A generalization of Khintchine's inequality and its application in the theory of operator ideals

by

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Abstract. We prove a generalization of Khintchine's inequality which can be used to estimate the absolutely $r$-summing norm and the $r$-factorable norm of the identity map from $\ell^n_u$ into $\ell_v^s$ for certain exponents $u$ and $v$. This result fills in the remaining gaps in the limit order diagrams of the operator ideals $\mathfrak{I}_r$ and $\mathfrak{I}_s$.

In the following $\mathcal{U}(E, F)$ denotes the set of all (bounded linear) operators from $E$ into $F$, where $E$ and $F$ are arbitrary Banach spaces.

An operator $S \in \mathcal{U}(E, F)$ is called absolutely $r$-summing ($1 \leq r < \infty$) if there exists a constant $C$ such that

$$\left( \sum_{n=1}^s \| S e_n \| r \right)^{1/r} \leq \sup \left( \left\{ \left( \sum_{n=1}^s | \langle \lambda_n, e_n \rangle | \right)^{1/r} : \| \lambda_n \| \leq 1 \right\} \right)$$

for all finite families of elements $x_1, \ldots, x_n \in E$. The class $\mathcal{U}_r$ of these operators is an ideal with the norm $\| S \|_r := \inf C$. An operator $S \in \mathcal{U}(E, F)$ is called $r$-factorable ($1 \leq r < \infty$) if there exists a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{L} & F \\
\downarrow & & \downarrow \\
A & \xrightarrow{K} & \mathfrak{I} \\
\end{array}
$$

with $A \in \mathcal{U}(E, L^1_\mu)$ and $Y \in \mathcal{U}(L_\mu, F^{**})$. Here $(\Omega, \mu)$ is a measure space and $E^*$ denotes the evaluation map from $E$ into $F^{**}$. The class $\mathcal{U}_r$ of these operators is an ideal with the norm $L_\mu(S) := \inf \| Y \|_{L_\mu}$, where the infimum is taken over all admissible factorizations.

Let us denote by $J$ the identity map from $\ell^u_v$ into $\ell^v_u$, where $\ell^u_v$ and $\ell^v_u$ are the Minkowski spaces with $1 \leq u, v < \infty$. It is well known that the asymptotic properties of $A(I: \ell^u_v \to \ell^v_u)$ give important information about the operator ideal $\mathfrak{I}$ with the norm $A$. In particular, we are interested to know the so-called limit order $\lambda(A, u, v)$ which is defined to be the infimum...
of all $\lambda \geq 0$ such that
\[ A(\lambda; \xi) \leq C \left| \xi_r \right|^2 \]
for $n = 1, 2, \ldots$
with some constant $C$.

For every normed operator ideal the behaviour of $\lambda(A, u, v)$ can
be graphically represented by means of diagrams in the unit square.
The coordinates are $1/u$ and $1/v$. In the left-hand diagram we plot the
level curves, while the algebraic expressions of $\lambda(A, u, v)$ are indicated
on the right.

For the ideals of absolutely $r$-summing and $r$-factorable operators
with $2 < r < \infty$ the following results are known:

The purpose of this paper is to fill in the remaining gaps. We shall
prove that
\[ \xi = \frac{1}{r} + \frac{(1/r' - 1)/s}{1/2 - 1/r} \]
and
\[ \sigma = \frac{(1/2 - 1/2/(1/r' - 1/s)}{1/2 - 1/r} \]

Therefore it turns out that the corresponding level curves are hyperbolas.

Finally, let us mention that $\lambda(P, u, v)$ with $1 < r < 2$ is completely
known, while the limit order of $\xi_r$ with $1 < r < 2$ is given by the formula
$\lambda(\xi_r, u, v) = \lambda(\xi_r, v', u')$.

In the sequel we shall use the notation introduced in [3]. In particular, $r'$
denotes the conjugate exponent of $r$ defined by $1/r + 1/r' = 1$.
\[ \sum_{n_i^*} |\langle x_i, \phi \rangle|^{2m} = \sum_{n_i^*} \cdots \sum_{n_i^*} \xi_{i_1} \cdots \xi_{i_{2m}} \sum_{n_i^*} \xi_{i_1} \cdots \xi_{i_{2m}} \]

\[ = \sum_{n_i^*} \sum_{n_i^*} (2m)! \sum_{(2n_1) \ldots (2n_k)} \xi_{i_1} \cdots \xi_{i_{2m}} \sum_{n_i^*} \xi_{i_1} \cdots \xi_{i_{2m}} \]

\[ \leq \sum_{n_i^*} \sum_{n_i^*} (2m)! \sum_{(2n_1) \ldots (2n_k)} \|x_i\|_{l^1} \cdots \|x_i\|_{l^1} N \left( \frac{k}{n} \right). \]

