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Finite-dimensional subspaces of uniformly convex and uniformly smooth Banach lattices and trace classes S_p

by

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Abstract. It is proved that if X is a Banach lattice whose moduli of convexity and smoothness admit the estimations $\delta_x(\varepsilon) \geqslant K_1\varepsilon^p$ and $\varrho_x(\tau) \leqslant K_2\tau^q$, where $q = 2$ if $p > 2$ and $1 < q \leqslant 2$ if $p = 2$, then there is a constant K such that for every n-dimensional subspace $F \subset X$ one has $d(F, l_2^n) \leqslant Kn^{|1/p-1/q|}$. Here $d(F, l_2^n)$ denotes the Banach–Mazur distance. If F is any n-dimensional subspace of the trace class S_p, then $d(F, l_2^n) \leqslant n^{|1/p-1/2|}$.

Introduction. In [14] Lewis proved that for $1 < p < \infty$ the space $L_p(\Omega, \mu)$ has the following property:

(*) *there is a constant C such that every n-dimensional subspace E satisfies* $d(E, l_2^n) \leqslant C\cdot n^{|1/p-1/2|}$.

In this paper we will prove that some general classes of Banach lattices satisfy (*). Moreover, it will be shown that "a noncommutative generalization" of l_p also has this property. In the proofs of ours basic results (Theorems 1.1, 1.2 and 2.2, 2.3) we follow the ideas of Lewis from [14].

For the convenience of the reader some notions and facts which will be used in both sections are put together in Section 0.

In Section 1 we examine uniformly convex and uniformly smooth Banach lattices. We shall show that any Banach lattice X whose moduli of convexity and smoothness admit the estimations

$$\delta_X(\varepsilon) \geqslant K_1\varepsilon^p \quad \text{and} \quad \varrho_X(\tau) \leqslant K_2\tau^2, \quad \text{whenever} \quad p > 2, \text{ or}$$
$$\delta_X(\varepsilon) \geqslant K_1\varepsilon^2 \quad \text{and} \quad \varrho_X(\tau) \leqslant K_2\tau^p, \quad \text{whenever} \quad 1 < p \leqslant 2,$$

has property (*) (Corollary 1.10). This already implies that any Banach lattice X of type 2 and cotype p whenever $p > 2$, or of type 2 and cotype p whenever $1 < p \leqslant 2$, satisfies (*). Let us note that the reader can more easily understand proofs considering them for atomic lattices, i.e. spaces with unconditional bases.

In Section 2 we prove that the trace classes S_p of operators acting in a Hilbert space H have property $(*)$ (Corollary 2.10). Let us recall that the spaces S_p are uniformly convex and uniformly smooth and their moduli of convexity and smoothness have the same order as the corresponding moduli of L_p ([19]). However, if $p \neq 2$ and $\dim H = \infty$, S_p is not isomorphic to complemented subspace of any Banach lattice ([7]). It has also been recently proved in [17] that for $p \neq 2$ and $\dim H = \infty$ the space S_p is not isomorphic to a quotient of a subspace of a uniformly convex Banach lattice.

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§ 0. Preliminaries. The notation and terminology in the theory of Banach spaces is standard. If X is a Banach space and X^* its dual, the value of a functional $y^* \in X^*$ on an element $x \in X$ is denoted by (x, y^*). For $x \in X$, $x \neq 0$ we say that the functional $x^* \in X^*$ with $\|x^*\| = 1$ is the *supporting functional at* x whenever $(x, x^*) = \|x\|$.

Let X and Y be Banach spaces. By $\mathscr{F}(X, Y)$ we denote the space of all finite rank linear operators from X to Y. For $u \in \mathscr{F}(X, X^{**})$ we define the trace of u as a trace $u = \sum_{i=1}^{\infty} (e_i^*, u(e_i))$, where $\{(e_i), (e_i^*)\}$ is some basis in X. Let α be a norm on the space $\mathscr{F}(X, Y)$. The *completion* of $\mathscr{F}(X, Y)$ under the norm α is denoted by $\alpha(X, Y)$. By α^* we mean the norm on $\mathscr{F}(Y, X^{**})$ defined by

$$\alpha^*(v) = \sup |\text{trace } v \circ u| \tag{1}$$

for $v \in \mathscr{F}(Y, X)$, with the supremum taken over all $u \in \mathscr{F}(X, Y)$ with $\alpha(u) \leqslant 1$. Then the dual of $\alpha(X, Y)$ is isometric to $\alpha^*(Y, X^{**})$. The duality is given via the trace: $(u, w) = \text{trace } w \circ u$ for $u \in \alpha(X, Y)$ and $w \in \alpha^*(Y, X^{**})$. Moreover, the following equality holds

$$\alpha(u) = \sup |\text{trace } v \circ u| \tag{2}$$

for $u \in \mathscr{F}(X, Y)$, with the supremum taken over all $v \in \mathscr{F}(Y, X^{**})$ with $\alpha^*(v) \leqslant 1$.

As an example let us recall the space $\gamma_2(X, Y)$ of all operators from X to Y factorizable through the Hilbert space, with the norm $\gamma_2(u) = \inf \|w_1\| \cdot \|w_2\|$ where the infimum is taken over all operators $w_1: X \to H$, $w_2: H \to Y$ such that $u = w_2 \circ w_1$ [11].

The *Banach–Mazur distance* between isomorphic Banach spaces X and Y is defined by $d(X, Y) = \inf \|T\| \cdot \|T^{-1}\|$ with the infimum taken over all isomorphisms T from X onto Y.

In the proofs of ours main results the following generalization of John's theorem on ellipsoids of minimum volume, due to Lewis [13], is a crucial fact.

THEOREM 0.1. *Let E and F be n-dimensional (real or complex) Banach spaces. Let α be some norm on the space $\mathscr{F}(E, F)$. There is an isomorphism $u: E \to F$ such that $\alpha(u) = 1$ and $\alpha^*(u^{-1}) = n$.*

Actually Lewis proved this theorem in the real case. But it is not difficult to extend this proof to the complex case and we omit this.

We shall also use the following fact from the theory of Banach ideals, the particular case of which was considered in [14].

