STUDIA MATHEMATICA, T. LXVI. (1980)

# Sets with the Radon–Nikodým property in conjugate Banach space

by

J. BOURGAIN* (Brussel)

**Abstract.** We prove that if $l^1$ does not imbed in $X$, then every subset $C$ of $X^*$ with the Radon–Nikodým property is $w^*$-dentable. From this we deduce that if, moreover, $X$ is separable, those sets in $X^*$ are separable. However, if $X$ is separable and $X^*$ is not separable, then there exists a set in $X^*$ with the Radon–Nikodým property and with non-separable $w^*$-closure.

Let $E$, $\| \ \|$ be a real Banach space. If $x \in E$ and $\varepsilon > 0$, then $B(x, \varepsilon)$ denotes the open ball with midpoint $x$ and radius $\varepsilon$. We establish some notations, referring to [6]. Let $A$ be a non-empty and bounded subset of $E$. If $x^* \in E^*$, we define $M(x^*, A) = \sup x^*(A)$ and if $a > 0$, we let $S(x^*, a, A) = \{x \in A;\ x^*(x) \geqslant M(x^*, A) - a\}$, which will be called a *slice of* $A$. Let $F$ be a subspace of $E^*$ separating the points of $E$. The set $A$ is said to be *$F$-dentable* if for each $\varepsilon > 0$ there is a point $x \in A$ such that $x$ does not belong to the $\sigma(E, F)$-closed convex hull of $A \setminus B(x, \varepsilon)$. By the separation theorem, this is equivalent with the existence of slices $S(x^*, a, A)$ with $x^* \in F$, thus $F$-slices, with arbitrarily small diameter. If in particular $F = E^*$, we say that $A$ is *dentable*. If $E$ is a conjugate space $X^*$ and $F = X$, we say that $A$ is $w^*$-dentable. Obviously $w^*$-dentability implies dentability. This paper deals with the other implication.

A *set with the Radon–Nikodým property* is a bounded, closed and convex subset of $E$ such that each of its non-empty subsets is dentable. For some remarkable properties of these sets, I refer the reader to [1].

A *diadic tree* in $E$ is a bounded sequence $(x_n)_n$ in $E$ so that for some $\varepsilon > 0$ we have that each point $x_p$ is the midpoint of 2 points $x_q$ and $x_r$ with $\|x_q - x_r\| \geqslant \varepsilon$.

We let $X$ be a real Banach space and $X^*$ is dual.

In the first four lemmas, $C$ is a bounded, closed and convex set in $X^*$ and $\tilde{C}$ is its $w^*$-closure.

LEMMA 1. *If there is $\varepsilon > 0$ and a system $((S_{n,k})_{1 \leqslant k \leqslant 2^n})_n$ of $X$-slices of $\tilde{C}$ such that:*

* Aspirant, N.F.W.O., Belgium, Vrije Universiteit Brussel.

1° $S_{n+1,2k-1} \subset S_{n,k}$, $S_{n+1,2k} \subset S_{n,k}$,

2° $\operatorname{dist}(S_{n+1,2k-1}, S_{n+1,2k}) \geqslant \varepsilon$,

3° $\lim_n \operatorname{diam} \sum_{k=1}^{2^n} \frac{1}{2^n} S_{n,k} = 0$,

*then $C$ contains a diadic tree.*

**Proof.** Since for each integer $N$ we have that

$$\sum_{k=1}^{2^{N+n}} \frac{1}{2^{N+n}} S_{N+n,k} = \sum_{K=1}^{2^N} \frac{1}{2^N} \sum_{k=(K-1)2^n+1}^{K2^n} \frac{1}{2^n} S_{N+n,k}$$

it follows that $\lim_n \operatorname{diam} \sum_{k=(K-1)2^n+1}^{K2^n} \frac{1}{2^n} S_{N+n,k} = 0$ for each $K = 1, \ldots, 2^N$.

