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A generalization of Khintchine's inequality and its application in the theory of operator ideals

by

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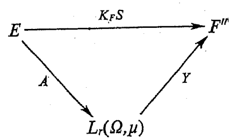
Abstract. We prove a generalization of Khintchine's inequality which can be used to estimate the absolutely r-summing norm and the r-factorable norm of the identity map from l_u^n into l_v^n for certain exponents u and v. This results fill in the remaining gaps in the limit order diagrams of the operator ideals $\mathfrak{P}_r$ and $\mathfrak{L}_r$.

In the following $\mathfrak{L}(E, F)$ denotes the set of all (bounded linear) operators from E into F, where E and F are arbitrary Banach spaces.

An operator $S \in \mathfrak{L}(E, F)$ is called *absolutely r-summing* $(1 \leqslant r < \infty)$ if there exists a constant σ such that

$$\Big\{\sum_1^n \|Sx_i\|^r\Big\}^{1/r} \leqslant \sigma \sup\Big[\Big\{\sum_1^n |\langle x_i, a\rangle|^r\Big\}^{1/r}: \|a\| \leqslant 1\Big]$$

for all finite families of elements $x_1, \ldots, x_n \in E$. The class $\mathfrak{P}_r$ of these operators is an ideal with the norm $\boldsymbol{P}_r(S) := \inf \sigma$. An operator $S \in \mathfrak{L}(E, F)$ is called *r-factorable* $(1 \leqslant r \leqslant \infty)$ if there exists a commutative diagram



with $A \in \mathfrak{L}(E, L_r(\Omega, \mu))$ and $Y \in \mathfrak{L}(L_r(\Omega, \mu), F'')$. Here (Ω, μ) is a measure space and K_F denotes the evaluation map from F into F''. The class $\mathfrak{L}_r$ of these operators is an ideal with the norm $\boldsymbol{L}_r(S) := \inf \|Y\|\,\|A\|$, where the infimum is taken over all admissible factorizations.

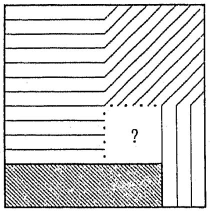
Let us denote by I the identity map from l_u^n into l_v^n, where l_u^n and l_v^n are the Minkowski spaces with $1 \leqslant u, v \leqslant \infty$. It is well known that the asymptotic properties of $\boldsymbol{A}(I: l_u^n \to l_v^n)$ give important information about the operator ideal $\mathfrak{A}$ with the norm $\boldsymbol{A}$. In particular, we are interested to know the so-called *limit order* $\lambda(\boldsymbol{A}, u, v)$ which is defined to be the infimum of all $\lambda \geqslant 0$ such that

$$\boldsymbol{A}(I:\ l_u^n \to l_v^n) \leqslant cn^{\lambda} \quad \text{for } n = 1, 2, \ldots$$

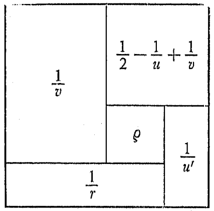
with some constant c.

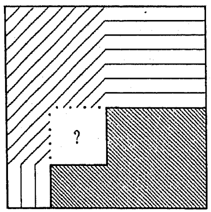
For every normed operator ideal the behaviour of $\lambda(\boldsymbol{A}, u, v)$ can be graphically represented by means of diagrams in the unit square. The coordinates are $1/u$ and $1/v$. In the left-hand diagram we plot the level curves, while the algebraic expressions of $\lambda(\boldsymbol{A}, u, v)$ are indicated on the right.

For the ideals of absolutely r-summing and r-factorable operators with $2 < r < \infty$ the following results are known:

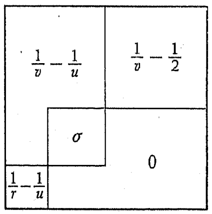


$\mathfrak{P}_r$





$\mathfrak{L}_r$



The purpose of this paper is to fill in the remaining gaps. We shall prove that

$$\varrho = \frac{1}{r} + \frac{(1/r' - 1/u)(1/v - 1/r)}{1/2 - 1/r} \quad \text{and} \quad \sigma = \frac{(1/2 - 1/u)(1/v - 1/r)}{1/2 - 1/r}.$$

Therefore it turns out that the corresponding level curves are hyperbolas.

Finally, let us mention that $\lambda(\boldsymbol{P}_r, u, v)$ with $1 < r < 2$ is completely known, while the limit order of $\mathfrak{L}_r$ with $1 < r < 2$ is given by the formula $\lambda(\boldsymbol{L}_r, u, v) = \lambda(\boldsymbol{L}_{r'}, v', u')$.

In the sequel we shall use the notation introduced in [3]. In particular, r' denotes the conjugate exponent defined by $1/r + 1/r' = 1$.