PROPOSITION 0.2. *For every reflexive Banach space X and every natural number n the following conditions are equivalent:*

(i) *there is a constant C such that for any n-dimensional subspace $E \subset X$ there is a projection P from X onto E with $\gamma_2(P) \leqslant C$;*

(ii) *there is a constant C such that for any n-dimensional Banach space G and any subspace $E \subset X$ every operator $u: E \to G$ can be extended to the operator $\tilde{u}: X \to G$ with $\gamma_2(\tilde{u}) \leqslant C\|u\|$.*

Proof. Recall that if G is a finite dimensional Banach space and Y is a reflexive one, then the space $\mathscr{F}(Y, G)$ of all operators from Y to G, with the operator norm, is reflexive and its dual is isometric to the space $N(G, Y)$ of all nuclear operators from G to Y, with the nuclear norm $n(\cdot)$. Similarly, the space $\gamma_2(Y, G)$ of all operators from Y to G factorizable through the Hilbert space is reflexive and its dual, $\gamma_2^*(G, Y)$, is the space of all operators of the form $w = v \circ u$, where $u: G \to H$ is a 2-absolutely summing operator from G to the Hilbert space H and $v: H \to Y$ is an operator with a 2-absolutely summing adjoint $v^*: Y^* \to H$. We define the norm $\gamma_2^*(w) = \inf \pi_2(u)\, \pi_2(v^*)$, with the infimum taken over all such factorizations of w [11].

The standard duality argument implies that condition (ii) is equivalent to

(ii)* *there is a constant C such that for any n-dimensional Banach space G and any subspace $E \subset X$ the following inequality holds for every operator $w: G \to E$*

$$n(w) \leqslant C\gamma_2^*(jw);$$

here $j: E \to X$ denotes the inclusion map.

We shall show that (i)$\Rightarrow$(ii)*. Pick $w: G \to E$ with $\gamma_2^*(jw) = 1$ and consider the best representation of jw in the form $jw = v \circ u$ where $u: G \to H$, $v: H \to X$ with $\pi_2(u) \cdot \pi_2(v^*) = 1$. Consider the projection $P: X \to w(G)$ with $\gamma_2(P) \leqslant C$, as in (i), and its best factorization through the Hilbert space H, $P = T \circ S$ with $S: X \to H$, $T: H \to w(G)$ and $\|T\| \cdot \|S\| = \gamma_2(P)$. Moreover, $i: w(G) \to E$ denotes the inclusion map. Thus $w = i \circ P \circ j \circ w = i \circ T \circ (S \circ v) \circ u = i \circ T \circ v_1 \circ u$, where $v_1 = S \circ v$ is the operator from H to H. The adjoint $v_1^* = v^* \cdot S^*$ is a 2-absolutely summing operator, since so is v^*. Since v_1 acts in the Hilbert space, v_1 is also 2-absolutely summing. We have $\pi_2(v_1) = \pi_2(v_1^*) \leqslant \pi_2(v^*) \cdot \|S\|$. Hence w, as the composition of two 2-absolutely summing operators, is nuclear [16], and $n(w) \leqslant \|i\|\cdot\|T\| \times \pi_2(v_1)\cdot\pi_2(u) \leqslant \|i\|\cdot\|T\|\cdot\|S\|\pi_2(v^*)\cdot\pi_2(u) = \gamma_2(P)\cdot\gamma_2^*(jw) \leqslant C$. This establishes condition (ii)*.

The implication (ii) ⇒(i) is obvious. ■

§ 1. Uniformly convex and uniformly smooth Banach lattices. In this section, for the sake of convenience, we consider only the real case. Let us recall some notions. A Banach space X which is a lattice under $\leqslant$ is called a *Banach lattice* provided $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$, where $|x| = \sup(x, -x)$. We say that the elements $x, y \in X$ are *disjointly supported* whenever $\inf(|x|, |y|) = 0$. We say that $\operatorname{supp} x \subset \operatorname{supp} y$ whenever $\inf(|y|, |z|) = 0$ implies $\inf(|x|, |z|) = 0$ for every $z \in X$. One can prove ([18]) that if for a Banach lattice X no subspace of X is isomorphic either to c_0 or to l_1, then X and X^* are order isomorphic to some function lattices on the same measure space (Ω, μ), with the duality given by $\langle f, x^*\rangle = \int f\cdot x^* d\mu$, for $f \in X$ and $x^* \in X^*$.

Now we recall some geometric properties of Banach spaces. For a Banach space X the *modulus of convexity* (resp. *of smoothness*) is defined by

$$\delta_X(\varepsilon) = \inf\{1-\|\tfrac{1}{2}(x+y)\|:\ \|x\| \leqslant 1,\ \|y\| \leqslant 1,\ \|x-y\| \geqslant \varepsilon\}$$

(resp.

$$\varrho_X(\tau) = \sup\{2^{-1}(\|x+y\|+\|x-y\|-2):\ \|x\| = 1,\ \|y\| = \tau\}).$$

We say that the space X is *uniformly convex* (resp. *smooth*) if $\delta_X(\varepsilon) > 0$ (resp. $\lim\limits_{\tau\to 0} \varrho_X(\tau)\cdot\tau^{-1} = 0$). It is well known that if a Banach space X is uniformly smooth, then for every $x \in X$ the supporting functional x^* at x is unique; if X is uniformly convex, then $(y, x^*) \neq 1$ for $y \neq x$ [8].

Since the uniform smoothness is preserved for subspaces, no uniformly smooth Banach lattice contains either c_0 or l_1. Thus, X and X^* are order isomorphic to some function lattices. Therefore, for $f, g \in X$ $(x^*, y^* \in X^*)$, by fg (x^*y^*) we denote a pointwise multiplication of functions. The action of a functional $x^* \in X^*$ on an element $f \in X$ will be denoted by $\langle f, x^*\rangle$ and after the above identification we have $\langle f, x^*\rangle = \int f\cdot x^*$.

The main result of this section is

THEOREM 1.1. *Let $p > 2$ and let X be a complete Banach lattice with $\delta_X(\varepsilon) \geqslant K_1\varepsilon^p$ and $\varrho_X(\tau) \leqslant K_2\tau^2$ for some constants K_1 and K_2. Then there is a constant K such that for every n-dimensional subspace $E \subset X$ there is a projection P from X onto E with $\gamma_2(P) \leqslant Kn^{1/2-1/p}$.*

To get this theorem we shall prove some auxiliary results. We start with a theorem analogous to the theorem due to Lewis [14].

THEOREM 1.2. *Let X be a uniformly smooth Banach lattice, and let $E \subset X$ be a n-dimensional subspace. Then there is a basis in E $f_1, \ldots, f_n$ such that for $f = \left(\sum_{i=1}^n f_i^2\right)^{1/2}$ we have $\|f\| = 1$ and for every sequence of scalars $\alpha_1, \ldots, \alpha_n$ we have*

$$n^{-1}\sum_{i=1}^n \alpha_i^2 = \left\langle f^{-1}\left(\sum_{i=1}^n \alpha_i f_i\right)^2, f^*\right\rangle, \tag{4}$$

where f^* denotes the supporting functional at f.

