,1]^d \to \mathbb{C}\) be a measurable kernel, which is \((n-1)\) times continuously differentiable with respect to the second variable and absolutely continuous \((n-1)\) at derivative. Assume
\[ K_j(y) = \int \frac{\partial}{\partial y_j} K(x, y) f(y) \, dy. \]

defines continuous linear operators \(K_j: L_p \to L_q\) for \(j = 0, \ldots, n\). Let further \(0 < q < \infty\) and \(K \in S^0_p(J_{\mu})\), for \(q = \infty\) we get a special condition. Then the eigenvalues of \(K = K_0\) in \(L_p\) obey
\[ \left\{ \sum_{k=1}^{n} |\lambda_k(K)|^{2/n} \right\}^{1/n} < \infty, \]

which for \(q < \infty\) means especially \(|\lambda_k(K)| = O(k^{-1+1/g})\) and for \(q = \infty\) should read \(|\lambda_k(K)| = O(k^{-n+1}).\)

This generalizes a result in I. Z. Gohberg-M.G. Krein [3], p. 120 for \(p = q = 2\).

**Proof.** We indicate only the case \(n = 1\). Partial integration yields
\[ \int_0^1 K(x, y) f(y) \, dy = \int_0^1 f(t) dt - \int_0^1 \frac{\partial}{\partial y} K(x, y) \left( \int_0^1 f(t) \, dt \right) \, dy. \]

Hence \(K = L - KJ_0\) and \(J_0: L_p \to L_q\) is the integral operator
\(J_0 \equiv \int_0^1 f(t) dt\) and \(J: L_p \to L_q\) is the trace operator \(J = \left( \int_0^1 f(t) dt \right) \times \mathbb{K}(1)\). Since \(L\) does not influence the asymptotic order of the approximation numbers, we have
\[ a_{n-1}(K) \sim a_{n-1}(K_0 \cdot J_0) \leq a_1(K_0) a_1(J_0) \leq a_q(K) a_q(J), \]
where \(a_q(J) \leq e\) by the previous proposition. Since by assumption \(K_0 \in S^0_p(J_{\mu})\), we get
\[ \left\{ \sum_{k=1}^{n} |\lambda_k(K_0)|^{2/n} \right\}^{1/n} < \infty, \]
i.e. \(\{a_n(K)\} \in l(S)\) with \(1/\gamma = 1-1/\gamma\). The result now follows directly from Theorem 1.

3. **The trace formula for \(S_q\)-type operators.** V. B. Lidskij [10] proved for operators of trace class in Hilbert spaces \(T = S_q(\mathcal{H}) = S_q(\mathcal{H})\) the trace formula
\[ \text{tr}(T) = \sum_{k=1}^{\infty} \lambda_k(T). \]

A. Grothendieck [4] showed that the notion of "trace" can be defined for nuclear operators in Banach spaces with the approximation property. However, formula (3.1) no longer makes sense for these operators, since the eigenvalues of nuclear operators in Banach spaces are only square summable in general. On the other side, A.S. Markus and V. I. Macserv [12] proved (3.1) for any operator \(T\) of type \(S_q^0(X)\) with \(p < 1\) in general Banach spaces, without requiring the approximation property. We show in the following that (3.1) remains valid for any operator \(T\) in the larger class \(S_q^0(X)\) and thus answer a problem of A. S. Markus and V. I. Macserv affirmatively.

Let \(X\) be a Banach space and \(T \in \mathcal{F}(X)\) a finite rank operator
\[ T = \sum_{i=0}^{n} x_i \otimes y_i, \quad x_i \in X', \quad y_i \in X. \]
Then $x_i(y_i)$ is independent of the special representation of $T$ and called the trace of $T$, denoted $\text{tr}(T)$. One has $\text{tr}(T) = \sum \lambda_j(T)$. Any such $T \in \mathcal{S}(X)$ has a representation (3.2) with

$$\sum_{n \in \mathbb{N}} |x_n| |y_n| \leq 8 \sigma_1(T),$$

cf. A. Pietsch [14]. Hence the linear functional trace can be uniquely and continuously extended to $(\mathcal{S}(X), \sigma_1^2) = \mathcal{S}_1^2(X)$, again denoted by $\text{tr}$.

