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Interpolation of multipliers of L_Y^p

by

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Abstract. Let G be a Hausdorff compact abelian group with character group Γ. For $Y \subset \Gamma$ denote by L_Y^p the translation invariant subspace of $L^p(G)$ of functions whose Fourier transforms vanish off Y. It is shown that if Γ is infinite and q is an even integer greater than 2, then Y can be chosen so that not every multiplier of $\widehat{L_Y^q}$ is a multiplier of $\widehat{L_Y^p}$, where $p \in [1, q)$ is not an even integer.

Throughout G will denote an infinite Hausdorff compact abelian group with character group Γ. Given $p \in [1, \infty]$, $L^p(G)$ will refer to the usual Lebesgue space with respect to normalised Haar measure λ on G. For $Y \subset \Gamma$ denote by L_Y^p the translation invariant subspace of $L^p(G)$ consisting of those functions whose Fourier transforms vanish off Y. Write M_Y^p for the set of functions in $l^\infty(Y)$ which are multipliers for $\widehat{L_Y^p}$; that is, $\varphi \in M_Y^p$ if and only if for every $f \in L_Y^p$ there exists $g \in L_Y^p$ with $\hat{g}(\gamma) = \varphi(\gamma)\hat{f}(\gamma)$ for all $\gamma \in Y$. To each such φ will correspond a bounded operator T_φ of L_Y^p into itself, given by $T_\varphi f = g$. The norm of T_φ will be written as $\|\varphi\|_{p,p}$.

The special symbols R, T, Z and $Z(r)$ will be reserved for the real line, the circle group, the group of integers and the finite cyclic group of r elements, respectively.

We are interested in the following question:

For $1 \leqslant q < p < 2$ or $2 < p < q \leqslant \infty$, is $M_Y^q \subset M_Y^p$?

This question is posed in [5] and, when $1 \leqslant q < p < 2$, has an affirmative answer for any G and every $Y \subset \Gamma$; see [6]. On the other hand, it has been shown (see [5]) that in the particular case where G is a product of cyclic groups of order n $(n \geqslant 3)$ and $q = 2n-2$ the answer can be negative. We show here that this latter phenomenon persists for any G whenever q is an even integer greater than 2.

THEOREM. *Let G be an infinite Hausdorff compact abelian group, $s > 1$ an integer and $p \in [1, 2s)$ not an even integer. Then there exists $Y \subset \Gamma$ and $\varphi \in l^\infty(Y)$ with $|\varphi| = 1$ such that $\varphi \in M_Y^{2s}$ but $\varphi \notin M_Y^p$.*

Our proof of the theorem uses, in part, techniques developed previously in the investigation of the upper majorant property for $L^p(G)$. In particular, the trigonometric polynomials we choose are precisely those first introduced by Hardy and Littlewood (see [3]) and later generalized by Boas [1] and Fournier [2]. We consider the following four cases: (i) $\Gamma = Z$; (ii) Γ has an element of infinite order; (iii) Γ has elements of arbitrarily large finite order; and (iv) Γ is of bounded order.

(i) $\Gamma = Z$. First consider $Y_1 = \{0, 1, s+1\}$ and $\varphi_1 \in l^\infty(Y_1)$ defined by $\varphi_1(0) = \varphi_1(1) = 1$, $\varphi_1(s+1) = -1$. Then $\|\varphi_1\|_{2s,2s} = 1$. However, the proof of Theorem 1 in [1] can be adapted to show that $\|\varphi_1\|_{p,p} = A > 1$.

For each $f = 1 + ae_1 + be_{s+1}$ and positive integer m write $f^{(m)} = 1 + ae_m + be_{m(s+1)}$; here $e_m\colon x \to e^{imx}$. It is well known that, for any $q \in [1, \infty)$,

$$\lim_{m\to\infty} \|f\cdot f^{(m)}\|_q = \|f\|_q^2.$$

Fix $\delta > 0$ and let $f \in L^p_{Y_1}$, $\varepsilon \in (0, 1)$ be chosen so that

$$\|T_{\varphi_1} f\|_p > (A-\delta)\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/2} \|f\|_p.$$

