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Lévy–Khinchine representation and Banach spaces of type and cotype

by

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Abstract. In this paper we prove the following two results: (1) For $1 \leqslant p \leqslant 2$ the Lévy–Khinchine representation, with Lévy measure having p-th moment finite in the neighbourhood of zero, gives an i.d. law on a real separable Banach space E iff E is of type p. (2) Let i.d. law μ be a weak limit of shifts of "exponentials" of non-decreasing sequence of finite measures F_n on E and $F = \lim_n F_n$. Then $\int_E \|x\|^p F(dx)$ is finite ($p \geqslant 2$) iff E is of cotype p. For $p = 2$, these give some recent results of Mandrekar and de Acosta and Samur.

0. Introduction. This work extends some recent works of Mandrekar [14] and de Acosta and Samur [2]. More specifically, we characterize the Banach spaces for which the following problems have solutions. Here E is a real separable Banach space.

I. Every non-Gaussian *infinitely divisible* (i.d., for short) law on E is a limit of a sequence of probability measures of the type $e(F_n) * \delta_{x_n}$ for $\{F_n\}$ non-decreasing sequence of finite measures with $\int_{\|x\| \leqslant 1} \|x\|^p F(dx) < \infty$ ($p \geqslant 2$), where $F = \lim_n F_n$ and F finite outside the neighbourhood of zero and;

II. Every function Ψ on E' (topological dual of E) of the form

$$\exp \int K(y, x) F(dx), \tag{0.1}$$

where

(0.2)
(a) $K(y, x) = \exp\{i\langle y, x\rangle\} - \left(1 + \dfrac{i\langle y, x\rangle}{1 + \|x\|^p}\right)$,

(b) F is σ-finite on E, $F\{0\} = 0$ and F is finite outside the neighbourhood of zero,

(c) $\int_{\|x\| \leqslant 1} \|x\|^p$ is finite $(1 \leqslant p \leqslant 2)$

is the *characteristic functional* (c.f., for short) of a measure (necessarily, i.d.) on E.

For $p = 2$, it was shown [14] that Problem II has solution iff E is of type 2. For $p = 2$, following arguments of [16] (Theorem 4.7, p. 176) we can show that Problem I has solution iff every i.d. law has c.f. of the form (0.1). Thus our result for $p = 2$ ([16], Theorem 7.1, p. 103) give the result of ([2], Theorem 5.2). In general, however, for $p > 2$, the exact analogue of the result of [2] is not valid. This is shown by means of an example in Section 3.

1. Preliminaries and notation. Let E be a real separable Banach space and $\mathscr{B}(E)$, the σ-field generated by all open subsets of E. A probability measure μ on $\mathscr{B}(E)$ is said to be i.d. if for each integer n there exists a probability measure μ_n such that $\mu = \mu_n^{*n}$. It is well known ([18], [16]) that each i.d. probability measure μ on E can be decomposed as $\mu = \gamma * \nu * \delta_x$, where γ is Gaussian in the sense that every continuous linear functional on E has Gaussian distribution under γ, ν is non-Gaussian i.d. and δ_x is point mass at x. Also, every non-Gaussian i.d. law is a weak limit of shifts of measures of type $e(F_n)$ with $\{F_n\}$ non-decreasing sequence of finite measures ([16], p. 103–104).

A sequence of finite measures $\{\mu_n\}$ on $(E, \mathscr{B}(E))$ is said to *converge weakly* to a finite measure μ if $\int g d\mu_n \to \int g d\mu$ for every bounded continuous function g on E. We say that sequence $\{\mu_n\}$ of probability measures is *shift-compact* if there exists a sequence $\{x_n\} \subset E$ such that $\{\mu_n * \delta_{x_n}\}$ is weakly compact. We note that $\{\mu_n\}$ is shift-compact if and only if $\{\mu_n * \overline{\mu}_n\}$ is weakly compact ([16], p. 58–59), where $\overline{\mu}_n(A) = \mu_n(-A)$. Given a finite measure G on $(E, \mathscr{B}(E))$, we denote by $e(G)$ the exponential of G defined by

$$e(G) = \exp\big(-G(E)\big)\left\{\delta_0 + \sum_{n=1}^{\infty} \frac{G^{*n}}{n!}\right\}.$$

We denote by $\langle\ \rangle$ the duality function on $E' \times E$. The c.f. of a probability measure μ is defined to be $\varphi_\mu(y) = \int \exp(i\langle y, x\rangle)\mu(dx)$ for $y \in E'$. It is known that φ_μ determines μ uniquely.