**Lemma 3.** Let $(\lambda_i)$ and $(\gamma_i)$ be two sequences of positive, monotone non-increasing numbers with $\prod_{i=1}^n (1 + r\lambda_i) \leq \prod_{i=1}^n (1 + r\gamma_i)$ for all $n \in \mathbb{N}$. Then for all $r \in \mathbb{R}^+$ and any $n \in \mathbb{N}$

$$\prod_{j=1}^n (1 + r\lambda_j) \leq \prod_{j=1}^n (1 + r\gamma_j).$$

This was shown by H. Weyl [16], cf. also [3].

**Theorem 2 (Trace formula).** Let $X$ be a Banach space and $T \in \mathcal{S}_1^2(X)$. Then the previously defined trace coincides with the sum of the eigenvalues,

$$\text{tr}(T) = \sum_{n \in \mathbb{N}} \lambda_n(T).$$

**Proof.** The sum exists by Theorem 1. We will use the fundamental ideas of A. S. Markus and V. I. Macaev [12]. However, the central estimate of the characteristic polynomials defined below proceeds differently. First, there are operators $T_n \in \mathcal{S}(X)$ of finite rank with $\sigma_1^2(T - T_n) \to 0$, since $\mathcal{S}(X)$ is $\sigma$-dense. For these $T_n$, the characteristic polynomials

$$D_n(\lambda) = \prod_{j=1}^n (1 - \lambda \beta_j(T_n)) = \det(I - \lambda T_n)$$

are well-defined, since only finitely many eigenvalues $\lambda_j(T_n)$ are non-zero. Let $r < 1$ and $n_0$ such that $\sigma_1^2(T - T_n) < 1$ for $n > n_0$. By (3.1) there is $c_r$ such that for all $S \in \mathcal{S}(X)$

$$|\lambda_j(S)| \leq c_r \left( \sum_{k=1}^j a_k(T_n^j)^1 \right) |\beta_j(S)|.$$

Lemma 1 implies for $r < p = q = 1$ that the $|\beta_j(S)|$ are summable for any $S \in \mathcal{S}_1^2(X)$ with

$$\sum_{j=1}^\infty |\beta_j(S)| \leq d \sum_{n \in \mathbb{N}} |a_n(S)|,$$

$d$ independent of $S$. Let $|\lambda| = r$. Weyl's Lemma 3 yields

$$D_n(\lambda) = \prod_{j=1}^n (1 + r|\beta_j(T_n)|) \leq \prod_{j=1}^n (1 + r|\beta_j(T_n)|),$$

$$\ln |D_n(\lambda)| \leq \sum_{j=1}^n \ln (1 + r|\beta_j(T_n)|) \leq \sum_{j=1}^n r|\beta_j(T_n)| \leq d \sum_{n \in \mathbb{N}} |a_n(T_n)| \leq d \sum_{n \in \mathbb{N}} \sigma_1^2(T_n) + 1.$$

By Montel's theorem, this implies as in [12] that the sequence of polynomials $D_n$ converges compactly to an entire function $D$. We will show now that $D$ is a function of order one, i.e.

$$\max_{|\lambda| \leq r} |D(\lambda)| = o(r) \text{ for } r \to \infty.$$
i.e. $D(\lambda)$ is an entire function of order one. The last part in the proof is now the same as in (12); By a theorem of Hurwitz the zeros of $D(\lambda)$ coincide with the limits of the zeros of the polynomials $D_n(\lambda)$, i.e. with $\lim_{n \to \infty} \lambda_j(T_n)^{-1}$, which is equal to $\lambda_j(T)^{-1}$ for an appropriate order of the $\lambda_j(T_n)$. Since $\sum_{n \in \mathbb{N}} |\lambda_j(T_n)| < \infty$ by Weyl’s generalized inequality, one gets for the order one function $D(\lambda)$

$$D(\lambda) = \prod_{j \in \mathbb{N}} (1 - \lambda_j(T))$$

and

$$\sum_{j \in \mathbb{N}} \lambda_j(T) = D'(0)D(0) = \lim_{n \to \infty} D'_n(0)D_n(0) = \lim_{n \to \infty} \left[ - \frac{\text{tr}(T_n)}{n} \right] = - \text{tr}(T).$$

This proves Theorem 2.

Corollary 2. Let $S'$, $T \in S'_1(X)$. Then we have the eigenvalue equality

$$\sum_{j \in \mathbb{N}} \lambda_j(S + T) = \sum_{j \in \mathbb{N}} \lambda_j(S) + \sum_{j \in \mathbb{N}} \lambda_j(T).$$

Proposition (12). Assume the Banach space $X$ has an unconditional basis $\{e_j\}$ with biorthogonal coefficient functionals $\{f_j\}$. Let $T \in X'$ be a nuclear operator. Then $\sum f_j(T e_j)$ converges absolutely and the sum is equal to the trace of $T$.