1. Preliminaries. Let $\boldsymbol{K}^n$ denote the n-dimensional real or complex linear space of all scalar vectors $x = (\xi_i)$. We write l_r^n if $\boldsymbol{K}^n$ is equipped with the norm

$$\|x\|_r := \Big\{\sum_1^n |\xi_i|^r\Big\}^{1/r} \text{ if } 1 \leqslant r < \infty \quad \text{and} \quad \|x\|_r := \sup|\xi_i| \text{ if } r = \infty.$$

As an easy consequence of Hölder's inequality we get the

LEMMA. *If* $1 \leqslant r_0 \leqslant r \leqslant r_1 \leqslant \infty$ *and* $c \geqslant 0$, *then*

$$c^{1/r}\|x\|_r \leqslant \max\big[c^{1/r_0}\|x\|_{r_0}, c^{1/r_1}\|x\|_{r_1}\big]$$

for all $x \in \boldsymbol{K}^n$.

2. A generalization of Khintchine's inequality. Let E_k^n denote the collection of all vectors $e = (\varepsilon_i)$ such that $\varepsilon_i \in \{-1, 0, +1\}$ and $\sum_1^n |\varepsilon_i| = k$. Put $N := \operatorname{card}(E_k^n) = 2^k\binom{n}{k}$. We now state a generalization of

KHINTCHINE'S INEQUALITY. *If* $r = 2m$ *with* $m = 1, 2, \ldots$, *then there exists a constant* c_r *not depending on* $n = 1, 2, \ldots$ *and* $k = 1, \ldots, n$ *such that*

$$\Big\{N^{-1}\sum_{E_k^n} |\langle x, e\rangle|^r\Big\}^{1/r} \leqslant c_r \max\Big[\Big(\frac{k}{n}\Big)^{1/2}\|x\|_2, \Big(\frac{k}{n}\Big)^{1/r}\|x\|_r\Big]$$

for all $x \in \boldsymbol{K}^n$.

Proof. In order to check the desired estimate we need some preliminary considerations.

Let $(j_1, \ldots, j_h)$ be a multi-index having different coordinates such that $j_\beta \in \{1, \ldots, n\}$. Then

$$\sum_{E_k^n} \varepsilon_{j_1}^{t_1} \ldots \varepsilon_{j_h}^{t_h} = 0 \tag{1}$$

if at least one of the exponents $t_\beta \in \{1, 2, \ldots\}$ is odd. On the other hand, it follows that

$$\sum_{E_k^n} \varepsilon_{j_1}^{2s_1} \ldots \varepsilon_{j_h}^{2s_h} = 2^k\binom{n-h}{k-h} \leqslant N\Big(\frac{k}{n}\Big)^h, \tag{2}$$

where $s_\beta \in \{1, 2, \ldots\}$ and $h = 1, \ldots, k$.

Let J_h^n denote the set of all multi-indices $(j_1, \ldots, j_h)$ described above. Furthermore, let S_h^m be the set of all $(s_1, \ldots, s_h)$ with $s_\beta \in \{1, 2, \ldots\}$ and $\sum_1^h s_\beta = m$. Put $h_0 := \min(k, m)$. Then, for every real vector $x \in \boldsymbol{K}^n$, we get

$$\sum_{E_k^n} |\langle x, e\rangle|^{2m} = \sum_{i_1=1}^{n} \dots \sum_{i_{2m}=1}^{n} \xi_{i_1} \dots \xi_{i_{2m}} \sum_{E_k^n} \varepsilon_{i_1} \dots \varepsilon_{i_{2m}}$$

$$= \sum_{h=1}^{h_0} \sum_{S_h^m} \frac{(2m)!}{(2s_1)!\dots(2s_h)!} \sum_{J_h^n} \xi_{j_1}^{2s_1} \dots \xi_{j_h}^{2s_h} \sum_{E_k^n} \varepsilon_{j_1}^{2s_1} \dots \varepsilon_{j_h}^{2s_h}$$

$$\leqslant \sum_{h=1}^{h_0} \sum_{S_h^m} \frac{(2m)!}{(2s_1)!\dots(2s_h)!} \|x\|_{2s_1}^{2s_1} \dots \|x\|_{2s_h}^{2s_h} N \left(\frac{k}{n}\right)^h.$$

Now, by the preceding lemma, it follows that

$$N^{-1} \sum_{E_k^n} |\langle x, e\rangle|^{2m} \leqslant \sum_{h=1}^{h_0} \sum_{S_h^m} \frac{(2m)!}{(2s_1)!\dots(2s_h)!} \max\left[\left(\frac{k}{n}\right)^{1/2} \|x\|_2, \left(\frac{k}{n}\right)^{1/2m} \|x\|_{2m}\right]^{2m}.$$

This yields the desired estimate, since

$$\sum_{h=1}^{h_0} \sum_{S_h^m} \frac{(2m)!}{(2s_1)!\dots(2s_h)!} \leqslant h_0^{2m} \leqslant m^{2m}.$$

The complex case can be derived from the real one in the usual way. So, the assertion is proved.