Let us observe that $f \in X$. Indeed, one can estimate

$$f = \left(\sum_{i=1}^n f_i^2\right)^{1/2} \leqslant \sqrt{n}\max(|f_1|, \ldots, |f_n|).$$

Moreover, $\max(|f_1|, \ldots, |f_n|) \in X$; thus $f \in X$. From Schwarz's inequality it follows that

$$\left|\sum_{i=1}^n \alpha_i f_i\right| \leqslant \left(\sum_{i=1}^n \alpha_i^2\right)^{1/2} f \quad \text{for all scalars } \alpha_1, \ldots, \alpha_n.$$

So

$$\left|f^{-1}\left(\sum_{i=1}^n \alpha_i f_i\right)^2\right| \leqslant \left(\sum_{i=1}^n \alpha_i^2\right) f$$

and $f \in X$ implies $f^{-1}\left(\sum_{i=1}^n \alpha_i f_i\right)^2 \in X$.

To prove Theorem 1.2 we need the notion of X-summing operators.

DEFINITION 1.3. Let Y be a Banach space. The operator $u\colon Y\to X$ is *X-summing* if the image of the unit ball of Y is order bounded in X. Its *X-summing norm* is denoted by $\pi_X(u)$ and equals

$$\pi_X(u) = \Big\|\sup_{\|y\|\leqslant 1} |u(y)|\Big\|_X.$$

If $E \subset X$ is a subspace, then an operator $u\colon Y\to E$ is (E, X)-summing if the composition $ju\colon Y\to X$ is X-summing, where $j\colon E\to X$ is the inclusion map. We define $\pi_{E,X}(u) = \pi_X(ju)$.

Let us notice that in the case $X = L_p(\Omega, \mu)$ this notion is exactly the same as that of p-decomposable operators ([10], [15]). Moreover, if Y is isomorphic to a quotient of $L_{p^*}(\Omega_1, \mu_1)$ $(p^* = p(p-1))$, then $u\colon Y\to L_p$ is L_p-summing if and only if u is absolutely summing. In our case the last condition is equivalent to the fact that u^* is p-absolutely summing ([10]).

In the sequel we shall consider only operators from l_2^n. An easy corollary of the definition is

COROLLARY 1.4. *Let $u\colon l_2^n \to X$ be an operator and let (e_i) denote the unit vector basis in l_2^n. Then*

$$\pi_X(u) = \Big\|\Big(\sum_{i=1}^n u(e_i)^2\Big)^{1/2}\Big\|_X. \tag{5}$$

PROPOSITION 1.5. *On the space $\mathscr{F}(X, l_2^n)$ of all operators, the norm dual to π_X is defined by the formula*

$$(\pi_X)^*(w) = \pi_{X^*}(w^*) \tag{6}$$

for $w\colon X \to l_2^n$. By $\pi_{X^*}(w^*)$ we denote the X^*-summing norm of the operator $w^*\colon l_2^n \to X^*$.

Proof. Let $w\colon X \to l_2^n$ be an operator. We have to prove (see [1]) that

$$\pi_{X^*}(w^*) = \sup_{\substack{u\colon l_2^n \to X\\ \pi_X(u)\leqslant 1}} |\mathrm{trace}\, w\circ u|. \tag{7}$$

Let (e_i) denote the unit vector basis in l_2^n. From the definition of the trace and Schwarz's inequality we obtain for $u\colon l_2^n \to X$ with $\pi_X(u) \leqslant 1$

$$\begin{aligned}|\mathrm{trace}\, w\circ u| &= \Big|\sum_{i=1}^n \big(wu(e_i), e_i\big)\Big| \\ &= \Big|\sum_{i=1}^n \langle u(e_i), w^*(e_i)\rangle\Big| \leqslant \int \Big(\sum_{i=1}^n u(e_i)^2\Big)^{1/2}\cdot\Big(\sum_{i=1}^n w^*(e_i)^2\Big)^{1/2} \\ &\leqslant \Big\|\Big(\sum_{i=1}^n u(e_i)^2\Big)^{1/2}\Big\|_X \Big\|\Big(\sum_{i=1}^n w^*(e_i)^2\Big)^{1/2}\Big\|_{X^*} \leqslant \pi_{X^*}(w^*).\end{aligned}$$

To show the equality let us define $y_i^* = w^*(e_i)$ for $i = 1, \ldots, n$, $y^* = \big(\sum_{i=1}^n y_i^{*2}\big)^{1/2}$. For arbitrary $\varepsilon > 0$ let us choose $x \in X$ with $\|x\| = 1$ and $\langle x, y^*\rangle \geqslant (1-\varepsilon)\|y\|$. Let us define

$$x_i = y_i^* x/y^* \quad \text{for} \quad i = 1, \ldots, n.$$

We have $\big(\sum_{i=1}^n x_i^2\big)^{1/2} = x$; thus $x_i \in X$ for $i = 1, \ldots, n$, because $|x_i| \leqslant x$. Let us define the operator $u\colon l_2^n \to X$ by

$$u(e_i) = x_i \quad \text{for} \quad i = 1, \ldots, n.$$

Then $\pi_X(u) = \|x\| = 1$. Moreover,

$$\begin{aligned}|\mathrm{trace}\, w\circ u| &= \Big|\sum_{i=1}^n \langle u(e_i), w^*(e_i)\rangle\Big| = \Big|\sum_{i=1}^n \langle x_i, y_i^*\rangle\Big| \\ &= \int \sum_{i=1}^n x y_i^{*2}/y^* = \int xy^* \\ &= \langle x, y^*\rangle \geqslant (1-\varepsilon)\|y\| = (1-\varepsilon)\pi_{X^*}(w^*).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies

$$\pi_{X^*}(w^*) \leqslant \sup_{\substack{u\colon l_2^n \to X\\ \pi_X(u)\leqslant 1}} |\mathrm{trace}\, w\circ u|;$$

thus we have equality in (7). This completes the proof. ∎

For the space $\Pi_{E,X}(l_2^n, E)$ of (E, X)-summing operators we have the following

COROLLARY 1.6. *On the space $\mathscr{F}(E, l_2^n)$ of all operators the norm dual to $\pi_{E,X}$ is defined by*

$$(\pi_{E,X})^*(w) = \inf \pi_{X^*}(v^*)$$

for $w\colon E \to l_2^n$, where the infimum is taken over all extensions $v\colon X \to l_2^n$ of w, i.e. $v|E = w$.