In view of $S'_1(X) \subset X'$, this result and Theorem 2 imply

Corollary 3. Let $X$ be a Banach space with an unconditional basis $\{e_j\}$ with biorthogonal coefficient functionals $\{f_j\}$. Assume $T$ is in $S'_1(X)$. Then the “matrix trace” is equal to the “spectral trace”;

$$\sum_{j \in \mathbb{N}} f_j(T e_j) = \sum_{j \in \mathbb{N}} \lambda_j(T).$$

4. $s$-numbers and eigenvalues. In Section 2 we considered estimates for sequence space norms of all eigenvalues of a compact operator against all $s$-numbers. Sometimes it may be more interesting to have an estimate for single eigenvalues against single $s$-numbers. In [9] it was shown that for any compact operator $T \in X'$ in a Banach space $X$ and $j \in \mathbb{N}$

$$|\lambda_j(T)| = \lim_{m \to \infty} s_j(T^{m+1})^{-1},$$

where $s_j$ is an arbitrary $s$-number sequence. But as the Littlewood matrices show easily, an estimate of the type

$$|\lambda_j(T)| \leq c s_j(T),$$

$c$ independent of $j \in \mathbb{N}$, is false in general. This is true, however, for a certain class of operators in Banach spaces which were studied by A. S. Markus [11]:

A compact operator in a Banach space $T \in \mathcal{X}(X)$ is called $\mathcal{X}$-operator, if the spectrum of $T$ is real and if the resolvent of $T$ fulfills

$$||(I - T)^{-1}|| \leq c||\text{Im} T||, \quad \lambda \in C \sim R.$$

Any self-adjoint operator in a Hilbert space is an $\mathcal{X}$-operator with $c = 1$. For self-adjoint operators, $|\lambda_j(T)| \leq s_j(T)$. The following result generalizes this. For the definition of the Bernstein- and Mittag-Leffler-numbers we refer to [13].

Proposition 3. Let $s_n$ be a sequence which is larger than or equal to the Bernstein- or Mittag-Leffler-numbers. Then for any compact $\mathcal{X}$-operator $T$ with constant $c$ in a Banach space $X$

$$\sum_{n} (2c)^{-1} s_{n}(T) \leq |\lambda_j(T)| \leq 2(c + 1) s_{n}(T).$$

Especially, $s_n(T) \leq 4c(c + 1) s_{n}(T)$: all $s$-numbers considered have the same asymptotic order as the eigenvalues.

Proof. A. S. Markus [11] showed this for the Kolmogorov and approximation numbers. Hence the left inequality is clear in view of $s_n(T) \leq c_n(T)$; instead of (9) $c^{-1}$ as in [11], $(2c)^{-1}$ is sufficient. If $T \in \mathcal{X}(X)$ is a $\mathcal{X}$-operator with constant $c$, the same is true for $T' \in \mathcal{X}(X')$. By A. Pietsch [13] the Mittag-Leffler and Bernstein-numbers are dual, $s_n(T) = s_n(T')$. Therefore it is enough to prove the right inequality for the Bernstein-numbers $s_n$. We will use similar ideas as in [11].

Assume without loss of generality $\lambda_j(T) = 0$ and let $k \in \mathbb{N}$ be minimal with $|\lambda_{k+1}(T)| < |\lambda_j(T)|$. Let $P_k$ be the spectral projection of $T$ relative to $A = \{\lambda_j(T), \ldots, \lambda_{k+1}(T)\}$. Then $\dim P_k(T) = k$ and the restriction $T_k : P_k(X) \to P_k(X)$ of $T$ to $P_k(X)$ is an isomorphism. Choose $r > 0$ with

$$|\lambda_j(T)|^{-1} < r < |\lambda_{k+1}(T)|^{-1}.$$

The spectrum of $T_k^{-1} \in \mathcal{X}(P_k(X))$ is equal to $|\lambda_j(T)|^{-1}$, $\ldots$, $|\lambda_{k+1}(T)|^{-1}$ and contained in the interior of the circle $\Gamma = \{\lambda \in C : |\lambda| = r\}$. Hence the identity on $P_k(X)$ has a representation by the Dunford integral

$$I_n = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I_n - T_k^{-1})^{-1} d\lambda,$$

with $(\lambda I_n - T_k^{-1}) \in \mathcal{X}(P_k(X))$, cf. [2], chap. 7. Since

$$I_n = \frac{1}{2\pi i} \int_{\Gamma} |\lambda| d\lambda,$$

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we get by subtractions

\[ 0 = -T_n^{-1}/(2\pi i) \int \frac{1}{\lambda} (\lambda I_n - T_n^{-1})^{-1} d\lambda. \]

