STUDIA MATHEMATICA, T. LXVII. (1980)

Spectral characterization of two-sided ideals in Banach algebras

by

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Abstract. We describe all two-sided ideals in Banach algebras that admit characterization by means of spectral properties of the elements. The characterization can be simply expressed even in terms of the spectral radius.

1. Introduction. We consider an arbitrary Banach algebra A over the complex field. For $x \in A$ denote by $\sigma(x)$ the spectrum and by $|x|_\sigma$ the spectral radius of the element x. Let I be a closed two-sided ideal in A. The spectrum of a class $x+I$ in the Banach algebra A/I will be denoted by $\sigma(x+I)$ and the corresponding spectral radius by $|x+I|_\sigma$. If the original algebra A does not have a unit, the spectra are to be understood with respect to A_1 or A_1/I where A_1 is the unitization of A, even if A/I might have its own unit; this convention will simplify our language and does not effect the essence of the results which all the same will be expressible in terms of the spectral radius. It is thus obvious that $\sigma(x+I) \subset \sigma(x)$ for all $x \in A$. Hence if $r \in I$, then

$$\sigma(a+I) \subset \sigma(a+r) \quad \text{for all } a \in A. \tag{1}$$

Now the natural question arises whether, conversely, an element $r \in A$ obeying condition (1) must belong to I. For a general ideal the answer may be no: the zero ideal in a non-zero radical algebra can serve as the simplest example. A less trivial example is the ideal of compact operators relative to the algebra of all bounded operators on the Banach space $L(0,1)$; cf. [3], p. 135 and Section 4 below. Nevertheless, it turns out there is an important class of ideals that admit this natural spectral characterization. Namely, we shall show that, in general, the set of all elements $r \in A$ satisfying condition (1) coincides with $\ker(\mathrm{hul}(I))$, the intersection of all primitive ideals containing I. Moreover, we shall see that an element $r \in A$ satisfies (1) if and only if it satisfies merely

$$|a+I|_\sigma \leqslant |a+r|_\sigma \quad \text{for all } a \in A, \tag{2}$$

the corresponding inequality for the spectral radii.

Thus the ideal I *admits spectral characterization* (that means, it coincides with the set of all elements $r \in A$ satisfying condition (1) or, which is equivalent, condition (2)) if and only if the algebra A has spectral synthesis at I in the sense that $I = \bigcap P$ where P runs over all primitive ideals containing I. In other words, this is the same as to say that the algebra A/I is semi-simple. This happens, for instance, if A is a C^*-algebra with I arbitrary. So our results give also, in particular, a spectral characterization of compact operators on Hilbert spaces as well as indicate the class of Banach spaces where compact operators can be described by means of their spectral behaviour (the corresponding Calkin algebra is to be semi-simple). In the case of operators similar characterizations in terms of the essential spectrum were known as so called converse Weyl's theorems [7], [8]; they will be considerably improved in the present general setting (cf. Section 5 for a more detailed comparison).

Our methods, however, entirely differ from those developed for operators. We incline rather to the previous paper [15] and its sequels [12], [13], [16], [17]. In the course of the proof we shall need spectral characterizations of elements belonging to the Jacobson radical or to the centre (modulo the radical) of a Banach algebra; these results, being announced first in [17], represent here (in Section 2) a local version of our earlier work [15]. They have interesting applications to operator algebras on Banach spaces as well. Since the radical is the intersection of all primitive ideals, the present characterizations can be viewed as the natural extension of those obtained in [16] for the radical.

2. Perturbations of the spectrum. For each $x \in A$ put

$$s(x) = \sup\{|x+y|_\sigma - |y|_\sigma :\ y \in A\}.$$

This function has been introduced by E. A. Gorin in his study of various continuity properties of the spectral radius, and also in [12]. The function s is a semi-norm with possibly infinite values, and $|x|_\sigma \leqslant s(x)$ for all $x \in A$. Moreover, for each invertible element c (in A or A_1) we have

$$s(cxc^{-1}) = s(x)$$

since

$$\begin{aligned}|cxc^{-1}+y|_\sigma - |y|_\sigma &= |c(x+c^{-1}yc)c^{-1}|_\sigma - |y|_\sigma \\ &= |x+c^{-1}yc|_\sigma - |c^{-1}yc|_\sigma\end{aligned}$$

and any element of A can be written as $c^{-1}yc$ for some $y \in A$.