Therefore, because it is decreasing, the intersection

$$\bigcap_n \sum_{k=(K-1)2^n+1}^{K2^n} \frac{1}{2^n} S_{N+n,k}$$

consists of a unique point $x^*_{N,K}$ which is clearly in $S_{N,K} \cap C$. It is easily verified that $x^*_{N,K} = \frac{1}{2}(x^*_{N+1,2K-1} + x^*_{N+1,2K})$, where $\|x^*_{N+1,2K-1} - x^*_{N+1,2k}\| \geqslant \varepsilon$. Hence $\big((x^*_{N,K})_{1\leqslant K\leqslant 2^N}\big)_N$ is the required diadic tree.

Lemma 2. *Let $(x_n)_n$ be a sequence in $X$ and $\big((S_{n,k})_{1\leqslant k\leqslant 2^n}\big)_n$ a system of $X$-slices of $\tilde{C}$, such that*

1° $\|x_n\| = 1$,

2° $S_{n+1,2k-1} \subset S_{n,k}$, $S_{n+1,2k} \subset S_{n,k}$,

3° $S_{n+1,2k-1} \cap S_{n+1,2k} = \emptyset$,

4° $\lim_n \sup_{1\leqslant k\leqslant 2^n} o(x_m | S_{n,k}) = 0$ *for each integer $m$.*

*If $l^1$ does not inbed in $X$, then*

$$\overline{\lim_n} \sum_{k=1}^{2^n} \frac{1}{2^n} [\inf x_n(S_{n+1,2k-1}) - \sup x_n(S_{n+1,2k})] \leqslant 0.$$

**Proof.** Let $\Delta = \bigcap_n \bigcup_{k=1}^{2^n} S_{n,k}$, which is a $w^*$-compact subset of $\tilde{C}$, and $\Delta_{n,k} = \Delta \cap S_{n,k}$ for each $n \in \boldsymbol{N}$ and $k = 1, \ldots, 2^n$. We consider the $\sigma$-algebra $\mathfrak{S}$ of subsets of $\boldsymbol{N}$, generated by $\{\Delta_{n,k};\ 1 \leqslant k \leqslant 2^n,\ n \in \boldsymbol{N}\}$. Defining $\lambda(\Delta_{n,k}) = 1/2^n$, a probability $\lambda$ on $\mathfrak{S}$ is obtained. By (4), each $x_n$ viewed as a function on $\Delta$ is $\mathfrak{S}$-measurable. Assume the claim untrue. Then there is $\varepsilon > 0$ and an infinite subset $N$ of $\boldsymbol{N}$ with

$$\sum_{k=1}^{2^n} \frac{1}{2^n} [\inf x_n(S_{n+1,2k-1}) - \sup x_n(S_{n+1,2k})] > \varepsilon \quad \text{for each } n \in \boldsymbol{N}.$$



If $l^1$ does not inbed in $X$, then by Rosenthal's theorem [7], $(x_n)_{n\in N}$ has a $w^*$-converging subsequence $(x_n)_{n\in M}$. Applying the Lebesgue theorem, we find some $m \in M$ satisfying $\int |x_m - x_n|\, d\lambda < \varepsilon/3$ whenever $n \in M$ and $n \geqslant m$. Take then some $n \in M$ such that $n \geqslant m$ and $\sup_{1\leqslant k\leqslant 2^n} o(x_m | S_{n,k}) < \varepsilon/3$.