Let $\{\varepsilon_j: j \in N\}$ be a sequence of i.i.d. Bernoulli random variables ($P(\varepsilon_j = -1) = P(\varepsilon_j = 1) = 1/2$). A Banach space E is of (Rademacher) type p if there exists a constant $C > 0$ such that for every finite set $\{x_1, x_2, \ldots, x_n\} \subset E$,

$$E\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|^p \leqslant C \sum_{j=1}^{n} \cdot \|x_j\|^p.$$

It is known that E is of type p if and only if $\sum_j \|x_j\|^p < \infty$ implies $\sum_j \varepsilon_j x_j$ converges *almost surely* (a.s.); and E of type p implies $1 \leqslant p \leqslant 2$ [7].

A Banach space E is of (Rademacher) cotype p if there exists a constant $C > 0$ such that, for every finite set $\{x_1, x_2, \ldots, x_n\} \subset E$,

$$\sum_{j=1}^{n} \|x_j\|^p \leqslant CE\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|^p.$$

It is known that E is of cotype p if and only if $\sum_j \varepsilon_j x_j$ converges a.s. implies $\sum_j \|x_j\|^p < \infty$; and E of cotype p implies $p \geqslant 2$ [7].

2. Lévy–Khinchine representation and spaces of type p. We first prove the following theorem.

2.1. THEOREM. *Problem II has solution in a Banach space E if and only if E is of type p.*

We note that, [6], E is of type p if and only if for any independent E-valued, symmetric random variables $X_1, X_2, \ldots, X_n$,

$$E\Big\|\sum_{j=1}^{n} X_j\Big\|^p \leqslant a \sum_{j=1}^{n} E\|X_j\|^p \tag{2.2}$$

for a universal constant a. In view of (2.2), we get the following lemma with arguments exactly as in ([14], Lemma 1.3).

2.3. LEMMA. *Let G be a finite symmetric measure on $(E, \mathscr{B}(E))$ and E of type p. Then $\int \|x\|^p e(G)(dx) \leqslant a\int \|x\|^p G(dx)$.*

Also as in ([14], Theorem 2.5) we get:

2.4. LEMMA. *Problem II has solution implies E is of type p.*

Proof. Let $\{x_j\}$ be a sequence in E satisfying $\sum_j \|x_j\|^p < \infty$. Write $F = \lim_n \sum_{j=1}^{n} \frac{1}{2}(\delta_{x_j} + \delta_{-x_j})$. Then F satisfies (0.2) (b), (c). Hence

$$\exp\left[\int\left\{e^{i\langle y, x\rangle} - 1 - \frac{i\langle y, x\rangle}{1 + \|x\|^p}\right\}F(dx)\right]$$

is a characteristic functional of a measure ν on E.

$$\varphi_{\nu * \bar{\nu}}(y) = \exp\left\{2\int (e^{i\langle y, x\rangle} - 1)F(dx)\right\} = \lim_n \prod_{j=1}^{n} \varphi_{\nu_j}(y),$$

where ν_j is the law of $2\pi_j x_j$, $\{\pi_j\}$ independent symmetric Poisson with parameter 1. By ([8], p. 40) $\sum_j 2\pi_j x_j$ converges a.s. giving by ([10], Theorem 5.1) $\sum_j x_j \varepsilon_j$ converges a.s., i.e., E is of type p.

To prove the sufficiency part of Theorem 2.1, we note that in view of ([16], p. 58) we can (and will) assume without loss of generality that F, satisfying (0.2) (b) and (c) is zero outside of unit ball of E. Also as in [14] we need the following two lemmas. The proofs being similar to the ones in [16], Theorem 4.7, p. 176, and Theorem 4.5, p. 171, are omitted. But we note that $\int_{\|x\| \leqslant 1} \|x\|^p F(dx) < \infty$ implies $\int_{\|x\| \leqslant 1} \|x\|^2 F(dx) < \infty$, since $p \leqslant 2$.