Now, by the preceding lemma, it follows that

\[ N^{-1} \sum_{n_i^*} |\langle x_i, \phi \rangle|^{2m} \leq \sum_{n_i^*} \sum_{n_i^*} (2m)! \max \left( \left( \frac{k}{n} \right)^{1/2} \|x_i\|_{l^1} \left( \frac{k}{n} \right)^{1/2m} \|x_i\|_{l^m} \right)^{2m}. \]

This yields the desired estimate, since

\[ \sum_{n_i^*} \sum_{n_i^*} (2m)! \leq k^m \leq m^m. \]

The complex case can be derived from the real one in the usual way. So, the assertion is proved.

Remark 1. Another proof has been given by the first named author in [2].

Remark 2. It seems very likely that the above inequality remains true for all exponents \( r > 2 \).

Remark 3. The classical estimate which is known as Khintchine's inequality appears in the case where \( k = n \).

Remark 4. A famous theorem of H. E. Rosenthal [5] yields another generalization of Khintchine's inequality which can also be used to prove the following results, cf. [4].

3. An operator in Minkowski spaces. In the sequel \( l_q(E_0) \) denotes the Banach spaces of all scalar families \( y = (y_n) \) equipped with the norm

\[ \|y\|_q := \left( \sum_{E_0} |y_n|^q \right)^{1/q}. \]

Then \( Ax = (\langle x, \phi \rangle) \) defines an operator \( A \) from \( l_q(E_0) \) into \( l_q(E_0) \).
4. The limit orders of $\mathfrak{P}_0$ and $\mathfrak{U}_0$.

**Proposition 1.** If $2 < u, v < r < \infty$, then

$$
\lambda(P_r, u', v') \leq c' := \frac{1}{v} + \frac{(1/r - 1/u)(1/v - 1/r)}{1/2 - 1/r}.
$$

**Proof.** If $r$ is defined by $1/v = (1 - 0)/r + 0/2$, then $c' = (1 - 0)|r + 0/u$.

Since we have

$$
P_r(I: E_r^u \to E_r^v) \leq P_r(I: E_r^u \to E_r^v) \leq w^{\lambda(v)}
$$

and

$$
P_r(I: E_r^u \to E_r^v) \leq P_r(I: E_r^u \to E_r^v) \leq w^{\lambda(v)},
$$

an interpolation theorem of B. Carl [1] yields

$$
P_r(I: E_r^u \to E_r^v) \leq P_r(I: E_r^u \to E_r^v) \leq w^{\lambda(v)}.
$$

**Proposition 2.** If $2 < u, v < r < \infty$, then

$$
\lambda(I_r, u, v) \leq \sigma := \frac{(1/2 - 1/|u|)1/|v| - 1/|r|}{1/2 - 1/|r|}.
$$

**Proof.** We consider $A \in \mathfrak{P}_0^n(I_r, E_r^u)$ and $B \in \mathfrak{P}_0^n(I_r, E_r^v)$ defined by $x \mapsto (x, \phi)$. Lemmas 1 and 2 imply

$$
\|A: E_r^u \to \mathfrak{P}_0^n(I_r, E_r^u)\| \leq \alpha \max\{k^{1/2}, k^{1/2}|u|^{1/2n}, 1\}
$$

and

$$
\|B: E_r^v \to \mathfrak{P}_0^n(I_r, E_r^v)\| \leq \alpha \max\{k^{1/2}, k^{1/2}|v|^{1/2n}, 1\}.
$$

Using

$$
\sum_{E_r^u} \sum_{E_r^v} \frac{\alpha \beta}{E_r^u} \quad \text{for } \beta = j,
$$

$$
\sum_{E_r^v} \sum_{E_r^u} \frac{\alpha \beta}{E_r^v} \quad \text{for } \beta \neq j,
$$

we get the commutative diagram

$$
\begin{array}{ccc}
E_r^u & \overset{2\lambda(E_r^u)^2}{\longrightarrow} & E_r^v \\
\downarrow & & \downarrow \\
\lambda(E_r^u) & \overset{\lambda(E_r^v)^2}{\longrightarrow} & \lambda(E_r^v)
\end{array}
$$

Therefore,

$$
\lambda(\mathfrak{P}_0^n(I_r: E_r^u \to E_r^v)) \leq \alpha \max\{k^{1/2}, k^{1/2}|u|^{1/2n}, 1\}.
$$

**References**