Since

$$\int_{\Delta_{n+1,2k-1}} |x_m - x_n|\, d\lambda \geqslant \frac{1}{2^{n+1}} [\inf x_n(S_{n+1,2k-1}) - \sup x_m(S_{n,k})],$$

$$\int_{\Delta_{n+1,2k}} |x_m - x_n|\, d\lambda \geqslant \frac{1}{2^{n+1}} [\inf x_m(S_{n,k}) - \sup x_n(S_{n+1,2k})],$$

it follows

$$\frac{\varepsilon}{3} > \sum_{k=1}^{2^n} \Big( \int_{\Delta_{n+1,2k-1}} |x_m - x_n|\, d\lambda + \int_{\Delta_{n+1,2k}} |x_m - x_n|\, d\lambda \Big)$$

$$\geqslant \frac{1}{2^{n+1}} \sum_{k=1}^{2^n} [\inf x_n(S_{n+1,2k-1}) - \sup x_n(S_{n+1,2k}) - o(x_m | S_{n,k})]$$

$$> \tfrac{1}{2} \sum_{k=1}^{2^n} \frac{1}{2^n} [\inf x_n(S_{n+1,2k-1}) - \sup x_n(S_{n+1,2k})] - \frac{\varepsilon}{6}.$$

Hence $\sum_{k=1}^{2^n} \frac{1}{2^n} [\inf x_n(S_{n+1,2k-1}) - \sup x_n(S_{n+1,2k})] < \varepsilon$, which is the required contradiction.

Lemma 3. *$E$ will denote the extreme points of $\tilde{C}$. If $S = S(x, \alpha, \tilde{C})$ is an $X$-slice of $\tilde{C}$, we let*

$$\mathring{S} = \mathring{S}(x, \alpha, C) = \{x^* \in \tilde{C};\ x^*(x) > M(x, \tilde{C}) - \alpha\}.$$

*For each $\varepsilon > 0$, there exists an $X$-slice $T$ of $\tilde{C}$, satisfying:*

1° $T \subset S$,

2° $T \subset \bar{o}(\mathring{S} \cap E) + B(0, \varepsilon)$.

**Proof.** Let $d = \operatorname{diam} \tilde{C}$ and take $T = S(x, \beta, \tilde{C})$, where $\beta = \min(\alpha, \varepsilon\alpha/2d)$. Obviously $T \subset S$ and we show that also 2° is verified. We first remark that $\tilde{C} = o\big(\bar{o}(\mathring{S} \cap E) \cup (\tilde{C} \setminus \mathring{S})\big)$. Take $x^* \in T$ and consider $x_1^* \in \bar{o}(\mathring{S} \cap E)$, $x_2^* \in \tilde{C} \setminus \mathring{S}$ and $\lambda \in [0, 1]$ with $x^* = (1-\lambda)x_1^* + \lambda x_2^*$. Since $M(x, \tilde{C}) - \beta \leqslant x^*(x) = (1-\lambda)x_1^*(x) + \lambda x_2^*(x) \leqslant M(x, \tilde{C}) - \lambda\alpha$, it follows that $\lambda \leqslant \varepsilon/2d$ and therefore $\|x^* - x_1^*\| < \varepsilon$. Thus $T \subset \bar{o}(\mathring{S} \cap E) + B(0, \varepsilon)$.

Lemma 4. *If $C$ is not $w^*$-dentable, then there exist $\varepsilon > 0$, a sequence $(x_n)_n$ in $X$ and a system $\big((S_{n,k})_{1\leqslant k\leqslant 2^n}\big)_n$ of slices of $\tilde{C}$, such that:*

1° $\|x_n\| = 1$,

2° $S_{n+1,2k-1}\subset S_{n,k}$, $S_{n+1,2k}\subset S_{n,k}$,
3° $\operatorname{dist}(S_{n+1,2k-1}, S_{n+1,2k})\geqslant \varepsilon$,
4° $\sup_{1\leqslant k\leqslant 2^n} o(x_m|S_{n,k})\leqslant 1/n$ *for each* $m=0,\ldots,n-1$,
5° $\operatorname{diam}\sum_{k=1}^{2^{n+1}}\frac{1}{2^{n+1}}S_{n+1,k}\leqslant \sum_{k=1}^{2^n}\frac{1}{2^n}[\inf x_n(S_{n+1,2k-1})-\sup x_n(S_{n+1,2k})]+$
$+1/(n+1)$