Let N denote the set of all quasi-nilpotent elements of the algebra A, i.e. $N = \{x \in A:\ |x|_\sigma = 0\}$, and $\operatorname{rad} A$ the Jacobson radical of A. Recall that $N \supset \operatorname{rad} A$. Also, the following set

$$Z(A) = \{x \in A:\ ax - xa \in \operatorname{rad} A \text{ for all } a \in A\}$$

arises naturally [17] in spectral characterizations of commutativity properties of Banach algebras. Clearly, $Z(A)$ is a closed subalgebra containing $\operatorname{rad} A$ and we call it the centre modulo the radical.

Before stating the main theorem of this section let us quote two lemmas which we shall need in the proof. The first one is due to C. Le Page [10].

LEMMA 1. *Let $b \in A$ be such that $ab - ba \in N$ for all $a \in A$. Then $ab - ba \in \operatorname{rad} A$ for all $a \in A$, i.e. $b \in Z(A)$.*

The second lemma is a slight modification of the first and in this very form it was proved in paper [13] by Z. Słodkowski, W. Wojtyński and the present author.

LEMMA 2. *Let $r \in N$ be such that $ar - ra \in N$ for all $a \in A$. Then $r \in \operatorname{rad} A$.*

Now we are able to prove the first principal result (cf. also [17]). It will be useful to remember the familiar fact that an element in A has the same spectrum as the corresponding class in $A/\operatorname{rad} A$, in other words, that $\sigma(a+r) = \sigma(a)$ for all $a \in A$, $r \in \operatorname{rad} A$.

THEOREM 1. *Let A be an arbitrary Banach algebra. Let $x \in A$ be a fixed element. Then*

(i) *$x \in Z(A)$ if and only if $s(x) < \infty$;*

(ii) *$x \in \operatorname{rad} A$ if and only if $s(x) = 0$.*

In fact, there are only two possibilities:

$$\begin{aligned} &\textit{either}\ \ s(x) = |x|_\sigma, && \textit{if } x \in Z(A), \\ &\textit{or}\ \ s(x) = \infty, && \textit{if } x \notin Z(A). \end{aligned}$$

Moreover,

$$s(x) = \sup\{|x-y|_\sigma - |x|_\sigma :\ y \in E(x) \cup \{-x\}\}$$

where

$$E(x) = \{e^a x e^{-a}:\ a \in A\}.$$

Proof. Assertion (ii) is an immediate consequence of (i) and of Lemma 2. Let us therefore prove (i).

If $c \in Z(A)$, then $|c+y|_\sigma \leqslant |c|_\sigma + |y|_\sigma$ for all $y \in A$ (consider the corresponding classes of c and y in $A/\operatorname{rad} A$ where they commute) so that $s(c) = |c|_\sigma$ is finite.

Conversely, take an element $b \in A$ such that $s(b) < \infty$ and let us prove that $b \in Z(A)$. To this end let $a \in A$ be an arbitrary fixed element with no restriction on $s(a)$. Consider the entire function

$$f(\lambda) = e^{\lambda a} b e^{-\lambda a} = b + \lambda\delta_a(b) + \frac{\lambda^2}{2!}\delta_a^2(b) + \ldots$$

where

$$\delta_a(b) = ab - ba;$$

cf. [3], p. 88. Then

$$h(\lambda) = \frac{f(\lambda)-b}{\lambda} = \delta_a(b) + \frac{\lambda}{2!}\delta_a^2(b) + \ldots$$

is again an entire function in λ. Since

$$s\big(f(\lambda)\big) = s(b) < \infty,$$

we get

$$|h(\lambda)|_\sigma \leqslant s\big(h(\lambda)\big) \leqslant \frac{2s(b)}{|\lambda|} \to 0 \quad \text{if} \quad |\lambda| \to \infty.$$

According to a theorem of E. Vesentini [14] the function $|h(\lambda)|_\sigma$ is subharmonic on the whole complex plane. Hence it follows that $|h(\lambda)|_\sigma = 0$ for all λ. In particular, for $\lambda = 0$ this gives $|ab-ba|_\sigma = 0$. This conclusion being true for all $a \in A$ we infer, by Lemma 1, that $b \in Z(A)$ as was to be proved. The rest of Theorem 1 is now obvious.