Pick m_1 sufficiently large (and, in any case, greater than $s(s+1)$) so that $\|f\cdot f^{(m_1)}\|_p < (1+\varepsilon)\|f\|_p^2$ and $\|T_{\varphi_1} f \cdot T_{\varphi_1} f^{(m_1)}\|_p > (1-\varepsilon)\|T_{\varphi_1} f\|_p^2$. Write $Y_2 = Y_1 + m_1 Y_1$ and define φ_2 on Y_2 by $\varphi_2(q + m_1 q') = \varphi_1(q)\varphi_1(q')$, $q, q' \in Y_1$. This definition of φ_2 makes sense since $m_1 > s+1$ implies that every number in Y_2 has a unique representation as $q + m_1 q'$ for some $q, q' \in Y_1$. It follows that

$$\begin{aligned}\|T_{\varphi_2}(f\cdot f^{(m_1)})\|_p &= \|T_{\varphi_1} f \cdot T_{\varphi_1} f^{(m_1)}\|_p\\ &> (1-\varepsilon)\|T_{\varphi_1} f\|_p^2\\ &> (A-\delta)^2(1+\varepsilon)\|f\|_p^2\\ &> (A-\delta)^2\|f\cdot f^{(m_1)}\|_p\end{aligned}$$

so that, with $\delta = \dfrac{A-1}{2}$, $\|\varphi_2\|_{p,p} > \left(\dfrac{A+1}{2}\right)^2$. Furthermore, for any $g \in L^{2s}_{Y_2}$, overlaps in the frequencies when the product

$$g^s = \Big(\sum_{n\in Y_2} \hat{g}(n) e_n\Big)^s$$

is expanded occur in such a way that

$$\|g\|_{2s}^{2s} = \int_T \Big|\Big(\sum_{n\in Y_2} \hat{g}(n) e_n\Big)^s\Big|^2 d\lambda$$

is the same as $\|T_{\varphi_2} g\|_{2s}^{2s}$ (use the property $m_1 > s(s+1)$). Hence $\|\varphi_2\|_{2s,2s} = 1$.

Iterating the above procedure, we obtain an increasing sequence (Y_n) of subsets of Z, where $Y_{n+1} = Y_n + m_n Y_n$, and multipliers $\varphi_{n+1} \in l^\infty(Y_{n+1})$ defined by $\varphi_{n+1}(q + m_n q') = \varphi_n(q)\varphi_n(q')$, $q, q' \in Y_n$. Here m_n is chosen sufficiently large so that $\|\varphi_n\|_{p,p} > \left(\dfrac{A+1}{2}\right)^{2^{n-1}}$, and in any case larger than $s(s+1)\prod_{i=1}^{n-1}(m_i+1)$ which guarantees us that $\|\varphi_n\|_{2s,2s} = 1$; for this equality use the result $p_i, q_i \in Y_n$, $\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i$ implies $\prod_{i=1}^{s}\varphi_n(p_i) = \prod_{i=1}^{s}\varphi_n(q_i)$. Now put $Y = \bigcup_{n=1}^{\infty} Y_n$ and define φ on Y by $\varphi|_{Y_n} = \varphi_n$. It is easy to check that φ is well defined, $|\varphi| = 1$, $\|\varphi\|_{2s,2s} = 1$ and $\|\varphi\|_{p,p} > \left(\dfrac{A+1}{2}\right)^{2^{n-1}}$ for all n. Consequently, since $A > 1$, $\varphi \notin M^p_Y$.

(ii) *Γ has an element of infinite order.* Here Γ contains a copy of Z, so there is a continuous surjection $\alpha\colon G \to T$. For any trigonometric polynomial f on T, $f\circ\alpha$ is a trigonometric polynomial on G with the same set of Fourier coefficients and the same q-norm for all $q \in [1, \infty)$ (for the latter see [4], (28.54)). The results of the first case now apply.

(iii) *Γ has elements of arbitrarily large finite order.* Using the results of the proof of case (i), choose $Y_n, \varphi_n \in M^{2s}_{Y_n}$, $f_n \in L^p_{Y_n}$ and $A > 1$ such that

$$\|T_{\varphi_n} f_n\|_p > \left(\frac{A+1}{2}\right)^{2^{n-1}} \|f_n\|_p.$$

Let $\gamma_n \in \Gamma$ have order r_n, where r_n is to be chosen. Then $\gamma_n(G)$ is the set $\{\omega_j\}_{j=1}^{r_n}$ of r_n-th roots of unity; for each j the set of x in G for which $\gamma_n(x) = \omega_j$ has measure r_n^{-1}. Now

$$F_n = f_n \circ \gamma_n = \sum_{j\in Y_n} \hat{f}_n(j)\gamma_n^j$$

and

$$G_n = T_{\varphi_n} f_n \circ \gamma_n = \sum_{j\in Y_n} \varphi_n(j)\hat{f}_n(j)\gamma_n^j$$