Proof. By Lemma 3, since $C$ is not $w^*$-dentable, there is $\delta>0$ such that $\operatorname{diam}(\mathring{S}\cap E)>\delta$ whenever $S$ is a slice of $\tilde{C}$. Take $\varepsilon=\delta/2$. Starting from $S_{0,1}=\tilde{C}$, we will define $x_n$ and $(S_{n+1,k})_{1\leqslant k\leqslant 2^{n+1}}$ inductively. We use the fact that a $w^*$-neighbourhood of a point in $E$ contains a $w^*$-neighbourhood of that point which is an $X$-slice.

Take $x_0\in X$ with $\|x_0\|=1$ satisfying

$$o(x_0|\mathring{S}_{0,1}\cap E)\geqslant \operatorname{diam}\ (\mathring{S}_{0,1}\cap E)-\tfrac{1}{3}.$$

We consider slices $S'_{1,1}\subset S_{0,1}$, $S'_{1,2}\subset S_{0,1}$ such that

$$\inf x_0(S'_{1,1})\geqslant \sup x_0(\mathring{S}_{0,1}\cap E)-\tfrac{1}{6},\qquad \sup x_0(S'_{1,2})\leqslant \inf x_0(\mathring{S}_{0,1}\cap E)+\tfrac{1}{6}.$$

By Lemma 3, there are slices $S''_{1,1}\subset S'_{1,1}$, $S''_{1,2}\subset S'_{1,2}$ with

$$S''_{1,1}\subset \tilde{c}(\mathring{S}'_{1,1}\cap E)+B(0,\tfrac{1}{6}),\qquad S''_{1,2}\subset \tilde{c}(\mathring{S}'_{1,2}\cap E)+B(0,\tfrac{1}{6}).$$

Finally, it is easily seen how to obtain slices $S_{1,1}\subset S''_{1,1}$, $S_{1,2}\subset S''_{1,2}$ so that $\operatorname{dist}(S_{1,1},S_{1,2})\geqslant\varepsilon$ and $o(x_0|S_{1,1})\leqslant 1$, $o(x_0|S_{1,2})\leqslant 1$. Since

$$\begin{aligned}\operatorname{diam}(\tfrac{1}{2}S_{1,1}+\tfrac{1}{2}S_{1,2})&\leqslant \operatorname{diam}(\tfrac{1}{2}S''_{1,1}+\tfrac{1}{2}S''_{1,2})\\&\leqslant \operatorname{diam}[\tfrac{1}{2}\tilde{c}(\mathring{S}'_{1,1}\cap E)+\tfrac{1}{2}\tilde{c}(\mathring{S}'_{1,2}\cap E)]+\tfrac{1}{3}\leqslant \operatorname{diam}(\mathring{S}_{0,1}\cap E)+\tfrac{1}{3}\\&\leqslant o(x_0|\mathring{S}_{0,1}\cap E)+\tfrac{2}{3}\leqslant \inf x_0(S_{1,1})-\sup x_0(S_{1,2})+1,\end{aligned}$$

all conditions are satisfied.

Assume now $(S_{n,k})_{1\leqslant k\leqslant 2^n}$ obtained. Take $x_n\in X$ with $\|x_n\|=1$ satisfying

$$\sum_{k=1}^{2^n}\frac{1}{2^n}o(x_n|\mathring{S}_{n,k}\cap E)\geqslant \operatorname{diam}\sum_{k=1}^{2^n}\frac{1}{2^n}\left(\mathring{S}_{n,k}\cap E-\frac{1}{3(n+1)}\right).$$

Let $k=1,\ldots,2^n$ be fixed. We consider slices $S'_{n+1,2k-1}\subset S_{n,k}$, $S'_{n+1,2k}\subset S_{n,k}$ such that

$$\inf x_n(S'_{n+1,2k-1})\geqslant \sup x_n(\mathring{S}_{n,k}\cap E)-\frac{1}{6(n+1)},$$
$$\sup x_n(S'_{n+1,2k})\leqslant \inf x_n(\mathring{S}_{n,k}\cap E)+\frac{1}{6(n+1)}.$$