Theorem 1 when applied to all $x \in A$ gives the global spectral characterizations of commutative Banach algebras (subadditivity, submultiplicativity or uniform continuity of the spectral radius) recently obtained in [1], [15], [12]. Another formal consequence of Theorem 1 is the characterization of algebras in which $N = \operatorname{rad} A$, although the original proof in [13] as well as the present proof of Theorem 1 are based on essentially the same ideas (the set N is to be invariant under sums or, which is equivalent, under products). Assertion (ii) seems to be of interest even in its weaker form as follows.

COROLLARY 1. *Let A be a Banach algebra. Then an element $r \in A$ belongs to the radical if and only if $\sigma(a+r) = \sigma(a)$ for all $a \in A$.*

This result we have first observed in [16]. In Sections 4 and 5 analogous spectral characterizations will be found for plenty of other two-sided ideals.

It may be instructive to insert at this point an alternative proof of Corollary 1. Let $r \in A$ be given such that $\sigma(a+r) = \sigma(a)$ for all $a \in A$. Taking an arbitrary element $c \in A$ we show that the element cr satisfies the same condition. Indeed, with λ sufficiently large ($|\lambda| > |c|_\sigma$) we have the decomposition (cf. [7], p. 89)

$$a+cr = (\lambda+c)[(\lambda+c)^{-1}(a-\lambda r)+r]$$

from which it follows, twice using the assumption, that the element $a+cr$ is invertible if and only if a is. In other words, $\sigma(b+cr) = \sigma(b)$ for all $b \in A$. Putting here $b = 0$, we obtain $|cr|_\sigma = 0$. This conclusion being true for any $c \in A$, we infer that $r \in \operatorname{rad} A$ by the classical characterization of the radical. In fact, we have used only a weaker part of the assumption: $a+r$ is invertible whenever a is.

An analogous argument was used by K. Gustafson [7] in his characterization of compact operators on Banach spaces in terms of the essential spectrum (cf. Section 5). Both these proofs are quite elementary but depend strongly on the set condition imposed on the spectra. When we wish to obtain a characterization in terms of the spectral radius only, which is just the main purpose of the present paper, we are obliged to use more ingenious methods. (Added in proof: see [19] or [20] for another proof of Theorem 1.)

We have thus seen that the radical of a Banach algebra is characterized as the set of exactly those elements perturbations by which leave the spectrum completely invariant while the centre, a larger set, consists of just those elements perturbations by which can give rise to merely bounded changes of the spectrum. Every element outside the centre can cause arbitrarily large changes of the spectrum; in particular, this refers to any element from $N \setminus \operatorname{rad} A$.

The concepts of the radical and of the centre arise in the general theory of abstract rings without topology where the notion of spectrum does not make a good sense. It is therefore all the more surprising that in the environment of Banach algebras these two general concepts, the centre and the radical, admit characterizations even in terms of the spectral radius, a phenomenon that seems to disclose the central role which the notion of spectrum or even the spectral radius itself play in the theory of Banach algebras.

3. A formula for the spectrum. The following observation in the case of algebras with unit we owe to B. Aupetit [2]. It is the appropriate non-commutative analogy of the classical description of the spectrum known from Gelfand's commutative theory. I am indebted to Professor W. Żelazko who called my attention to the manuscript [2] when it was not yet published.

To make this formula available also for algebras without unit it will be convenient to allow the zero irreducible representation of such algebras (cf. the case of commutative algebras without unit). Thus, if A does not have a unit, then $P = A$ is regarded as a primitive ideal. If A possesses a unit, we require, as usual, that an irreducible representation of A take the unit onto a non-zero operator.