Remark 1. Another proof has been given by the first named author in [2].

Remark 2. It seems very likely that the above inequality remains true for all exponents $r \geqslant 2$.

Remark 3. The classical estimate which is known as Khintchine's inequality appears in the case where $k = n$.

Remark 4. A famous theorem of H. R. Rosenthal [5] yields another generalization of Khintchine's inequality which can also be used to prove the following results, cf. [4].

3. An operator in Minkowski spaces. In the sequel $l_r(E_k^n)$ denotes the Banach spaces of all scalar families $y = (\eta_e)$ equipped with the norm

$$\|y\|_r := \Big\{\sum_{E_k^n} |\eta_e|^r\Big\}^{1/r}.$$

Then $Ax = (\langle x, e\rangle)$ defines an operator A from l_u^n into $l_r(E_k^n)$.

LEMMA 1. *If $2 \leqslant u \leqslant r < \infty$, then*

$$\|A: l_u^n \to l_r(E_k^n)\| \leqslant c_r N^{1/r} \left(\frac{k}{n}\right)^{1/r} \max[k^{1/2-1/r} n^{1/r-1/u}, 1].$$

Here the constant c_r does not depend on $n = 1, 2, \dots$ and $k = 1, \dots, n$.

Proof. Choose a natural number m with $r < r_0 = 2m$. Then there are θ and u_0 such that

$$1/r = (1-\theta)/r_0 + \theta/2 \quad \text{and} \quad 1/u = (1-\theta)/u_0 + \theta/2.$$

The preceding generalization of Khintchine's inequality yields

$$\|A: l_{u_0}^n \to l_{r_0}(E_k^n)\| \leqslant c_{r_0} N^{1/r_0} \left(\frac{k}{n}\right)^{1/r_0} \max[k^{1/2-1/r_0} n^{1/r_0-1/u_0}, 1].$$

In particular, we have

$$\|A: l_2^n \to l_2(E_k^n)\| \leqslant c_2 N^{1/2} \left(\frac{k}{n}\right)^{1/2}, \quad \text{where} \quad c_2 = 1.$$

Therefore, the assertion follows from

$$\|A: l_u^n \to l_r(E_k^n)\| \leqslant \|A: l_{u_0}^n \to l_{r_0}(E_k^n)\|^{1-\theta} \|A: l_2^n \to l_2(E_k^n)\|^{\theta}.$$

LEMMA 2. *If $1 \leqslant r \leqslant u \leqslant 2$, then*

$$\|A: l_u^n \to l_r(E_k^n)\| \leqslant N^{1/r} \left(\frac{k}{n}\right)^{1/u}.$$

Proof. Choose θ and r_0 such that

$$1/u = (1-\theta)/2 + \theta/1 \quad \text{and} \quad 1/r = (1-\theta)/r_0 + \theta/1.$$

It follows from

$$\|A: l_2^n \to l_2(E_k^n)\| \leqslant N^{1/2} \left(\frac{k}{n}\right)^{1/2}$$

and

$$\|I: l_2(E_k^n) \to l_{r_0}(E_k^n)\| \leqslant N^{1/r_0-1/2}$$

that

$$\|A: l_2^n \to l_{r_0}(E_k^n)\| \leqslant N^{1/r_0} \left(\frac{k}{n}\right)^{1/2}.$$

Since we obviously have

$$\|A: l_1^n \to l_1(E_k^n)\| \leqslant N \left(\frac{k}{n}\right)$$

the assertion is implied by

$$\|A: l_u^n \to l_r(E_k^n)\| \leqslant \|A: l_2^n \to l_{r_0}(E_k^n)\|^{1-\theta} \|A: l_1^n \to l_1(E_k^n)\|^{\theta}.$$

4. The limit orders of $\mathfrak{P}_r$ and $\mathfrak{L}_r$.

PROPOSITION 1. *If $2 < u, v < r < \infty$, then*

$$\lambda(\boldsymbol{P}_r, u', v) \leqslant \varrho' := \frac{1}{r} + \frac{(1/u - 1/r)(1/v - 1/r)}{1/2 - 1/r}.$$