Again by Lemma 3, there are slices $S''_{n+1,2k-1}\subset S'_{n+1,2k-1}$, $S''_{n+1,2k}\subset S'_{n+1,2k}$ with

$$S''_{n+1,2k-1}\subset \tilde{c}(\mathring{S}'_{n+1,2k-1}\cap E)+B\left(0,\frac{1}{6(n+1)}\right),$$
$$S''_{n+1,2k}\subset \tilde{c}(\mathring{S}'_{n+1,2k}\cap E)+B\left(0,\frac{1}{6(n+1)}\right).$$

There are slices $S_{n+1,2k-1}\subset S''_{n+1,2k-1}$, $S_{n+1,2k}\subset S''_{n+1,2k}$ so that

$$\operatorname{dist}(S_{n+1,2k-1},S_{n+1,2k})\geqslant\varepsilon\quad\text{and}\quad o(x_m|S_{n+1,2k-1})\leqslant\frac{1}{n+1},$$
$$o(x_m|S_{n+1,2k})\leqslant\frac{1}{n+1}\quad\text{for each } m=0,\ldots,n.$$

We have that

$$\begin{aligned}\operatorname{diam}\sum_{k=1}^{2^{n+1}}\frac{1}{2^{n+1}}S_{n+1,k}&\leqslant \operatorname{diam}\sum_{k=1}^{2^{n+1}}\frac{1}{2^{n+1}}S''_{n+1,k}\\&\leqslant \operatorname{diam}\sum_{k=1}^{2^{n+1}}\frac{1}{2^{n+1}}\tilde{c}(\mathring{S}'_{n+1,k}\cap E)+\frac{1}{3(n+1)}\\&\leqslant \operatorname{diam}\sum_{k=1}^{2^n}\frac{1}{2^n}\tilde{c}(\mathring{S}_{n,k}\cap E)+\frac{1}{3(n+1)}=\operatorname{diam}\sum_{k=1}^{2^n}\frac{1}{2^n}(\mathring{S}_{n,k}\cap E)+\frac{1}{3(n+1)}\\&\leqslant \sum_{k=1}^{2^n}\frac{1}{2^n}o(x_n|\mathring{S}_{n,k}\cap E)+\frac{2}{3(n+1)}\\&\leqslant \sum_{k=1}^{2^n}\frac{1}{2^n}[\inf x_n(S_{n+1,2k-1})-\sup x_n(S_{n+1,2k})]+\frac{1}{n+1}.\end{aligned}$$

This completes the construction.

THEOREM 1. *If $C$ is a bounded, closed and convex set in $X^*$ which is not $w^*$-dentable, then either $l^1$ inbeds in $X$ or $C$ contains a diadic tree.*

Proof. We consider $\varepsilon$, $(x_n)_n$ and $((S_{n,k})_{1\leqslant k\leqslant 2^n})_n$ as in Lemma 4. If $l^1$ does not inbed in $X$, then

$$\varlimsup_n\sum_{k=1}^{2^n}\frac{1}{2^n}[\inf x_n(S_{n+1,2k-1})-\sup x_n(S_{n+1,2k})]\leqslant 0,$$

by Lemma 2. By (5) of Lemma 4, we find that

$$\varlimsup_n \operatorname{diam}\sum_{k=1}^{2^{n+1}}\frac{1}{2^{n+1}}S_{n+1,k}=0.$$

Now Lemma 1 applies and hence $C$ contains a diadic tree.