We multiply this by \( rT_n \) and take the sum with

\[ T_n^{-1} = (2\pi i)^{-1} \int \frac{\lambda^2 - r^2}{\lambda} (\lambda I_n - T_n^{-1})^{-1} d\lambda, \]

which yields

\[ (\lambda I_n - T_n^{-1})^{-1} = \lambda^{-2} I_n - \frac{r^2}{\lambda} (\lambda I_n - T_n^{-1})^{-1}. \]

The equality

\[ (\lambda I_n - T_n^{-1})^{-1} = \lambda^{-2} I_n - \frac{r^2}{\lambda} (\lambda I_n - T_n^{-1})^{-1} \]

implies that \( T_n^{-1} \) is a \( \mathcal{S} \)-operator with constant \( c + 1 \). Hence (4.2) enables the estimate

\[ \| T_n^{-1} \| \leq (c+1)^r \max_{k+i=\infty} \left| \frac{\lambda^2 - r^2}{\lambda(\lambda+i\delta)} \right| = 2(c+1)^r. \]

By (13), the isomorphism numbers are the smallest and the Bernstein numbers the smallest injective \( s \)-numbers. Factoring the identity on \( P_n(X) = T_n^{-1}T_nI_n \), we conclude by the definition of the isomorphism numbers

\[ 2(c+1)^{-1} \leq \| T_n^{-1} \| \leq \| I_n \| \leq \| P_n(X) \| \leq \| P_n(X) \| \leq \| u_n(T) : P_n(X) \rightarrow P_n(X) \| \leq u_n(T) \| X \rightarrow X \|, \]

using the injectivity of the \( u_n \)'s. Since \( r > |\alpha_n(T)|^{-1} \) was arbitrary, this shows \( |\alpha_n(T)| \leq 2(c+1)^r u_n(T) \).

Weyl's inequality in Banach spaces

\[ \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p \leq c_p \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p, \quad 0 < p < \infty \]

was formulated in Section 2 for the Gelfand, Kolmogorov or approximation numbers. We want to show now that in general this inequality does not hold for other \( s \)-number sequences. To do so, we need a fact on the isomorphism numbers which were considered in the previous proof.

**Lemma 4.** For \( 1 < n \leq m \), \((1/4)(m/n) \leq (\lambda_n(T_n^{-1})^{-1} \leq m/n\).

**Proof.** (a) Let \( \dim X_n \geq n \) and take a factorization of the identity on \( X_n \),

\[ X_n \cong \mathbb{R}^m \cong \mathbb{R}^n \cong X_n. \]

Then \( \| P \|^{-1} \| Q \|^{-1} \leq \| P \|_{s_n} \| Q \|_{s_n}^{-1} \). The norm of \( P \) is the maximum of the absolute values of the elements in the matrix representation, hence larger than or equal to \( 1/m \) times the trace,

\[ \| P \|^{-1} \| Q \|^{-1} \leq |m|/\text{tr}(|P Q|). \]

\( \text{Id} P Q : \mathbb{R}^m \to \mathbb{R}^m \) is a projection of rank \( n \). Therefore

\[ \text{tr}(|P Q|) = \sum_{\lambda \in \mathbb{C}} \lambda_n(\text{Id} P Q) \geq n. \]

Taking the supremum over all factorizations, we get

\[ \epsilon_n(\text{Id} ; \mathbb{R}^m \to \mathbb{R}^m) \leq m/n. \]

(b) Let \( m = 2^m \), \( n = 2^N \), \( M, N \in \mathbb{N} \). Let \( \hat{A} \) be the \((n \times n)\)-matrix of the first \( m \) rows of the Littlewood matrices \( A_M \) of order \( m = 2^m \) and \( Q : \mathbb{R}^n \to \mathbb{R}^n \) the operator defined by \( 1/\sqrt{m} \hat{A} \). Let \( P : \mathbb{R}^m \to \mathbb{R}^n \) be defined by \( 1/\sqrt{m} \hat{A} \). Then we have a factorization of the identity on \( \mathbb{R}^m \) as

\[ \ldots \mathbb{R}^m \to \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^m, \]

implying

\[ \epsilon_n(\text{Id} ; \mathbb{R}^m \to \mathbb{R}^m) = \| P \|^{-1} \| Q \|^{-1} = m/n. \]

The left inequality of Lemma 4 is now an easy consequence.