are trigonometric polynomials on G. Also

$$\int_G |F_n|^p\, d\lambda = \frac{1}{r_n}\sum_{j=1}^{r_n} |f_n(\omega_j)|^p,$$

and the right-hand side of this equation is a Riemann sum for

$$\frac{1}{2\pi}\int_0^{2\pi} |f_n(e^{i\theta})|^p\, d\theta = \|f_n\|_p^p.$$

Similarly, $\|G_n\|_p^p$ is equal to a Riemann sum for $\|T_{\varphi_n} f_n\|_p^p$. Hence for r_n

sufficiently large (and, in any case, larger than $s\max\{m\colon m\in Y_n\}$)

$$\|G_n\|_p > \left(\frac{A+1}{2}\right)^{2^{n-1}} \|F_n\|_p.$$

Define Φ_n on $\Omega_n = \{\gamma_n^j\colon j\in Y_n\}$ by $\Phi_n(\gamma_n^j) = \varphi_n(j)$ (the condition on r_n is more than sufficient to ensure that Φ_n is well defined). By the previous inequality, $\|\Phi_n\|_{p,p} > \left(\frac{A+1}{2}\right)^{2^{n-1}}$. We shall show that $\|\Phi_n\|_{2s,2s} = 1$. Consider $g\in L^{2s}_{\Omega_n}$; $g = \sum_{j\in Y_n} c_j\gamma_n^j$. Then

$$g^s = \sum_{j_t\in Y_n} c_{j_1}c_{j_2}\dots c_{j_s}\gamma_n^{j_1+j_2+\dots+j_s}.$$

An argument similar to that used in the proof of case (i) will show that $\|T_{\Phi_n}g\|_{2s} = \|g\|_{2s}$, since the inequalities

$$r_n > s\max\{m\colon m\in Y_n\} \geqslant \max\{j_1+j_2+\dots+j_s\colon j_t\in Y_n\}$$

imply that, for $j_t, j_t'\in Y_n$,

$$\gamma_n^{j_1+j_2+\dots+j_s} = \gamma_n^{j_1'+j_2'+\dots+j_s'}$$

if and only if $j_1+j_2+\dots+j_s = j_1'+j_2'+\dots+j_s'$.

To complete the proof we need to impose more restrictions on the choice of γ_n and r_n. Assume that $\gamma_1, \gamma_2, \dots, \gamma_{n-1}$ have been chosen and select γ_n with order $r_n > r_1r_2\dots r_{n-1}\, s\max\{m\colon m\in Y_n\}$; this condition guarantees that $\Omega_n\cap\Omega_{n'} = \{1\}$ for $n\neq n'$ and, since $\Phi_n(1) = 1$ for all n, we can define Φ on $\Omega = \bigcup_{n=1}^{\infty}\Omega_n$ by $\Phi|_{\Omega_n} = \Phi_n$. Clearly, $\|\Phi\|_{p,p} > \left(\frac{A+1}{2}\right)^{2^{n-1}}$ for all $n = 1, 2, \dots$, so that $\Phi\notin M^p_\Omega$. We shall show that the choice of r_n ensures that $\Phi\in M^{2s}_\Omega$. Let $g\in L^{2s}_\Omega$; $g = \sum_{n=1}^{\infty}\sum_{j\in Y_n} c_{j,n}\gamma_n^j$. Then

$$g^s = \sum_{j_t\in Y_{n_t}} c_{j_1,n_1}c_{j_2,n_2}\dots c_{j_s,n_s}\gamma_{n_1}^{j_1}\gamma_{n_2}^{j_2}\dots\gamma_{n_s}^{j_s}.$$

Now $\gamma_{n_1}^{j_1}\gamma_{n_2}^{j_2}\dots\gamma_{n_s}^{j_s} = \gamma_{m_1}^{k_1}\gamma_{m_2}^{k_2}\dots\gamma_{m_s}^{k_s}$ (with $j_t\in Y_{n_t}$, $k_t\in Y_{m_t}$, $1\leqslant t\leqslant s$) only holds if, for each positive integer n,

$$\sum_{n_t\in A_n} j_t = \sum_{m_t\in A_n'} k_t,$$

where $A_n = \{n_t\colon n_t = n\}$ and $A_n' = \{m_t\colon m_t = n\}$. This is easily deduced from the property $r_n > r_1r_2\dots r_{n-1}\, s\max\{m\colon m\in Y_n\}$. It follows, using the result that $\sum_{n_t\in A_n} j_t = \sum_{m_t\in A_n'} k_t$ implies $\prod_{n_t\in A_n}\Phi_n(\gamma_n^{j_t}) = \prod_{m_t\in A_n'}\Phi_n(\gamma_n^{k_t})$ for each positive integer n, that $\|\Phi\|_{2s,2s} = 1$.