2.5. LEMMA. *Let F be as in (0.2) (b), (c) and let F_n be the restriction of F to $\{x \mid \|x\| \geq 1/n\}$. Then*

$$\lim_n \sup_{y \in S} \left| \varphi_{e(F_n)*\delta_{z_n}}(y) - \exp\left\{ \int K(y, x) F(dx) \right\} \right| = 0,$$

where S is bounded ball in E' and $z_n = -\int \frac{x}{1+\|x\|^p} F_n(dx)$, integral being in the sense of Bochner.

2.6. LEMMA. *Let $\{\mu_n\}$ be a shift-compact sequence of probability measures on E and suppose $\varphi_{\mu_n}(y) \to \Psi(y)$ uniformly over bounded balls; then $\Psi(y) = \varphi_\mu(y)$ for some probability measure μ and μ_n converges weakly to μ.*

Proof of Theorem 2.1. In view of Lemma 2.4, it remains to prove the sufficiency. In view of Lemmas 2.5 and 2.6 it suffices to prove that $e(F_n+\overline{F}_n)$ is weakly relatively compact. We do this by using Theorem 2.3 of [1]. Let $\lambda_n = e(F_n) * \delta_{z_n}$. We note that ([16], p. 59) by Lemma 2.5, $\{\lambda_n \circ y^{-1}, n = 1, 2, \ldots\}$ is weakly relatively compact on R given by ([16], p. 76), $\{e(F_n+\overline{F}_n) \circ y^{-1}, n = 1, 2, \ldots\}$ weakly relatively compact. It therefore remains to prove that $e(F_n+\overline{F}_n)$ is flatly concentrated. Let $\varepsilon > 0$, choose a simple map ψ such that $\int \|x-\psi(x)\|^p F(dx) < \varepsilon^{p+1}/2a$ with a as in (2.2). This is possible by (0.2) (c) and ([4], p. 226). Then by Chebychev Inequality and Lemma 2.3 we get

$$\sup_n e(F_n+\overline{F}_n)\{x \mid \|x-\psi(x)\| \geq \varepsilon\} \tag{2.7}$$

$$\leq \frac{1}{\varepsilon^p} \sup_n 2a \int \|x-\psi(x)\|^p F_n(dx) < \varepsilon.$$

Let M be the linear subspace generated by the range of ψ. Then $\{x \mid \|x-M\| \geq \varepsilon\} \subset \{x \mid \|x-\psi(x)\| \geq \varepsilon\}$. Hence (2.7) implies $\sup_n e(F_n+\overline{F}_n)\{x \mid \|x-M\| \geq \varepsilon\} < \varepsilon$ giving the result.

Remark. The above proof is similar to the one in [14].

3. Infinitely divisible laws on spaces of cotype p. In this section we prove the following theorem.

3.1. THEOREM. *Problem* I *has solution in a Banach space E if and only if E is of cotype p.*

To prove Theorem 3.1 we follow the method given in [16] due to S. R. S. Varadhan [19].

3.2. DEFINITION. For a probability measure μ on E, concentration function $Q_\mu(t)$ is defined for $0 < t < \infty$, as

$$Q_\mu(t) = \sup_{x \in E} \mu(S_t + x),$$

where S_t is the ball of radius t and $S_t + x$ is the translate by an element x.

The proof of the following lemma is exactly as in [16], p. 166, and hence omitted.

3.3. LEMMA (Lévy inequality). *Let $X_1, X_2, \ldots, X_n$ be independent symmetric E-valued random variables and $S_j = X_1+X_2+\ldots+X_j$ ($j = 1, 2, \ldots, n$). Then for all $t > 0$*

$$P\{\sup_{1 \leq j \leq n} \|S_j\| > 4t\} \leq 2[1-Q_\mu(t)],$$

where μ is the distribution of S_n. (Note that in [16], for $S_t + x \subseteq S_{2t}$ for $\|x\| \leq t$, parallelogram law is used, however, the inclusion is valid in general.