Moreover, there is an operator $\overline{w} = X \to l_2^n$ with $\overline{w}|E = w$ such that

$$\pi_{X^*}(\overline{w}^*) = \inf \pi_{X^*}(v^*).$$

Proof. It is a standard consequence of Proposition 1.6. ∎

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. We apply Theorem 0.1 to the norms $\pi_{E,X}$ and $(\pi_{E,X})^*$. Thus there is an isomorphism $u\colon l_2^n \to E$ with $\pi_{E,X}(u) = 1$ and $(\pi_{E,X})^*(u^{-1}) = n$. By Corollary 1.7 there is an extension $w\colon X \to l_2^n$ of u^{-1} with $\pi_{X^*}(w^*) = n$. Let (e_i) denote the unit vector basis in l_2^n. We define $f_i = u(e_i)$ and $f = \big(\sum_{i=1}^n f_i^2\big)^{1/2}$. Hence $\|f\| = \pi_{E,X}(u) = 1$. Furthermore, we define $g_i = w^*(e_i) \in X^*$ and $g = \big(\sum_{i=1}^n g_i^2\big)^{1/2}$. Hence $\|g\| = \pi_{X^*}(w^*) = n$.

Since $w\circ u = \mathrm{id}_{l_2^n}$, then Schwarz's inequality yields

$$\begin{aligned}n = \mathrm{trace}\, w\circ u &= \sum_{i=1}^n \big(w\circ u(e_i), e_i\big) = \sum_{i=1}^n \langle f_i, g_i\rangle \\ &\leqslant \int \Big(\sum_{i=1}^n f_i^2\Big)^{1/2}\cdot\Big(\sum_{i=1}^n g_i^2\Big)^{1/2} = \langle f, g\rangle \leqslant \|f\|\cdot\|g\| = n.\end{aligned}$$

Thus

$$\sum_{i=1}^{n}\langle f_i, g_i\rangle = \int \Big(\sum_{i=1}^{n} f_i^2\Big)^{1/2}\cdot\Big(\sum_{i=1}^{n} g_i^2\Big)^{1/2}$$

and

$$\langle f, g\rangle = \|f\|\cdot\|g\|.$$

The last equality gives, by the uniform smoothness of the space X, $g = nf^*$. On the other hand, the first equality gives

$$g_i = f_i\cdot g/f \quad \text{for} \quad i = 1, \ldots, n.$$

Combining the above expressions for g and g_i, we get

(8) $$g_i = nf_i\cdot f^*/f \quad \text{for} \quad i = 1, \ldots, n.$$

Formula (8) immediately implies

$$\delta_{ij} = \langle f_i, g_j\rangle = n\langle f_i f_j/f, f^*\rangle.$$

Since for every sequence of scalars $a_1, \ldots, a_n$ we have

$$f^{-1}\Big(\sum_{i=1}^{n} a_i f_i\Big)^2 = \sum_{i,j} a_i a_j f_i f_j/f,$$

we finally get

$$\Big\langle f^{-1}\Big(\sum_{i=1}^{n} a_i f_i\Big)^2, f\Big\rangle = n\cdot\sum_{i=1}^{n} a_i^2\langle f_i^2/f, f^*\rangle + n\sum_{i\neq j} a_i a_j\langle f_i f_j/f, f^*\rangle = n\sum_{i=1}^{n} a_i^2,$$

and this completes the proof. ■

We need also some information on the connections between the moduli of convexity, smoothness and the norms in Banach lattices. We begin with the notion of a 2-convex norm. We say that the norm in a Banach lattice X is *2-convex* provided $\|(x^2+y^2)^{1/2}\|^2 \leqslant \|x\|^2+\|y\|^2$ for $x, y \in X$. The notion of 2-convexity was introduced in [3] for Banach spaces with an unconditional basis, and independently, under another name, for Banach lattices in [9]. We have the following possibility of renormalization.

PROPOSITION 1.7. *Let X be a Banach lattice with the modulus of smoothness satisfying $\varrho_X(\tau) \leqslant K\tau^2$ for some constant K. Then there exists a norm $|\cdot|$ on X, equivalent to the original one and 2-convex.*

Proof. See [2]. For the space with an unconditional basis the proposition follows also from the fact that, by the duality between the modulus of smoothness of X and the modulus of convexity of X^* ([1]), we have $\delta_{X^*}(\varepsilon) \geqslant K_1\varepsilon^2$. Thus X is of cotype 2 ([5]) and every operator from c_0 to X is 2-absolutely summing ([15]). Then the result follows from [3] (the proof of Lemma 3.1). ■

Thus, without loss of generality, we may assume that the original norm on X is 2-convex. This allows us to construct some new Banach lattice, denoted by Z. Namely we define new operations on X:

$$x\oplus y = |x^2\operatorname{sgn} x + y^2\operatorname{sgn} y|^{1/2}\operatorname{sgn}(x^2\operatorname{sgn} x + y^2\operatorname{sgn} y) \quad \text{for} \quad x, y \in X,$$

$$\lambda\odot x = |\lambda|^{1/2}(\operatorname{sgn}\lambda)x \quad \text{for} \quad \lambda \in \boldsymbol{R} \text{ and } x \in X.$$

We write $Z = (X, \oplus, \odot)$. On Z we define a norm $|||z||| = \|z\|^2$ for $z \in Z$. The 2-convexity of the norm $\|\cdot\|$ implies that the norm $|||\cdot|||$ satisfies the triangle inequality. If $\varphi^* \in Z^*$, then the value of φ^* on an element $z \in Z$ is denoted by $\langle\langle z, \varphi^*\rangle\rangle$. The space Z will be used in the proof of the following auxiliary result.

PROPOSITION 1.8. *Let X be a uniformly smooth Banach lattice with a 2-convex norm. Let $f \in X$, $f > 0$, be a positive element with $\|f\| = 1$, and let $f^* \in X^*$ denote the supporting functional at f. Then, for every $y_1, y_2 \in X$ with* $\operatorname{supp} y_1, \operatorname{supp} y_2 \subset \operatorname{supp} f$,

(9) $$|\langle y_1 y_2/f, f^*\rangle| \leqslant \|y_1\|\cdot\|y_2\|.$$

Proof. Let us regard f as an element of Z. By $\varphi^* \in Z^*$ we denote the supporting functional at f. Since $f > 0$, we have $\varphi^* > 0$. We define the functional $x^* \in X^*_+$ by

$$\langle y, x^*\rangle = \langle\langle |fy|^{1/2}\operatorname{sgn}(fy), \varphi^*\rangle\rangle \quad \text{for} \quad y \in X.$$

It is easy to see that the formula above really defines the linear functional on X. Moreover, $\langle f, x^*\rangle = 1$. We shall show that $|\langle y, x^*\rangle| \leqslant 1$ whenever $\|y\| = 1$. Indeed, for any reals $a, b \in \boldsymbol{R}$ we have

$$|ab|^{1/2} \leqslant 2^{-1}(|a|+|b|).$$

Thus

$$|fy|^{1/2} \leqslant 2^{-1}(f+|y|)$$

and we have

$$\big|\big|\big|\,|fy|^{1/2}\operatorname{sgn}(fy)\big|\big|\big| = \big|\big|\big|\,|fy|^{1/2}\big|\big|\big| = \big\|\,|fy|^{1/2}\big\|^2 \leqslant [2^{-1}(\|f\|+\|y\|)]^2 = 1.$$