PROPOSITION 1. *Let A be a Banach algebra. For each $x \in A$ we have*

$$\sigma(x) = \bigcup_P \sigma(x+P)$$

where P runs over all primitive ideals of the algebra A. In particular,

$$|x|_\sigma = \max_P |x+P|_\sigma .$$

Proof. The inclusion "$\supset$" is obvious as already remarked in the introduction. Conversely, take a $\lambda \in \sigma(x)$ and show that $\lambda \in \sigma(x+P)$ for a suitable primitive ideal P. Assume first that A has a unit. Since $\lambda \in \sigma(x)$, the element $\lambda - x$ does not have, say, a left inverse in A. So the set $A(\lambda - x)$ is a proper left ideal in A and hence it is contained in a maximal left ideal L. Put

$$P = L:A = \{a \in A:\ aA \subset L\}.$$

If $\lambda \notin \sigma(x+P)$, then there exists a $y \in A$ such that

$$y(\lambda - x) - 1 \in P \subset L$$

and since also $y(\lambda - x) \in L$, we get $1 \in L$, a contradiction. Thus it is $\lambda \in \sigma(x+P)$.

If A does not have a unit, we proceed similarly. If $\lambda = 0$, put $P = A$. If $\lambda \neq 0$ and $\lambda - x$ does not have, say, a left inverse in A_1, let L_1 be a maximal left ideal in A_1 that contains $\lambda - x$. As $\lambda \neq 0$ we have $L_1 \not\supset A$. Then $L = A \cap L_1$ is a maximal modular left ideal in A (any element $u \in A$ such that $u - 1 \in L_1$ can serve as a right unit modulo L in A). If we put $P = A \cap P_1$ where $P_1 = L_1 : A_1$, then P $(= L:A)$ is a primitive ideal in A and $\lambda \in \sigma(x+P_1) \subset \sigma(x+P)$.

To prove the formula for the spectral radius, take a point $\lambda \in \sigma(x)$ with $|\lambda| = |x|_\sigma$. Then there is a P such that $\lambda \in \sigma(x+P)$. Thus $|x+P|_\sigma \geqslant |\lambda| = |x|_\sigma$. The converse inequality $|x+P|_\sigma \leqslant |x|_\sigma$ being obvious, we have $|x|_\sigma = |x+P|_\sigma$ as desired.

Remark 1. In [15] we have raised the problem of characterizing the Banach algebras with continuous spectral radius. Proposition 1 yields the following observation: if the spectral radius is continuous on all the quotients A/P, then so is on the whole of A. Indeed, let $x_n \to x$ in A. Take a P such that $|x+P|_\sigma = |x|_\sigma$. With this P we have $x_n + P \to x + P$ and so, by assumption, $|x_n + P|_\sigma \to |x+P|_\sigma = |x|_\sigma$. Since $|x_n + P|_\sigma \leqslant |x_n|_\sigma$, we have $\liminf |x_n|_\sigma \geqslant |x|_\sigma$. But $\limsup |x_n|_\sigma \leqslant |x|_\sigma$ by upper semi-continuity of the spectrum. It follows $\lim |x_n|_\sigma = |x|_\sigma$ as claimed. Similar remark concerns the problem of continuity of the spectrum.

We do not know whether, conversely, continuity of the spectrum or of the spectral radius on A forces that on the quotients A/P. If yes, it would reduce the problem to the case of primitive algebras.

We shall apply Proposition 1 rather in the following form.

PROPOSITION 2. *Let I be a closed two-sided ideal of a Banach algebra A. Then for each $x \in A$ we have*

$$\sigma(x+I) = \bigcup_{P \supset I} \sigma(x+P)$$

where P runs over all primitive ideals of A that contain I. Hence again

$$|x+I|_\sigma = \max_{P \supset I} |x+P|_\sigma .$$

4. Spectral characterization of two-sided ideals. We first consider the case of a primitive ideal. The result is as follows.

THEOREM 2. *Let P be a primitive ideal of a Banach algebra A. Then an element $r \in A$ belongs to P if and only if the inequality*

$$|a+P|_\sigma \leqslant |a+r|_\sigma$$

holds for all $a \in A$.

Proof. If $r \in P$, then the inequality holds as we have already mentioned several times. Conversely, let $r \in A$ be an element that satisfies the above inequality for all $a \in A$. For $a = -r$ the condition gives $|r+P|_\sigma = 0$.