(iv) *Γ is of bounded order.* Now ([4], (A.25))

$$\Gamma \simeq \Gamma_0\times\prod_{j=1}^{\infty}{}^{*}Z(r),$$

where the asterisk indicates a weak direct product. We first note that it suffices to find $n\geqslant 1$, $Y'\subset\prod_{j=1}^{n}Z(r)$ and $\varphi\in l^\infty(Y')$ such that $\|\varphi\|_{2s,2s} = 1$ and $\|\varphi\|_{p,p} > 1$. For then we have ([5], Lemma 2.4) that Φ defined on $Y = \{1\}\times\prod_{k=1}^{\infty}{}^{*}Y'$ by

$$\Phi\big((1, \chi_1, \chi_2, \dots, \chi_n, \chi_1', \chi_2', \dots, \chi_n', \dots)\big) = \varphi\big((\chi_1, \chi_2, \dots, \chi_n)\big)\varphi\big((\chi_1', \chi_2', \dots, \chi_n')\big)$$

(note that all but a finite number of the entries are unity) is in M^{2s}_Y but not in M^p_Y.

Fix n and, for each positive integer $k\leqslant n$, let γ_k denote that element of $\prod_{j=1}^{n}Z(r)$ which has 1 in the k-th entry and zero elsewhere. Consider the trigonometric polynomial

$$f = 1+t(\gamma_1+\gamma_2+\dots+\gamma_n)+\omega t^n\gamma_1\gamma_2\dots\gamma_n,$$

where $t\in R^+$ and $|\omega| = 1$. For t suitably small,

$$f^{p/2} = \sum_{m=0}^{\infty}\binom{p/2}{m}[t(\gamma_1+\gamma_2+\dots+\gamma_n)+\omega t^n\gamma_1\gamma_2\dots\gamma_n]^m.$$

Put $\chi = \gamma_1^{d_1}\gamma_2^{d_2}\dots\gamma_n^{d_n}$, where $d_j\in\{0, 1, \dots, r-1\}$, and let l denote the cardinality of the set of non-zero d_j's.

In the binomial expansion of $f^{p/2}$ terms involving χ can arise as follows. For those not containing ω the coefficient of χ is given by $\binom{p/2}{d}Kt^d+o(t^d)$, where $d\geqslant l$ and K is a positive integer depending on χ. For those containing ω the coefficient is $\omega\left[\binom{p/2}{d'+1}K't^{n+d'}+o(t^{n+d'})\right]$, where $d'\geqslant(n-l)(r-1)$ and K' is a positive integer depending on χ. It follows that the portion of the coefficient of χ involving ω contributes

$$(*)\qquad \binom{p/2}{d}K\binom{p/2}{d'+1}K'(\omega+\bar\omega)t^{n+d+d'}+o(t^{n+d+d'})$$

to the evaluation of $\|f\|_p^p = \int|f^{p/2}|^2\,d\lambda$. Every character χ in the subgroup of Γ generated by $\gamma_1, \gamma_2, \dots, \gamma_n$ gives rise to a term of the above form. Furthermore, $n+d+d'\geqslant n+l+(n-l)(r-1) = nr+l(2-r)$. We examine such terms which are minimal with respect to the power of t; there are two cases to consider.

If $r > 2$, then $n+d+d' \geqslant nr+n(2-r) = 2n$ (recall that $l \leqslant n$). It follows that the term giving the minimum power of t (namely, t^{2n}) occurs when $d = l = n$ and $d' = 0$, in which case $\chi = \gamma_1\gamma_2\ldots\gamma_n$. Here $K = n!$, $K' = 1$, and (*) becomes

$$\binom{p/2}{n} n! \binom{p/2}{1} (\omega+\bar{\omega}) t^{2n} + o(t^{2n}).$$

For p not an even integer, $\binom{p/2}{n} n! \binom{p/2}{1} \neq 0$.