Then, by the definition of x^*,

$$|\langle y, x^*\rangle| = |\langle\langle |fy|^{1/2}\operatorname{sgn}(fy), \varphi^*\rangle\rangle| \leqslant \big\|\,|fy|^{1/2}\operatorname{sgn}(fy)\big\| = 1.$$

Thus the uniform smoothness of X implies that $x^* = f^*$. To prove inequality (9) pick $y_1, y_2 \in X$ with $\|y_1\| = \|y_2\| = 1$ and put $y = y_1 y_2/f$. We have

$$|fy|^{1/2}\operatorname{sgn}(fy) = |y_1 y_2|^{1/2}\operatorname{sgn}(y_1 y_2),$$

hence, as above, $\big|\big|\big||y_1y_2|^{1/2}\operatorname{sgn}(y_1y_2)\big|\big|\big| \leqslant 1$. Thus

$$|\langle y_1y_2/f, f^*\rangle| = |\langle y, x^*\rangle| = \big|\langle\!\langle |y_1y_2|^{1/2}\operatorname{sgn}(y_1y_2), \varphi^*\rangle\!\rangle\big| \leqslant 1.$$

By the homogeneity of (9) this completes the proof. ■

PROPOSITION 1.9. *Let X be a Banach lattice with a 2-convex norm and with the modulus of convexity satisfying $\delta_X(\varepsilon) \geqslant K_1\varepsilon^p$ $(p > 2)$ for some constant K_1. Then there is a constant L such that for any $f \in X$, $f > 0$, $\|f\| = 1$ the estimation holds*

$$\|y\|^p \leqslant L\langle y^2/f, f^*\rangle \quad \textit{whenever} \quad |y| \leqslant f. \tag{10}$$

Proof. First let us observe that, since $\|y\| = \big\||y|\big\|$ and $y^2/f = |y|^2/f$, we can assume that $y \geqslant 0$. We shall prove that there are constants K_2 and K_3 such that

$$\delta_X(K_2\|y\|) \leqslant K_3\langle y^2/f, f^*\rangle \quad \text{whenever} \quad 0 \leqslant y \leqslant f. \tag{11}$$

Next, applying the estimation for δ_X we shall get the required inequality (10).

If $y = 0$, then (11) is trivial. Assume $y \neq 0$. Inequality (11) will be proved in two steps.

Step 1. Assume that there is a $z \in X$ with $0 \leqslant z \leqslant f$, disjointly supported with y, such that $y+z = f$. Then $\|z\| \leqslant 1$ and $y^2/f = y$. From the definition of δ_X we obtain

$$\delta_X(\|y\|) = \delta_X(\|f-z\|) \leqslant 1-2^{-1}\|f+z\|$$
$$\leqslant \langle f-2^{-1}(f+z), f^*\rangle = 2^{-1}\langle y, f^*\rangle = 2^{-1}\langle y^2/f, f^*\rangle.$$

Step 2 (general). Let us observe that the 2-convexity of the norm implies that if $x_1, \ldots, x_k \in X$ are disjointly supported, then

$$\Big\|\sum_{i=1}^k x_i\Big\| \leqslant \Big(\sum_{i=1}^k \|x_i\|^2\Big)^{1/2}. \tag{12}$$

Let m be a natural number such that $2^{-2m+2} \leqslant (y^2/f, f^*)$ $(\langle y^2/f, f^*\rangle > 0$ because $y^2/f > 0)$. One can represent $y = y_1+\ldots+y_m$ and $f = z_1+\ldots+z_m$ as the sums of disjointly supported elements satisfying

$$\begin{gathered} 2^{-i}z_i \leqslant y_i \leqslant 2^{-i+1}z_i \quad \text{for} \quad i = 1, \ldots, m-1, \\ 0 \leqslant y_m \leqslant 2^{-m+1}z_m \end{gathered} \tag{13}$$

Now, from (12) we get

$$\|y\|^2 = \Big\|\sum_{i=1}^m y_i\Big\|^2 \leqslant \sum_{i=1}^m \|y_i\|^2 \leqslant \sum_{i=1}^m 2^{-2i+2}\|z_i\|^2.$$

On the other hand, see [1] (Lemma 2 and Proposition 11), there exist

a convex increasing function $\hat\delta$ and a constant K_4 such that

$$\hat\delta(\varepsilon^2) \leqslant \delta_X(\varepsilon) \leqslant K_4\hat\delta(2\varepsilon^2).$$

Thus write $a = \sum_{i=1}^m 2^{-2i+2}$. The convexity of $\hat\delta$ gives

$$\begin{aligned} \delta_X\big((2\sqrt{a})^{-1}\|y\|\big) &\leqslant K_4\hat\delta(a^{-1}\|y\|^2) \\ &\leqslant K_4\hat\delta\Big(a^{-1}\sum_{i=1}^m 2^{-2i+2}\|z_i\|^2\Big) \leqslant K_4a^{-1}\sum_{i=1}^m 2^{-2i+2}\hat\delta(\|z_i\|^2) \\ &\leqslant K_4a^{-1}\sum_{i=1}^m 2^{-2i+2}\delta_X(\|z_i\|). \end{aligned} \tag{14}$$

To estimate $\delta_X(\|z_i\|)$ we can apply step 1. Thus

$$\delta_X(\|z_i\|) \leqslant 2^{-1}\langle z_i^2/f, f^*\rangle \quad \text{for} \quad i = 1, \ldots, m. \tag{15}$$

Moreover, from the choice of m and of y_i and z_i we obtain

$$\begin{aligned} \sum_{i=1}^m 2^{-2i+2}\langle z_i^2/f, f^*\rangle &\leqslant 4\sum_{i=1}^{m-1}\langle y_i^2/f, f^*\rangle + 2^{-2m+2} \\ &\leqslant 5\langle y^2/f, f^*\rangle. \end{aligned} \tag{16}$$

Thus, by adding up inequalities (15) and combining the result with (14) and (16), we get

$$\delta_X\big((2\sqrt{a})^{-1}\|y\|\big) \leqslant (2a)^{-1}5K_4\langle y^2/f, f^*\rangle.$$

This completes the proof of Proposition 1.9. ■

Proof of Theorem 1.1. Let $E \subset X$ be any n-dimensional subspace. Let $f_1, \ldots, f_n$ and f be as in Theorem 1.2. By $X_0 \subset X$ we denote the ideal generated by f, i.e. the least subspace with $f \in X_0$ satisfying two properties: (i) $|y| \leqslant |x|$ and $x \in X_0$ implies $y \in X_0$, (ii) for every subset $A \subset X_0$, $\sup A \in X_0$. It is well known, [16], that for every $x \in X_0$, $\operatorname{supp} x \subset \operatorname{supp} f$ and then there is a projection Q from X onto X_0 with $\|Q\| = 1$. Moreover, $E \subset X_0$, since $f_i \in X_0$ for $i = 1, \ldots, n$.