The following analogue of Theorem 3.3 ([16], p. 168) is immediate from the Kolmogorov inequality proved by de Acosta and Samur ([2], Theorem 4.1).

3.4. LEMMA. *Let $X_1, X_2, \ldots, X_n$ be n independent symmetric E-valued random variables uniformly bounded in norm by c. Let $S_j = X_1+X_2+\ldots+X_j$; then*

$$E\|S_n\|^p \leq \frac{d^p + 2^{p-1}(c+d)^p}{1-2^{p-1}P[\sup_{1 \leq j \leq n} \|S_j\| > d]}.$$

From Lemmas 3.3 and 3.4 we get:

3.5. THEOREM. *Let $X_1, X_2, \ldots, X_n$ be E-valued independent symmetric random variables uniformly bounded in norm by c, and let $Q_\mu(t)$ be the concentration function as above. Then*

$$E\|S_n\|^p \leq \frac{4^p t^p + (c+4t)^p 2^{p-1}}{2^p Q_\mu(t) - (2^p-1)}$$

for t satisfying $Q_\mu(t) > 1-1/2^p$.

3.6. THEOREM. *Let E be of cotype p and F_n a sequence of finite measures such that $e(F_n)$ is shift-compact. Then*

$$\sup_n \int_{\|x\| \leq 1} \|x\|^p F_n(dx) < \infty.$$

Proof. In view of Theorem 4.3 ([16], p. 80) we can (and will) assume without loss of generality that F_n is zero outside the unit ball. Let $M_n = F_n + \overline{F}_n$. Then $e(M_n)$ is weakly compact. Further we can assume that $M_n(E)$ is an integer for each n. As otherwise, we can write $M_n = M_n^{(1)} + M_n^{(2)}$ with $M_n^{(1)}$ with total mass an integer and $M_n^{(2)}(E) \leq 1$, consequently $\int_{\|x\| \leq 1} \|x\|^p M_n^{(2)}(dx) \leq 1$. It therefore suffices to prove the theorem for $M_n^{(1)}$. Let $M_n = k_n \mu_n$, where μ_n is a symmetric probability measure. Let $\nu_n = \mu_n^{*k_n}$. Since E is of cotype p, we get that there exists a universal constant β such that

$$\sum_{j=1}^n E\|Z_j\|^p \leq \beta E\Big\|\sum_{j=1}^n Z_j\Big\|^p, \tag{3.7}$$

where Z_j are i.i.d. with distribution μ_n. Hence it suffices to show that $\sup_n \int \|x\|^p d\nu_n < \infty$. If $Q_n(t)$ denotes the concentration function of ν_n, we have from Theorem 3.5

$$\int \|x\|^p d\nu_n \leqslant \frac{4^p t^p + (c+4t)^p 2^{p-1}}{2^p Q_{\nu_n}(t) - (2^p - 1)},$$

whenever $Q_{\nu_n}(t) > 1 - 1/2^p$. Therefore, it is enough to prove the existence of a t_0 such that $\inf_n Q_{\nu_n}(t_0) \geqslant 1 - \frac{1}{2^{p+1}}$. By ([16], Theorem 3.1), the above inequality will follow if we show that ν_n is conditionally compact. But this follows by LeCam ([12], see [11], p. 143).

Proof of Theorem 3.1. It is known ([16], p. 103–104) that on any complete separable metric group a non-Gaussian i.d. law μ is a limit of $e(F_n) * \delta_{x_n}$ with $F_n \uparrow$ to a σ-finite measure F. By ([16], Theorem 4.3, p. 80) we get that F is finite outside the neighbourhood of zero. From Theorem 3.6 we get the necessity. To prove sufficiency assume that for $\{x_j\} \subset E$, $\sum \pi_j x_j$ converges a.s. Then $\nu = \lim_n \prod_{j=1}^n e(\frac{1}{2}\delta_{x_j} + \frac{1}{2}\delta_{-x_j}) = \lim_n e(\sum_{j=1}^n \{\frac{1}{2}\delta_{x_j} + \frac{1}{2}\delta_{-x_j}\})$ is i.d. Since Problem I has solution, we get $\int_{\|x\| \leqslant 1} \|x\|^p F(dx)$ is finite with $F = \sum_{j=1}^\infty \frac{1}{2}(\delta_{x_j} + \delta_{-x_j})$ i.e. $\sum_{j=1}^\infty \|x_j\|^p < \infty$. Hence by closed graph theorem $\exists$ a constant C so that for all n,

$$\sum_{j=1}^n \|x_j\|^p \leqslant CE \Big\| \sum_{j=1}^n \pi_j x_j \Big\|^p \quad (p \geqslant 2). \tag{3.8}$$