Proof. If θ is defined by $1/v = (1-\theta)/r + \theta/2$, then $\varrho' = (1-\theta)/r + \theta/u$. Since we have

$$\boldsymbol{P}_r(I: l_{u'}^n \to l_r^n) \leqslant \|I: l_{u'}^n \to l_\infty^n\| \boldsymbol{P}_r(I: l_\infty^n \to l_r^n) \leqslant n^{1/r}$$

and

$$\boldsymbol{P}_r(I: l_{u'}^n \to l_2^n) \leqslant \|I: l_{u'}^n \to l_1^n\| \boldsymbol{P}_r(I: l_1^n \to l_2^n) \leqslant n^{1/r},$$

an interpolation theorem of B. Carl [1] yields

$$\boldsymbol{P}_r(I: l_{u'}^n \to l_v^n) \leqslant \boldsymbol{P}_r(I: l_{u'}^n \to l_r^n)^{1-\theta} \boldsymbol{P}_r(I: l_{u'}^n \to l_2^n)^{\theta} \leqslant n^{\varrho'}.$$

PROPOSITION 2. *If $2 < u, v < r < \infty$, then*

$$\lambda(\boldsymbol{L}_r, u, v) \leqslant \sigma := \frac{(1/2 - 1/u)(1/v - 1/r)}{1/2 - 1/r}.$$

Proof. We consider $A \in \mathfrak{L}(l_u^n, l_r(E_k^n))$ and $B \in \mathfrak{L}(l_{v'}^n, l_{r'}(E_k^n))$ defined by $x \to (\langle x, e \rangle)$. Lemmas 1 and 2 imply

$$\|A: l_u^n \to l_r(E_k^n)\| \leqslant c_r N^{1/r} \left(\frac{k}{n}\right)^{1/r} \max[k^{1/2-1/r} n^{1/r-1/u}, 1]$$

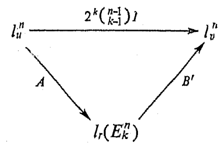
and

$$\|B: l_{v'}^n \to l_{r'}(E_k^n)\| \leqslant N^{1/r'} \left(\frac{k}{n}\right)^{1/v'}.$$

Using

$$\sum_{E_k^n} \varepsilon_i \varepsilon_j = \begin{cases} 2^k \binom{n-1}{k-1} & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

we get the commutative diagram



Therefore,

$$\boldsymbol{L}_r(I: l_u^n \to l_v^n) \leqslant c_r \left(\frac{k}{n}\right)^{1/r-1/v} \max[k^{1/2-1/r} n^{1/r-1/u}, 1].$$

If, in particular, k is the greatest integer not exceeding $n^{(1/u-1/r)/(1/2-1/r)}$, then it follows that

$$\boldsymbol{L}_r(I: l_u^n \to l_v^n) \leqslant c_r^0 n^{\sigma}.$$

Here c_r^0 denotes some constant not depending on $n = 1, 2, \ldots$

Let $\mathfrak{P}_r^{\mathrm{dual}}$ denote the ideal of all operators whose duals are absolutely r-summing.

PROPOSITION 3. *If $1 \leqslant u, v, w \leqslant \infty$ and $1 < r < \infty$, then*

$$1 \leqslant \lambda(\boldsymbol{P}_r^{\mathrm{dual}}, w, u) + \lambda(\boldsymbol{L}_{r'}, u, v) + \lambda(\boldsymbol{P}_{r'}, v, w).$$

Proof. By [3], 19.5.1, we have $\mathfrak{P}_{r'} \circ \mathfrak{L}_r \circ \mathfrak{P}_r^{\mathrm{dual}} \subseteq \mathfrak{I}$, where $\mathfrak{I}$ denotes the ideal of integral operators. Now the assertion follows from [3], 14.4.6, and $\lambda(\boldsymbol{I}, w, w) = 1$.

THEOREM. *If $2 < u, v < r < \infty$, then*

$$\lambda(\boldsymbol{P}_r, u', v) = \varrho' \quad \text{and} \quad \lambda(\boldsymbol{L}_r, u, v) = \sigma.$$

Proof. By Propositions 1 and 2 we have

$$(*) \qquad \lambda(\boldsymbol{P}_r, u', v) \leqslant \varrho' \quad \text{and} \quad \lambda(\boldsymbol{L}_r, u, v) \leqslant \sigma.$$

From [3], 22. 4. 11, we know that $\lambda(\boldsymbol{P}_{r'}, v, v') \leqslant 1/v'$. Obviously

$$\lambda(\boldsymbol{P}_r^{\mathrm{dual}}, v', u) = \lambda(\boldsymbol{P}_r, u', v).$$

Therefore,

$$1 \leqslant \lambda(\boldsymbol{P}_r^{\mathrm{dual}}, v', u) + \lambda(\boldsymbol{L}_r, u, v) + \lambda(\boldsymbol{P}_{r'}, v, v') \leqslant \varrho' + \sigma + 1/v' = 1.$$

This proves that identity holds in $(*)$.

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