On X_0 we define a scalar product by $\langle\!\langle y_1, y_2\rangle\!\rangle = \langle y_1y_2/f, f^*\rangle$. We write $\|y\|_2 = \langle\!\langle y, y\rangle\!\rangle^{1/2}$. Then the completion of $(X_0, \|\cdot\|_2)$ is a Hilbert space. Let us define $P = i_2\circ Q_1\circ i_1\circ Q$, where Q: $X \to X_0$ is defined as above, i_1: $(X_0, \|\cdot\|) \to (X_0, \|\cdot\|_2)$ is the inclusion map, Q_1: $(X_0, \|\cdot\|_2) \to i_1(E)$ is the orthogonal projection, i_2: $i_1(E) \to (E, \|\cdot\|)$ is the formal identity.

The estimation of the norm of i_1 follows immediately from Proposition 1.8

$$\|y\|_2 = \langle y^2/f, f^*\rangle^{1/2} \leqslant \|y\|$$

for all $y \in X_0$. Then $\|i_1\| \leqslant 1$.

For the estimation of the norm of i_2 pick a sequence of scalars $a_1, \ldots, a_n$ with $\sum_{i=1}^{n} a_i^2 = 1$ ant put $y = \sum_{i=1}^{n} a_i f_i \in i_1(E)$. Then Schwarz's inequality and property (a) imply that $|y| \leqslant f$. Moreover, from Theorem 1.2 we get

$$1 = \Big(\sum_{i=1}^{n} a_i^2\Big)^{p/2-1} = \Big[n\Big\langle f^{-1}\Big(\sum_{i=1}^{n} a_i f_i\Big)^2, f^*\Big\rangle\Big]^{p/2-1} = [n\langle y^2/f, f^*\rangle]^{p/2-1}.$$

Thus, by application of Proposition 1.9 to y, we obtain

$$\|y\|^p \leqslant L\langle y^2/f, f^*\rangle = Ln^{p/2-1}\langle y^2/f, f^*\rangle^{p/2} = Ln^{p/2-1}(\|y\|_2)^p.$$

Hence $\|i_2\| \leqslant L^{1/p} n^{1/2-1/p}$. Let us write $w_1 = i_1 \circ Q$, $w_2 = i_2 \circ Q_1$ and let us observe that the representation of P as the composition $P = w_2 \circ w_1$ determines some factorization of P through a Hilbert space. Thus $\gamma_2(P) \leqslant \|w_2\|\|w_1\| \leqslant Kn^{1/2-1/p}$. This completes the proof. ■

COROLLARY 1.10. *Let F be an n-dimensional Banach space isometric to a quotient of a subspace of a Banach lattice X whose moduli of convexity and smoothness admit the estimations $\delta_X(\varepsilon) \geqslant K_1 \varepsilon^p$ and $\varrho_X(\tau) \leqslant K_2 \tau^q$, where $q = 2$ if $p > 2$, and $1 < q \leqslant 2$ if $p = 2$. Then there is a constant K, depending on K_1 and K_2 only, such that $d(F, l_2^n) \leqslant K \cdot n^{|1/q-1/p|}$.*

Proof. Since a quotient of a subspace is isometric to a subspace of a quotient, by the duality argument it is sufficient to prove this statement in the case $p \geqslant 2$, $q = 2$. Therefore, let X be a Banach lattice satisfying the assumptions, let $E \subset X$ be a subspace, and let $Q: E \to F$ be a quotient map. Thus Theorem 1.1 and Proposition 0.2 yield an extension $\tilde{Q}: X \to F$ of Q with $\gamma_2(\tilde{Q}) \leqslant Kn^{1/2-1/p}$. Thus for an isometric embedding $Q^*: F^* \to E^*$ we have $\gamma_2(Q^*) = \gamma_2(Q) \leqslant \gamma_2(\tilde{Q}) \leqslant Kn^{1/2-1/p}$. Hence $d(F, l_2^n) \leqslant Kn^{1/2-1/p}$. ■

The next corollary improves an estimation given in [4]. For the definition of the type and the cotype of any Banach space see for example [2], [5].

COROLLARY 1.11. *Let X be a Banach lattice of type q and cotype p, where $q = 2$ if $p > 2$, and $1 < q \leqslant 2$ if $p = 2$. There exists a constant K such that, for any n-dimensional space F isometric to a quotient of a subspace of X, $d(F, l_2^n) \leqslant Kn^{|1/p-1/q|}$.*

Proof. It is proved by Figiel in [2] that the uniformly convex Banach lattice of type q has the modulus of smoothness admiting an estimation $\varrho_X(\tau) \leqslant K'\tau^q$ for some constant K'. Thus the Banach lattice X of type q and cotype p satisfies the assumption of Corollary 1.10. ■

§ 2. The trace classes S_p. In this section we shall prove the "non-commutative analogue" of Theorem 1.2. We begin with the definitions.

Let H be a complex Hilbert space. The basic notation from the theory of operators in a Hilbert space is the same as in [6]. Let $1 \leqslant p < \infty$ be any real number. For an operator $A \in \mathscr{F}(H, H)$ we put

$$\|A\|_p = \big(\operatorname{trace}(A \circ A^*)^{p/2}\big)^{1/p}.$$

The completion of $\mathscr{F}(H, H)$ under the norm $\|\cdot\|_p$ is denoted by S_p. It is well known (cf. [6]) that if $1 < p < \infty$, the dual space $(S_p)^*$ is isometric to S_{p^*} $(p^* = p/p-1)$. This duality is given by the trace: $(A, B) = \operatorname{trace} B \circ A$ for $A \in S_p$ and $B \in S_{p^*}$.

Subsequently the following fact will often be used.

LEMMA 2.1. *Let $A: H \to H$ be a compact operator, $A \geqslant 0$. Let $1 \leqslant r < \infty$ be a real number. For every $x \in H$ with $\|x\| = 1$ we have*

$$\big(A(x), x\big)^r \leqslant \big(A^r(x), x\big).$$

Moreover, in the case $r > 1$, the equality holds if and only if x is an eigenvector of the operator A, i.e. $A(x) = \lambda \cdot x$ for some $\lambda > 0$.