Since (3.8) is a super-property if we show

$$c_0 \text{ does not satisfy (3.8)}, \tag{3.9}$$

then we get that c_0 is not finitely representable in E and hence by Corollary 1.3 ([15], p. 25) we get (3.8) is equivalent to E being of cotype p. It thus remains to prove (3.9). Let $\{e_j\}$ be the usual basis in c_0; then for $\{a_j\} \subset R$

$$\Big\| \sum_m^n \pi_j a_j e_j \Big\|_{c_0} = \sup_{m \leqslant j \leqslant n} |\pi_j a_j|.$$

Now, with π Poisson with parameter 2

$$P(\sup_{m \leqslant j \leqslant n} |\pi_j a_j| > \varepsilon) \leqslant \sum_m^n P(|\pi_j a_j| > \varepsilon) = \sum_m^n P(\pi > \varepsilon/a_j). \tag{3.10}$$

But $P(\pi > \varepsilon/a_j) \leqslant \exp(-\varepsilon/a_j) E \exp(\pi)$. Hence from (3.10) we get, with $a_j = j^{-1/p}$, that $\sum \pi_j a_j e_j$ converges a.s. in c_0. But $\sum_{j=1}^\infty \|a_j e_j\|_{c_0}^p = \sum_{j=1}^\infty 1/j$ diverges. This implies (3.9) completing the proof.

Final remark. We now dwell on an example mentioned in the introduction. Let $F_n = \frac{1}{2}\sum_1^n (\delta_{x_j} + \delta_{-x_j})$, and $F = \lim_n F_n$. Suppose $\varphi_\nu(y) = \exp\left[\int [\cos(y,x) - 1] F(dx)\right]$ is c.f. of ν on E. This will imply that $\int_{\|x\| \leqslant 1} (y,x)^2 F(dx) < \infty$ for each y. Choose $x_j \in l_p$ $(p > 2)$ (cotype p) given by $x_j^{(i)} = 1/i^a j^a$, then clearly $\int \|x\|^p F(dx) < \infty$ if $ap > 1$. Further if we choose a, so that $2a < 1$, then $\sum_j \int_{\|x\| \leqslant 1} (x_j^{(i)})^2 F(dx) = \infty$. Hence for a so that $2a < 1 < pa$ we get $\int_{\|x\| \leqslant 1} \|x\|^p F(dx) < \infty$ but $\exp\left[\int (\cos(y,x) - 1) F(dx)\right]$ is not a c.f. In other words, to obtain Lévy–Khinchine type representation one has to assume at least $\int_{\|x\| \leqslant 1} (y,x)^2 F(dx) < \infty$ for $y \in E'$. This explains the conditions put in [5], [12] for l_p or L_p $(p > 2)$.

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Note added in proof. The results of Section 2 were obtained by E. Dettweiler and by E. Giné independently and by different methods.