We show that the class $r+P$ belongs to $Z(A/P)$, the centre of the semi-simple algebra A/P. In view of Theorem 1 it is enough to verify that the number $s(r+P)$ is finite. But this is obvious from the second expression of the function s (cf. Theorem 1). Indeed, let $q+P \in E(r+P)$. The canonical homomorphism $A \mapsto A/P$ being continuous, we may assume that $q = e^a r e^{-a}$ for some $a \in A$. Now, by assumption,

$$|r-q+P|_\sigma = |-r+q+P|_\sigma \leqslant |q|_\sigma = |r|_\sigma$$

so that

$$s(r+P) \leqslant |r|_\sigma$$

is finite.

From $|r+P|_\sigma = 0$ and $r+P \in Z(A/P)$ we conclude, using Lemma 2, that $r+P \in \operatorname{rad}(A/P) = 0$, i.e. $r \in P$ as claimed. Alternatively one could apply Schur's lemma to the irreducible representation with kernel P.

Now we are able to pass to the general case.

THEOREM 3. *Let I be a closed two-sided ideal in a Banach algebra A. Then the set of all elements $r \in A$ satisfying the condition*

$$|a+I|_\sigma \leqslant |a+r|_\sigma \quad \textit{for all } a \in A$$

coincides with $\ker(\operatorname{hul}(I))$, *the intersection of all primitive ideals containing I.*

Proof. If $r \in \bigcap P$, then we have

$$|a+P|_\sigma \leqslant |a+r|_\sigma, \quad a \in A,$$

for every such P. Hence by Proposition 2 also

$$|a+I|_\sigma \leqslant |a+r|_\sigma, \quad a \in A.$$

Conversely, let $r \in A$ be an element satisfying the above condition.

Since obviously $\sigma(a+P) \subset \sigma(a+I)$ for any $P \supset I$, we get

$$|a+P|_\sigma \leqslant |a+I|_\sigma \leqslant |a+r|_\sigma$$

and so by Theorem 2 we conclude that $r \in P$. This being true for every primitive ideal P containing I, we obtain $r \in \bigcap P$ as desired. Thus the theorem is proved.

COROLLARY 2. *Let I be a closed two-sided ideal in a Banach algebra A. Let $r \in A$ be a fixed element. Then the conditions*

$$\sigma(a+I) \subset \sigma(a+r) \quad \text{for all } a \in A \tag{1}$$

and

$$|a+I|_\sigma \leqslant |a+r|_\sigma \quad \text{for all } a \in A \tag{2}$$

are equivalent. They are fulfilled if and only if $r \in \ker(\text{hul}(I))$.

5. Operators on Banach spaces and the essential spectrum. In this section we apply the general results of Sections 2 and 4 to the algebra $B(X)$ of bounded operators on a complex Banach space X.

COROLLARY 3. *Let $C \in B(X)$ be such that*

$$\sup\{|C+T|_\sigma - |T|_\sigma \colon T \in B(X)\} < \infty$$

or merely

$$\sup\{|C-T|_\sigma \colon T \in E(C)\} < \infty .$$

Then C is a scalar multiple of the identity operator on X.

Proof. The algebra $B(X)$, acting irreducibly on X, is semi-simple. According to Theorem 1 the operator C belongs to the centre of $B(X)$. But by Schur's lemma this centre consists just of scalar multiples of the identity.

COROLLARY 4. *Let $C \in B(X)$ be given. Suppose there is a constant γ such that*

$$|CT|_\sigma \leqslant \gamma |C|_\sigma |T|_\sigma \quad \text{for all } T \in B(X).$$

Then C is a scalar multiple of the identity.

Proof. We shall verify the first condition of Corollary 3. Without loss of generality let $|C|_\sigma < 1$. Take a complex number λ with $|\lambda| > |T|_\sigma + \gamma$. Then $\text{dist}(\lambda, \sigma(T)) > \gamma$ and

$$\lambda - (C+T) = (\lambda - T) - C = \big(1 - C(\lambda - T)^{-1}\big)(\lambda - T)$$

where

$$|C(\lambda-T)^{-1}|_\sigma \leqslant \gamma |C|_\sigma |(\lambda-T)^{-1}|_\sigma = \frac{\gamma |C|_\sigma}{\text{dist}(\lambda, \sigma(T))} \leqslant |C|_\sigma < 1$$

so that $\lambda \notin \sigma(C+T)$. This means, however, that $|C+T|_\sigma \leqslant |T|_\sigma + \gamma$ or $|C+T|_\sigma - |T|_\sigma \leqslant \gamma$ for all $T \in B(X)$.