Proof. Let (e_i) denote the orthonormal basis in H of eigenvectors of the operator A, $A(e_i) = a_i e_i$ for $i = 1, \ldots$ For given $x \in H$, $x = \sum_{i=1}^{\infty} a_i e_i$, by Hölder's inequality we obtain

$$\big(A(x), x\big) = \sum_{i=1}^{\infty} a_i \alpha_i^2 = \sum_{i=1}^{\infty} a_i \alpha_i^{2/r} \alpha_i^{2/r^*} \leqslant \Big(\sum_{i=1}^{\infty} a_i^r \alpha_i^2\Big)^{1/r} \Big(\sum_{i=1}^{\infty} \alpha_i^2\Big)^{1/r^*} = \big(A^r(x), x\big).$$

The case where the equality holds is obvious. ■

The main result of this section is the following.

THEOREM 2.2. *Let $p > 2$ and let $E \subset S_p$ be an n-dimensional subspace. There is a projection P from S_p onto E with $\gamma_2(P) \leqslant n^{1/2-1/p}$.*

First we shall prove

THEOREM 2.3. *Let $1 < p < \infty$ and let $E \subset S_p$ be an n-dimensional subspace. There is a basis in E, $F_1, \ldots, F_n$ such that for a sequence of (complex) scalars $a_1, \ldots, a_n$ we have*

$$n^{-1} \sum_{i=1}^{n} |a_i|^2 = \operatorname{trace}\Big(\sum_{i=1}^{n} a_i F_i\Big) \circ \Big(\sum_{i=1}^{n} a_i F_i\Big)^* \circ F^{p-2} \tag{17}$$

where $F = \big(\sum_{i=1}^{n} F_i \circ F_i^*\big)^{1/2}$. Moreover, $\|F\|_p = 1$.

To establish this theorem we shall define some Banach space of operators from l_2 to subspaces of S_p.

DEFINITION 2.4. Let $1 < p < \infty$. The operator $u\colon l_2 \to S_p$ is R(right)-S_p-*summing* or L(left)-S_p-*summing* if

$$\sum_{i=1}^{\infty} \big(u(e_i)\big) \circ \big(u(e_i)\big)^* \in S_{p/2} \quad \text{or} \quad \sum_{i=1}^{\infty} \big(u(e_i)\big)^* \circ \big(u(e_i)\big) \in S_{p/2},$$

respectively. Here (e_i) denotes the unit vector basis in l_2. We define the norms by

$$\pi^{\mathrm{R}}_{S_p}(u) = \Big\| \Big[\sum_{i=1}^{\infty} \big(u(e_i)\big) \circ \big(u(e_i)\big)^* \Big]^{1/2} \Big\|_p, \tag{18}$$

$$\pi^{\mathrm{L}}_{S_p}(u) = \Big\| \Big[\sum_{i=1}^{\infty} \big(u(e_i)\big)^* \circ \big(u(e_i)\big) \Big]^{1/2} \Big\|_p,$$

respectively. If $E \subset S_p$ is a subspace, the operator $u\colon l_2 \to E$ is R-(E, S_p)-*summing* (resp. L-(E, S_p)-*summing*) if the composition $ju\colon l_2 \to S_p$ is R-S_p-summing (resp. L-S_p-summing), where $j\colon E \to S_p$ is the inclusion map. We define $\pi^{\mathrm{R}}_{E,S_p}(u) = \pi^{\mathrm{R}}_{S_p}(ju)$ (resp. $\pi^{\mathrm{L}}_{E,S_p}(u) = \pi^{\mathrm{L}}_{S_p}(ju)$).

Let us observe that the definition of the norm (18) is similar, in a sense, to formula (5). If $E \subset S_p$ is the subspace of diagonal operators with respect to a fixed orthonormal basis in H isometric to l_p, then R-(E, S_p)-summing and L-(E, S_p)-summing operators from l_2 to E are p-decomposable, which in this case is the same as p-summing [10].

Let us examine the dual norm to $\pi^{\mathrm{R}}_{S_p}$.

PROPOSITION 2.5. *Let $1 < p < \infty$, $p^* = p/p-1$. For an operator $w\colon S_p \to l_2^n$ with the norm dual to $\pi^{\mathrm{R}}_{S_p}$ the following equality holds:*

$$(\pi^{\mathrm{R}}_{S_p})^*(w) = \pi^{\mathrm{L}}_{S_p}(w^*). \tag{19}$$

Here $\pi^{\mathrm{L}}_{S_{p^*}}(w^*)$ denotes the L-S_{p^*}-*summing norm of the operator $w^*\colon l_2^n \to S_{p^*}$.*

Proof. Fix $w\colon S_p \to l_2^n$. First we shall prove that for every $u\colon l_2^n \to S_p$ with $\pi^{\mathrm{R}}_{S_p}(u) \leqslant 1$ we have

$$|\operatorname{trace} w \circ u| \leqslant \pi^{\mathrm{L}}_{S_{p^*}}(w^*).$$

Let (e_i) be the unit vector basis in l_2^n. For convenience we write $A_i = u(e_i)$, $B_i = w^*(e_i)$. By Hölder's inequality, for arbitrary orthonormal basis (φ_k) in H we have

$$|\operatorname{trace} w \circ u| = |\operatorname{trace} u^* \circ w^*| = \Big| \sum_{i=1}^{n} \big(u^* w^*(e_i), e_i\big) \Big| = \Big| \sum_{i=1}^{n} (B_i, A_i) \Big|$$

$$= \Big| \operatorname{trace} \sum_{i=1}^{n} A_i B_i \Big| = \Big| \sum_{k=1}^{\infty} \sum_{i=1}^{n} \big(A_i B_i(\varphi_k), \varphi_k\big) \Big|$$

$$= \Big| \sum_{k=1}^{\infty} \sum_{i=1}^{n} \big(B_i(\varphi_k), A_i^*(\varphi_k)\big) \Big| \leqslant \sum_{k=1}^{\infty} \sum_{i=1}^{n} \|B_i(\varphi_k)\| \cdot \|A_i^*(\varphi_k)\|$$

$$\leqslant \sum_{k=1}^{\infty} \Big(\sum_{i=1}^{n} \|B_i(\varphi_k)\|^2 \Big)^{1/2} \cdot \Big(\sum_{i=1}^{n} \|A_i^*(\varphi_k)\|^2 \Big)^{1/2}$$

$$\leqslant \Big[\sum_{k=1}^{\infty} \Big(\sum_{i=1}^{n} \|B_i(\varphi_k)\|^2 \Big)^{p^*/2} \Big]^{1/p^*} \Big[\sum_{k=1}^{\infty} \Big(\sum_{i=1}^{n} \|A_i^*(\varphi_k)\|^2 \Big)^{p/2} \Big]^{1/p}$$

$$= \Big[\sum_{k=1}^{\infty} \Big(\Big(\sum_{i=1}^{n} B_i^* B_i \Big)(\varphi_k), \varphi_k \Big)^{p^*/2} \Big]^{1/p^*} \Big[\sum_{k=1}^{\infty} \Big(\Big(\sum_{i=1}^{n} A_i A_i^* \Big)(\varphi_k), \varphi_k \Big)^{p/2} \Big]^{1/p}.$$

Thus, if $1 < p \leqslant 2$, we specify (φ_k) as the orthonormal basis of eigenvectors of the (nonnegative) operator $\sum_{i=1}^{n} A_i A_i^*$.