**Example 2.** We prove that there is no constant \( c_p \) such that for all \( T \in s_n^p \), one has

\[ \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p \leq c_p \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p, \quad 0 < p < \infty \]

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Let \( n = 2^N \) and \( T_n : T_n \to \mathbb{R}^n \) the operator defined by the Littlewood matrices \( A_M \) (or by imbedding, \( T_n : 1 \to 1 \)). Both eigenvalues \( \pm \sqrt{n} \) are of order \( n^{1/2} \). Hence

\[ \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p \leq n^{1+p/2}. \]

On the other hand,

\[ \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p \leq \| T_n \|_{s_n} \| T_n \|_{s_n}^p \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p. \]

But \( \| T_n \|_{s_n} \leq 1 \). Hence by Lemma 4

\[ \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p \leq n^p \sum_{\lambda \in \mathbb{C}} |\lambda_n(T)|^p. \]

For large \( n \), this and (4.4) contradict (4.3). We remark that it is not known whether the classes \( s_n^p \) are operator ideals.
EXAMPLE 3. By A. Pietsch [13] one has for any invertible operator $T \in \mathcal{L}(X, Y)$ with $\dim X = \dim Y = m$

\begin{equation}
\gamma_\alpha(T) \cdot a_{\frac{\alpha + 1}{2}}(T^{-1}) \leq 1.
\end{equation}

In general, however, there may be a strict inequality "<", as will follow from Lemma 4. This answers a question in [13] negatively and seems a bit surprising, since for other corresponding pairs of $\alpha$-numbers like Bernstein and Gelfand numbers there is an equality of type (4.5). Let $m = 2^k$, $n = m/2$. As is well known,

\begin{equation}
a_{\frac{\alpha + 1}{2}}(1) : l_2^{\infty} \rightarrow l_2^{\infty} \leq (\sqrt{\alpha} + 1)^{-1},
\end{equation}

cf. B. S. Kasin [8]. This and Lemma 4 imply for $T = 1 : l_2^{\infty} \rightarrow l_2^{\infty}$ and $m = m/2 > 1$

\begin{equation}
\gamma_\alpha(T) \cdot a_{\frac{\alpha + 1}{2}}(T^{-1}) \leq 2(\sqrt{\alpha} + 1) < 1.
\end{equation}

This shows also that the estimate $u_\alpha(T) \leq a_{\alpha + 1}(T)$ in [13] is of optimal order, since

\begin{equation}
u_\alpha(T) = \gamma_{\frac{\alpha + 1}{2}}(T^{-1})^{-1} = a_{\frac{\alpha + 1}{2}}(T^{-1})^{-1} \geq \frac{1}{\sqrt{\alpha}} \geq \frac{1}{\sqrt{\alpha}} \gamma_\alpha(T).
\end{equation}

Any $T \in S_2(X, Y)$ is nuclear. On the other hand, one can ask whether the $\alpha$-numbers imply to some $p$-summability classes of $\alpha$-numbers and whether $\sigma_\alpha(T) \approx a_{\alpha + 1}(T)$ holds. This is false in general for $p = 1$, as the identity $T = 1 : l_2^{\infty} \rightarrow l_2^{\infty}$ shows:

\begin{equation}
\min_{m = 1} m \approx \sigma_\alpha(T) \leq a_{\frac{\alpha + 1}{2}}(T) \leq \sigma_\alpha(T) = \sigma_\alpha(T).
\end{equation}

A weaker positive result is true for $p > 2$:

**Proposition 4.** For all Banach spaces $X$ and $Y$, $\mathcal{A}_p(X, Y) \subset S_{2p}(X, Y)$; i.e., the isomorphism numbers $i_\alpha$ of nuclear operators are of order $\alpha^{-1/2}$.