COROLLARY 1. *$X$ is an Asplund space if and only if $X^*$ has the RNP.*

Proof. The "only if" part was obtained in [4]. If conversely $X^*$ has RNP, then by Stegall's theorem [8], $l^1$ does not inbed in $X$. If $K$ is a $w^*$-compact convex subset of $X^*$, then $K$ does not contain a diadic tree. By Theorem 1, $K$ is $w^*$-dentable. Therefore $X$ is an Asplund space (see [4]).

COROLLARY 2. *If $l^1$ does not inbed in the separable Banach space $X$, then every subset of $X^*$ with the Radon–Nikodým property is separable.*

Proof. Let $C$ be a Radon–Nikodým set in $X^*$ and assume $C$ not separable. Then there exist $\varepsilon > 0$ and a subset $D$ of $C$ such that if $U$ is $\sigma(X^*, X)$-open and $U \cap D \neq \emptyset$, then $\operatorname{diam}(U \cap D) > \varepsilon$. Hence $\bar{c}(D)$ is not $w^*$-dentable. Thus $\bar{c}(D)$ contains a diadic tree, contradicting the fact that $C$ is Radon–Nikodým.

Remark. The last corollary fails if $l^1$ inbeds in $X$. Indeed, if $X = l^1$, then $l^1(c)$ inbeds in $X^*$ and the closed unit ball of $l^1(c)$ is a non-separable Radon–Nikodým subset of $X^*$.

It is untrue that if $l^1$ does not inbed in the separable Banach space $X$ and $C$ is a Radon–Nikodým set in $X^*$, then the $w^*$-closure $\tilde{C}$ is separable. In fact, we will prove the following:

THEOREM 2. *If $X$ is separable and $X^*$ not separable, then there exists a Radon–Nikodým subset of $X^*$, with non-separable $w^*$-closure.*

LEMMA 5. *Let $(x_n)_n$ be a bounded sequence in $X$ and $(x_n^*)_n$ a bounded sequence in $X^*$ with*

$$\limsup_{m} \sup_{n<m} |x_n^*(x_m)| = 0 \quad \text{and} \quad \underline{\lim}_{m} \inf_{n \geqslant m} x_n^*(x_m) > 0.$$

*Then $C = \bar{c}(x_n^*; n)$ is a Radon–Nikodým convex.*

Proof. Let $\varepsilon = \liminf_{m} \inf_{n \geqslant m} x_n^*(x_m)$ and let $d = \operatorname{diam} C$. Clearly the sequence $(x_n)_n$ is pointwise converging to 0 on $C$. Let now $A$ be a non-empty subset of $C$ and choose a point $a^* \in A$. Let $\delta > 0$ and $\iota = \varepsilon\delta/16d$. Consider $m \in \boldsymbol{N}$ such that $|a^*(x_m)| < \iota$, $\sup_{n<m} |x_n^*(x_m)| < \iota$ and $\inf_{n \geqslant m} x_n^*(x_m) > \varepsilon/2$. If $C_1 = \bar{c}(x_n^*;\ n < m)$ and $C_2 = \bar{c}(x_n^*;\ n \geqslant m)$, then $C = \bar{c}(C_1 \cup C_2)$. Clearly $M(-x_m, A) > -\iota$, $\inf x_m(C_1) \geqslant -\iota$ and $\inf x_m(C_2) > \varepsilon/2$. We will show that $\operatorname{dist}(x^*, C_1) < \delta$ if $x^* \in S(-x_m, \iota, A)$. Indeed, there is $y^* \in C_1$, $z^* \in C_2$ and $\lambda \in [0, 1]$ satisfying $\|x^* - (1-\lambda)y^* - \lambda z^*\| < \delta/2$ and $-(1-\lambda)y^*(x_m) - \lambda z^*(x_m) > M(-x_m, A) - 2\iota$. It follows that $-3\iota < (1-\lambda)\iota - \lambda\varepsilon/2$ and thus $\lambda < 8\iota/\varepsilon = \delta/2d$. Therefore $\operatorname{dist}(x^*, C_1) < \lambda\|y^* - z^*\| + \delta/2 < \delta$, proving the claim.