Remark 2. In this proof we have used the assumption only for T invertible. Thus we can reformulate the result as follows: if

$$\sup\{|CT|_\sigma / |T|_\sigma \colon |T|_\sigma \neq 0\} < \infty,$$

then C is a scalar multiple of the identity.

COROLLARY 5. *Let $R \in B(X)$ be such that $|R+Q|_\sigma = 0$ for all Q quasi-nilpotent. Then $R = 0$.*

Proof. Putting $Q = 0$ we obtain $|R|_\sigma = 0$. Now the conclusion follows by Theorem 1 since $B(X)$ has zero radical.

Let us now turn to the ideal $\mathscr{K}$ of compact operators on X. For an operator $T \in B(X)$ the essential spectrum (sometimes called the Weyl spectrum) is usually defined [7] as the set

$$\omega(T) = \bigcap_{R \in \mathscr{K}} \sigma(T+R).$$

Clearly,

$$\sigma(T+\mathscr{K}) \subset \omega(T) \subset \sigma(T).$$

Moreover, it follows from a theorem of R. Harte [9] (see also P. A. Fillmore, J. G. Stampfli and J. P. Williams [6]) that $\omega(T)$ is contained in the polynomially convex hull of $\sigma(T+\mathscr{K})$. Thus, in particular, $|T+\mathscr{K}|_\sigma = |T|_\omega$ for any $T \in B(X)$ while $|T|_\omega$ may be strictly less than $|T|_\sigma$.

If now $R \in \mathscr{K}$, then obviously

$$\omega(T+R) = \omega(T) \quad \text{for all } T \in B(X). \tag{3}$$

Conversely, but only in the case when X is a Hilbert space, K. Gustafson and J. Weidmann [8] proved a result originally suggested by P. Rejto that condition (3) forces R to be compact. A modification of this result for Banach spaces was later established by K. Gustafson in [7].

Let us compare these converse Weyl theorems with our Corollary 1: if

$$\sigma(T+R) = \sigma(T) \quad \text{for all } T \in B(X), \tag{4}$$

then $R = 0$. Thus we see that condition (4) is, in fact, much stronger than (3) although this actual relation is by no means obvious from the conditions alone.

In order to verify condition (3) one should know the spectra $\sigma(T+\mathscr{K})$ and $\sigma(T+R+\mathscr{K})$, then he should know which holes are to be filled (cf. [6]) to obtain $\omega(T)$ and $\omega(T+R)$ and, finally, he should compare the two resulting sets. Alternatively, to obtain $\omega(T)$ one could also exclude some eigenvalues from $\sigma(T)$, cf. [6]; but it is not again clear which points should be just excluded.

Our general approach, however, provides a considerably simpler procedure. Knowing the spectral radii $|T+\mathscr{K}|_\sigma$ and $|T+R|_\sigma$ only and verifying the inequality $|T+\mathscr{K}|_\sigma \leq |T+R|_\sigma$ between them, for each $T \in B(X)$, we can make the same conclusion about compactness of R. We need know neither the exact structure of the spectra nor which holes are to be filled or which eigenvalues are to be excluded. The precise statement is the following.

COROLLARY 6. *Let $R \in B(X)$ be such that* (3) *is fulfilled. Then R is inessential, that means its image in the algebra $B(X)/\mathscr{K}$ lies in the radical.*

It would be interesting to characterize those Banach spaces X for which the algebra $B(X)/\mathscr{K}$ is semi-simple. A partial information concerning this problem can be found in the book of D. Przeworska-Rolewicz and S. Rolewicz [11].

If we define similarly

$$\omega(a) = \bigcap_{j \in I} \sigma(a+j)$$

for any closed two-sided ideal I of a Banach algebra A, then we can state the following analogy of Corollary 1.