**Proof.** The factorization diagram for nuclear operators [14] and the properties of the $\alpha$-numbers imply that it is enough to consider diagonal operators $D_\sigma : l_\sigma \rightarrow l_\sigma$, $a_{\alpha + 1}(\sigma l_\sigma) \leq \sigma l_\sigma$ and show

\begin{equation}
\sup_{\alpha \in N} \sigma_\alpha(D_\sigma) \leq ||\sigma||_1 = \sigma_1(D_\sigma).
\end{equation}

We will assume without loss of generality that the $\sigma_i$ are positive, non-increasing. Let $\mu = \sqrt{\alpha}$. Then $\mu \geq \sigma_i$ and

\begin{equation}
\gamma_\alpha(D_\mu) \leq \gamma_\alpha(D_\sigma) : l_\mu \rightarrow l_\mu \parallel D_\sigma : l_\mu \rightarrow l_\mu \parallel = ||\sigma||_1^2 \gamma_\alpha(D_\mu) : l_\mu \rightarrow l_\mu.
\end{equation}

Let $D_\mu^p$ be the “restriction” of $D_\mu$ to $l_\mu^p$. Then

\begin{equation}
\gamma_\alpha(D_\mu) = \sup \{ \gamma_\alpha(D_\mu^p) : m \in \mathbb{N} \}
\end{equation}

since

\begin{equation}
\gamma_\alpha(D_\mu) - \gamma_\alpha(D_\mu^p) \leq ||D_\mu - D_\mu^p|| : l_\mu \rightarrow l_\mu \parallel \rightarrow 0 \text{ for } m \rightarrow \infty.
\end{equation}

Using the relation between the isomorphism and approximation numbers considered in Example 3 and the results of M. Z. Solomjak and V. M. Tihonov [15], one gets

\begin{equation}
\gamma_\alpha(D_\mu) \leq a_{\alpha + 1}(\sigma_\alpha(D_\mu)^{-1}) : l_\mu^p \rightarrow l_\mu^p \parallel^{-1} \leq \min_{m = 1} \sqrt{\sum_{i = m+1}^{N} \sigma_i^2} \sqrt{\frac{m}{(b - m + 1)} \leq ||\sigma_1||_1^2 \gamma_\alpha(D_\mu)}.
\end{equation}

This yields

\begin{equation}
\gamma_\alpha(D_\mu) : l_\mu \rightarrow l_\mu \parallel \leq ||\sigma_1||_1^2 \gamma_\alpha(D_\mu) \leq ||\sigma_1||_1^2 \gamma_\alpha(D_\mu).
\end{equation}

References


Differentiability of distributions at a single point

by

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Abstract. We develop tools needed to study the differentiability of distributions at a point of \( R^n \).

The purpose of this paper is to develop tools to study distributions at a point \( x_0 \). The integral expressions that we will be examining are closely related to the area integral of Lusin and the Littlewood–Paley \( g_1 \) functions. Rather than use the theory of harmonic functions we will consider analogous expressions that are independent of the mollifier that is used. The results of this paper are used to study tempered nontangential boundedness and convergence in [3].

Let \( \varphi \) be a Schwartz function with \( f \varphi(x) \, dx = 1 \). Using this as a mollifier, form

\[
\psi(x, t) = f \varphi_t(x) \quad \text{where} \quad \varphi_t(x) = t^{-n} \varphi \left( \frac{x}{t} \right).
\]

If \( f \) is a continuous function then \( u(x, t) \) approaches \( f(x) \) as \( t \to 0 \). It seems reasonable then that by examining \( u(x, t) \) in the set \( \Omega = \{x, t\}; \ x \in R^n, \ 0 < t < 1 \) we should be able to understand the behavior of \( f \) at a point, say \( x_0 \). We form a certain integral \( \mathcal{P}^{(2)}_{n,k}(f)(x_0) \) of \( u(x, t) \) over the set \( \Omega \).

Of the various parameters involved the most important is \( \gamma \in R \).

We will see that if \( \mathcal{P}^{(2)}_{n,k}(f)(x_0) < \infty \) then \( \gamma \) gives the order of differentiability of \( f \) at \( x_0 \).

If we add certain harmless terms to those \( \mathcal{G} \) functions we can form norms \( N_{\mathcal{G},k}^{(2)} \). Thus

\[
N_{\mathcal{G},k}^{(2)}(f)(x_0) = \mathcal{P}^{(2)}_{n,k}(f)(x_0) + \text{"other terms"}
\]

In the second and third sections we show that both the \( \mathcal{G} \) functions and the norm \( \mathcal{X} \) are essentially independent of the mollifier \( \varphi \). In addition, using a different \( k \) gives an equivalent norm.

With these norms we can define Banach spaces

\[
A^{n,k}_p(x_0) = \{ f \in \mathcal{G} : N_{\mathcal{G},k}^{(2)}(f)(x_0) < \infty \text{ for some } k > \gamma + n/p \}.
\]

Contrast these with the Sobolev spaces \( L^p_k(R^n) \) of functions that have