References

[1] A. de Acosta, *Existence and convergence of probability measures in Banach spaces*, Trans. Amer. Math. Soc. 152 (1970), pp. 273–298.
[2] — and J. D. Samur, *Infinitely divisible probability measures and converse Kolmogorov inequality in Banach space* (preprint).
[3] A. de Araujo, *On infinitely divisible laws in c[0, 1]*, Proc. Amer. Math. Soc. 51 (1975), pp. 179–185.
[4] N. Dinculeanu, *Vector measures*, Pergamon, New York 1967.
[5] G. Hamedani and V. Mandrekar, *Central limit problem on L_p $(p \geqslant 2)$ II; compactness of infinitely divisible laws*, J. Mult. Analysis 7 (1977), pp. 363–373.
[6] J. Hoffman-Jorgenson, *Sums of independent Banach space valued random variables*, Studia Math. 52 (1974), pp. 159–186.
[7] — *Sums of independent Banach space valued random variables*, Aarhus Preprint.
[8] K. Ito and M. Nisio, *On the convergence of sums of independent Banach space valued random variables*, Osaka J. Math. 5 (1968), pp. 35–48.
[9] N. Jain, *Central limit theorem in a Banach space*, Lecture Notes 526, pp. 114–130, Springer-Verlag, New York 1976.
[10] — and M. Marcus, *Integrability of infinite sums of independent vector-valued random variables*, Trans. Amer. Math. Soc. 212 (1975), pp. 1–36.
[11] J. Kuelbs and V. Mandrekar, *Harmonic analysis on F-spaces with a basis*, ibid. 169 (1972), pp. 113–152.

[12] L. LeCam, *Remarque sur le théorème limite central dans les espaces localement convexes*, Les Problémes sur les Structures Algébriques. Colloq. de la Recherche Scientifique, Paris 1970, pp. 233–249.
[13] V. Mandrekar, *Central limit problem on L_p ($p \geqslant 2$) I; Lévy-Khinchine representation*, Lecture Notes 526, pp. 159–166, Springer-Verlag, New York 1976.
[14] — *Lévy-Khinchine representation and Banach space type*, MSU-RM-374.
[15] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendentes et propriétés géométriques des espaces de Banach*, Studia Math. 58 (1976), pp. 45–90.
[16] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York 1967.
[17] A. Tortrat, *Sur la structure des lois infinitement divisibles dans les espaces vectoriels*, Z. Wahrscheinlichkeitstheorie 11 (1969), pp. 311–326.
[18] — *Structure de lois infinitement divisibles dans un espace vectoriel topologique*, Lecture Notes 31, pp. 299–328, Springer-Verlag, New York 1967.
[19] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space*, Sankhyā 24A (1962), pp. 213–238.

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The stability radius of a bundle of closed linear operators

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Abstract. Given a bundle of linear operators $T-\lambda S$, where T is closed and S is bounded, a sequence $\{\gamma_m(T\colon S)\}$ of extended real numbers is defined. When T is the identity operator, $\gamma_m(T\colon S)$ is equal to $\|S^{m-1}\|^{-1}$; when S is the identity operator, $\gamma_m(T\colon S)$ is the reduced minimum modulus $\gamma(T^m)$ of T^m. It is shown that in several important cases (including the case when T is a Fredholm operator and S is arbitrary)

$$\lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}$$

exists and is equal to the supremum of all positive r such that the ranges $R(T-\lambda S)$ are closed and $\dim N(T-\lambda S)$ and $\operatorname{codim} R(T-\lambda S)$ are constant on $0<|\lambda|<r$. This work generalizes the usual spectral radius formula, a recent theorem of K.-H. Förster and M. A. Kaashoek, and an earlier result of H. A. Gindler and A. E. Taylor.

0. Introduction. If S is a bounded linear operator on a Banach space, the usual spectral radius formula implies that

$$\lim_{m\to\infty} \|S^m\|^{-1/m} \tag{0.1}$$

exists and is equal to the supremum of all $r>0$ such that $I-\lambda S$ is a bijective operator on $|\lambda|<r$. Recently, K.-H. Förster and M.A. Kaashoek [6] studied a similar limit, namely

$$\lim_{m\to\infty} \gamma(T^m)^{1/m}, \tag{0.2}$$

where T is a (possibly unbounded) Fredholm operator and $\gamma(T^m)$ is the reduced minimum modulus of T^m. Förster and Kaashoek showed that the limit in (0.2) exists and equals the supremum of all $r>0$ such that the dimensions of the null spaces $N(T-\lambda I)$ and the codimensions of the ranges $R(T-\lambda I)$ are constant on $0<|\lambda|<r$.

In the present paper we describe a general setting which includes the results involving (0.1) and (0.2) as special cases. We consider an operator bundle $T-\lambda S$, where S is a bounded linear operator between two

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