COROLLARY 7. *An element $r \in A$ is inessential* (*i.e.* $r+I \in \operatorname{rad}(A/I)$) *if and only if $\omega(a+r) = \omega(a)$ for all $a \in A$.*

6. An open problem. We are thus returned again to a general Banach algebra A. For simplicity suppose A has a unit. In this last section we wish to compare the condition

$$(5) \qquad \sigma(a+r) = \sigma(a) \quad \text{for all } a \in A$$

with still another weakening of it, namely

$$(6) \qquad \sigma(a+r) \cap \sigma(a) \neq \emptyset \quad \text{for all } a \in A.$$

Condition (6) is satisfied whenever the element r belongs to some proper ($\neq A$) two-sided ideal since then the non-empty set $\sigma(a+I) = \sigma(a+r+I)$ is contained in both $\sigma(a)$ and $\sigma(a+r)$, for any $a \in A$.

These observations lead to the natural question whether, conversely, an element $r \in A$ satisfying condition (6) must belong to some proper two-sided ideal. An affirmative answer is easily seen when the algebra $A/\operatorname{rad} A$ is commutative (from the Gelfand representation). There was only one non-trivial case tried, namely, the case when $A = B(H)$, the algebra of bounded operators on Hilbert space. This is the work of J. A. Dyer, P. Porcelli and M. Rosenfeld [5]. In separable case their result was later strengthened by A. Brown, C. Pearcy and N. Salinas [4] but applying Proposition 7 of C. Le Page [10] to the algebra constructed by A. S. Nemirovskiĭ or J. Duncan and A. W. Tullo (cf. [3], p. 254) it becomes obvious

that the stronger version [4] does not allow transfering to general Banach algebras. However, especially in view of the already established characterization of the radical by condition (5) it seems reasonable to conjecture that condition (6) as a natural weakening of (5) might be a possible spectral characterization of the elements $r \in A$ belonging to (some, but unknown to which) two-sided ideals. (Added in proof: A negative answer is given in [21]. But the conjecture can be corrected taking into consideration the spectrum modulo the strong radical. We intend to explain the details elsewhere.)

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Received March 11, 1977 (1276)

STUDIA MATHEMATICA, T. LXVII. (1980)

Compacts de fonctions mesurables et filtres non mesurables

par

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Resumé. Désignons par $M(\Sigma)$ l'ensemble des fonctions réelles mesurables sur l'espace mesuré (X, Σ, μ) et par τ_p (resp: τ_m) la topologie de la convergence ponctuelle (resp: en mesure) sur $M(\Sigma)$.

On construit un espace pathologique (X, Σ, μ) et une suite de $M(\Sigma)$ qui est relativement compacte mais ne contient aucune sous suite convergente μ-p.p. Ce résultat répond à une question de D. H. Fremlin qui a montré que cette situation est impossible si (X, Σ, μ) est „parfait".

Nous montrons également qu'elle est impossible si Σ est la tribu complétée d'une tribu dénombrablement engendrée. Si $A \subset M(\Sigma)$ est compact pour τ_p et *séparé* pour τ_m, alors τ_p et τ_m coincident sur A, qui est donc métrisable. Ce résultat résoud un problème de A. Ionescu-Tulcea.

La construction de l'exemple est basée sur le fait frappant qu'une intersection dénombrable de filtres non-mesurables est non-mesurable. La méthode qui conduit à ce résultat est utilisée pour une étude systématique des filtres non-mesurables. On étudie également par analogie les filtres non-maigres dont on obtient une caractérisation très simple, qui généralise des résultats connus sur les filtres analytiques.

On étudie enfin l'indépendance pour les filtres de la propriété d'être maigre ou mesurable.

0. Introduction. Cet article développe les résultats annoncées dans deux notes aux Comptes Rendus ([10], [11]).

Étant donné un espace mesuré (X, Σ, μ), nous désignerons par $M(\Sigma)$ l'espace des fonctions réelles mesurables. Nous désignerons par τ_p la topologie produit de $\mathbf{R}^X$, ainsi que la topologie induite sur $M(\Sigma)$, que nous appellerons topologie de la convergence ponctuelle. L'identification d'une partie Y de X et de sa fonction indicatrice χ_Y, définit une topologie sur Σ qui sera encore notée τ_p.

Sur $M(\Sigma)$ nous désignerons par τ_m la topologie de la convergence en mesure, c'est-à-dire la topologie définie par les semi-normes

$$l_E(x) = \int_E \inf(1, |x(t)|)\, d\mu(t) \qquad \forall x \in M(\Sigma)$$

où $E \in \Sigma$ et $\mu(E) < +\infty$.