Hence $S(-x_m, \iota, A)$ admits a finite $2\delta$-covering. By the lemma of Huff and Morris [3], $A$ is dentable. Thus $C$ is a Radon–Nikodým convex.



From Lemma 5, we deduce immediately:

LEMMA 6. *Let $(x_n)_n$ be a bounded sequence in $X$ and $(y_n^*)_n$, $(z_n^*)_n$ bounded sequences in $X^*$, such that:*

*1° $\limsup_{m} \sup_{n} |y_n^*(x_m)| = 0$,*

*2° $|z_n^*(x_m)| < 1/m$ if $n < m$ and $z_n^*(x_m) > 1 - 1/m$ if $n \geqslant m$.*

*If $x_n^* = y_n^* + z_n^*$, then $\bar{c}(x_n^*; n)$ is a Radon–Nikodým convex.*

Proof of Theorem 2. Let $X$ be separable and $X^*$ not separable. Then, by Stegall's result [8], there is a subset $K$ of $X^*$ which is $w^*$-homeomorphic to the Cantor set and a system $((x_{n,k})_{1 \leqslant k \leqslant 2^n})_n$ in $X$ such that $\|x_{n,k}\| \leqslant 2$ and $|x^*(x_{n,k}) - \chi_{n,k}(x^*)| < 1/n$ if $x^* \in K$, where $\chi_{n,k}$ denotes the characteristic function of the Cantor subset $K_{n,k}$. Let $(y_n^*)_n$ be a $w^*$-dense sequence in $K_{1,1}$. For each integer $n$, take $x_n = x_{n,2^n}$ and let $z_n^*$ be some point in $K_{n+1,2^{n+1}-1}$. Let $m \in \boldsymbol{N}$ be fixed. Clearly $\sup_{n} |y_n^*(x_m)| \leqslant 1/m$. If $n < m$, then $K_{n+1,2^{n+1}-1} \cap K_{m,2^m} = \emptyset$ and thus $|z_n^*(x_m)| < 1/m$. If otherwise $n \geqslant m$, then $K_{n+1,2^{n+1}-1} \subset K_{m,2^m}$ and hence $z_n^*(x_m) > 1 - 1/m$. If we let $x_n^* = y_n^* + z_n^*$, then $C = \bar{c}(x_n^*; n)$ has the Radon–Nikodým property by Lemma 6. Denote by $\tilde{C}$ its $w^*$-closure. Because the sequence $(z_n^*)_n$ is $w^*$-converging, its $w^*$-closure $L$ is $\|\ \|$-separable. Since each point $y_n^*$ belongs to $\tilde{C} - L$, we have that $K_{1,1} \subset \tilde{C} - L$ and therefore $\tilde{C}$ is not $\|\ \|$-separable.

## References


[1] J. Bourgain, *On dentability and the Bishop–Phelps property*, Israel J. Math. 28, 4 (1977), pp. 265–271.

[2] J. Diestel, *Geometry of Banach spaces*, Springer, 1975.

[3] R. E. Huff and P. D. Morris, *Geometric characterizations of the Radon–Nikodým property in Banach spaces*, to appear.

[4] I. Namioka and R. R. Phelps, *Banach spaces which are Asplund spaces*, Duke Math. J. 42, 4 (1975).

[5] E. Odell and H. Rosenthal, *A double dual characterization of separable Banach spaces containing $l^1$*, Israel J. Math. 20 (1975), pp. 375–384.

[6] R. R. Phelps, *Dentability and extreme points in Banach spaces*, J. Functional Anal. 17 (1974), pp. 78–90.

[7] H. Rosenthal, *A characterization of Banach spaces containing $l^1$*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), pp. 2411–2413.

[8] C. Stegall, *The Radon-Nikodým property in conjugate Banach spaces*, Trans. Amer. Math. Soc. 206 (1975), pp. 213–223.


*Received April 29, 1977* (1314)