If $r = 2$, then $n+d+d' \geqslant 2n$ and, since $d \geqslant l$ and $d' \geqslant n-l$, the terms giving the minimum power of t (again, t^{2n}) occur when $d = l$ and $d' = n-l$; here l can take any value from $\{0, 1, \ldots, n\}$. Clearly, for each χ and corresponding l, $K = l!$, $K' = (n-l+1)!$, and (*) becomes

$$\binom{p/2}{l} l! \binom{p/2}{n-l+1} (n-l+1)! (\omega+\bar{\omega}) t^{2n} + o(t^{2n}).$$

Summing over all possible χ, we obtain

$$\sum_{l=0}^{n} \binom{n}{l} \binom{p/2}{l} l! \binom{p/2}{n-l+1} (n-l+1)! (\omega+\bar{\omega}) t^{2n} + o(t^{2n})$$

$$= n! \sum_{l=0}^{n} \binom{p/2}{l} \binom{p/2}{n-l+1} (n-l+1)(\omega+\bar{\omega}) t^{2n} + o(t^{2n})$$

$$= \frac{(n+1)!}{2} \binom{p}{n+1} (\omega+\bar{\omega}) t^{2n} + o(t^{2n}).$$

(The last equality can be proved by noting that $\sum_{l=0}^{n} \binom{p/2}{l} \binom{p/2}{n-l+1} (n-l+1)$ is the coefficient of z^{n+1} in the Maclaurin expansion of $(1+z)^{p/2} z\frac{d}{dz}(1+z)^{p/2}$.) For p not an integer, $\frac{(n+1)!}{2} \binom{p}{n+1} \neq 0$.

Write $g = 1+t(\gamma_1+\gamma_2+\ldots+\gamma_n)+t^n\gamma_1\gamma_2\ldots\gamma_n$. Then, for $\omega \neq 1$ and t suitably small, the working above shows that n can be chosen so that $\|f\|_p \neq \|g\|_p$. This means that the multiplier $\varphi \in l^\infty\big((1, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1\gamma_2\ldots\gamma_n)\big)$ defined by the sequence $(1, 1, \ldots, 1, \omega)$ has $\|\varphi\|_{p,p} > 1$. We now show that, for ω a primitive r-th root of unity and $n \geqslant 2s$, $\varphi \in M_Y^{2s}$, where $Y = \{1, \gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_1\gamma_2\ldots\gamma_n\}$. Consider any $h \in L_Y^{2s}$; $h = a_0+a_1\gamma_1+\ldots+a_n\gamma_n+a\gamma_1\gamma_2\ldots\gamma_n$. Then

$$T_\varphi h = a_0+a_1\gamma_1+\ldots+a_n\gamma_n+a\omega\gamma_1\gamma_2\ldots\gamma_n.$$

In the expansion of $(T_\varphi h)^s$ the coefficient of the character $\chi = \gamma_1^{d_1}\gamma_2^{d_2}\ldots\gamma_n^{d_n}$ $(0 \leqslant d_j < r)$ is a sum of terms $a_0^{i_0}a_1^{i_1}\ldots a_n^{i_n}(a\omega)^{i_{n+1}}$ for which $\sum_{j=0}^{n+1} i_j = s$ and $\gamma_1^{i_1}\gamma_2^{i_2}\ldots\gamma_n^{i_n}(\gamma_1\gamma_2\ldots\gamma_n)^{i_{n+1}} = \chi$. Thus it follows from the lemma below that T_φ is an isometry.

This completes the proof of case (iv), and hence of our theorem.

It remains to give a statement of the lemma promised above; the proof is straightforward, and will be omitted.

LEMMA. *Suppose that n and $d_1, d_2, \ldots, d_n$ are integers such that $n \geqslant 2s$ and $0 \leqslant d_j < r$, $1 \leqslant j \leqslant n$. There exists an integer c with $0 \leqslant c < r$ such that the following holds. If $i_1, i_2, \ldots, i_{n+1}$ are non-negative integers for which $i_j + i_{n+1} = d_j \pmod r$ $(1 \leqslant j \leqslant n)$ and $\sum_{j=1}^{n+1} i_j \leqslant s$, then $i_{n+1} = c \pmod r$.*

From this result we can deduce Theorem 4.1 of [5] which states that, for Γ infinite and $s > 1$ an integer, $Y \subset \Gamma$ can be chosen so that not every multiplier of $\widehat{L_Y^{2s}}$ extends to a multiplier of $\widehat{L^{2s}}(G)$. Indeed we just choose $p \in (2, 2s)$ not an even integer, and $Y \subset \Gamma$, $\varphi \in M_Y^{2s} \setminus M_Y^p$ as in our theorem. It follows, using the Riesz–Thorin convexity theorem, that φ is not the restriction of a multiplier of $\widehat{L^{2s}}(G)$.

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