We have $\big(\sum_{i=1}^{n} A_i A_i^*\big)(\varphi_k) = a_k^2 \varphi_k$. Hence, applying Lemma 2.1 to the exponent $p^*/2 > 1$, we obtain

$$|\operatorname{trace} w \circ u| \leqslant \Big[\sum_{k=1}^{\infty} \Big(\Big(\sum_{i=1}^{n} B_i^* B_i \Big)(\varphi_k), \varphi_k \Big)^{p^*/2} \Big]^{1/p^*} \Big[\sum_{k=1}^{\infty} a_k^p \Big]^{1/p}$$

$$\leqslant \Big[\sum_{k=1}^{\infty} \Big(\Big(\sum_{i=1}^{n} B_i^* B_i \Big)^{p^*/2} (\varphi_k), \varphi_k \Big) \Big]^{1/p^*} \Big[\operatorname{trace} \Big(\sum_{i=1}^{n} A_i A_i^* \Big)^{p/2} \Big]^{1/p}$$

$$= \Big[\operatorname{trace} \Big(\sum_{i=1}^{n} B_i^* B_i \Big)^{p^*/2} \Big]^{1/p^*} \Big[\operatorname{trace} \Big(\sum_{i=1}^{n} A_i A_i^* \Big)^{p/2} \Big]^{1/p}.$$

By the definition of the norm in S_p and S_p we get

$$|\operatorname{trace} w \circ u| \leqslant \Big\| \Big(\sum_{i=1}^{n} B_i^* B_i \Big)^{1/2} \Big\|_{p^*} \cdot \Big\| \Big(\sum_{i=1}^{n} A_i A_i^* \Big)^{1/2} \Big\|_p \leqslant \pi^{\mathrm{L}}_{S_{p^*}}(w^*). \tag{20}$$

If $2 < p < \infty$, we specify (φ_k) as the orthonormal basis of eigenvectors of the operator $\sum_{i=1}^{n} B_i^* B_i$ and we obtain the same inequality (20). Hence $\sup |\operatorname{trace} w \circ u| \leqslant \pi^{\mathrm{L}}_{S_{p^*}}(w^*)$.

To prove the equality for a given operator $w\colon S_p \to l_2^n$ we write $B_i = w^*(e_i)$ and we put

$$A_i = \Big\| \Big(\sum_{i=1}^{n} B_i^* B_i \Big)^{1/2} \Big\|_{p^*}^{-p^*/p} \cdot \Big(\sum_{i=1}^{n} B_i^* B_i \Big)^{p^*/2-1} \circ B_i^*.$$

Let us define $u\colon l_2^n \to S_p$ by

$$u(e_i) = A_i^* \quad \text{for} \quad i = 1, \ldots, n.$$

It is easy to see that $\pi^{\mathrm{R}}_{S_p}(u) = 1$ and that

$$\operatorname{trace} w \circ u = \pi^{\mathrm{L}}_{S_{p^*}}(w^*).$$

Thus $\pi^{\mathrm{L}}_{S_{p^*}}(w^*) \leqslant \sup |\operatorname{trace} w \circ u|$. This completes the proof. ■

A standard argument yields

COROLLARY 2.6. *Let $1 < p < \infty$ and $E \subset S_p$ be a subspace. On the space $L(E, l_2^n)$ of all operators from E to l_2^n the norm dual to π_{E,S_p}^{R} is defined by the formula*

$$(\pi_{E,S_p}^{\mathrm{R}})^*(w) = \inf \pi_{S_{p^*}}^{\mathrm{L}}(v^*), \tag{21}$$

where the infimum is taken over all extensions $v\colon S_p \to l_2^n$ of w.

Moreover, there is an operator $\overline{w}\colon S_p \to l_2^n$ with $\overline{w}|E = w$ such that

$$\pi_{S_{p^*}}^{\mathrm{L}}(\overline{w}^*) = \inf \pi_{S_{p^*}}^{\mathrm{L}}(v^*).$$

Proof of Theorem 2.3. We apply Theorem 0.1 to the norms π_{E,S_p}^{R} and $(\pi_{E,S_p}^{\mathrm{R}})^*$. Thus there is an isomorphism $u\colon l_2^n \to E$ with $\pi_{E,S_p}^{\mathrm{R}}(u) = 1$ and $(\pi_{E,S_p}^{\mathrm{R}})^*(u^{-1}) = n$. Corollary 2.6 implies that there is an extension $w\colon S_p \to l_2^n$ of u^{-1} with $\pi_{S_{p^*}}^{\mathrm{L}}(w^*) = n$. We define $F_i = u(e_i)$, $G_i = w(e_i)$ for $i = 1, \ldots, n$ and $F = (\sum_{i=1}^n F_i F_i^*)^{1/2}$, $G = (\sum_{i=1}^n G_i^* G_i)^{1/2}$. It is obvious that $\|F\|_p = 1$ and $\|G\|_{p^*} = n$.

Since $w \circ u = \mathrm{id}_{l_2^n}$, we have

$$n = \operatorname{trace} w \circ u = \sum_{i=1}^n (w \circ u(e_i), e_i) = \sum_{i=1}^n (F_i, G_i)$$
$$= \sum_{i=1}^n \operatorname{trace} G_i F_i = \operatorname{trace} \sum_{i=1}^n F_i G_i,$$

(for the last equality see [6], Theorem II. 8.2).

Let us assume $2 \leqslant p < \infty$ (this is the case used in the proof of Theorem 2.2). Let us denote by (φ_k) the orthonormal basis in H of eigenvectors of the operator G. We have $G(\varphi_k) = g_k \varphi_k$. Thus, applying Hölder's inequality and Lemma 2.1, we get

$$n = \operatorname{trace} \sum_{i=1}^n F_i G_i = \sum_{k=1}^\infty \sum_{i=1}^n (F_i G_i(\varphi_k), \varphi_k) = \sum_{k=1}^\infty \sum_{i=1}^n (G_i(\varphi_k), F_i(\varphi_k))$$
$$\leqslant \sum_{k=1}^\infty \Big(\sum_{i=1}^n \|G_i(\varphi_k)\|^2\Big)^{1/2} \Big(\sum_{i=1}^n \|F_i(\varphi_k)\|^2\Big)^{1/2}$$
$$= \sum_{k=1}^\infty \Big(\Big(\sum_{i=1}^n G_i^* G_i\Big)(\varphi_k), \varphi_k\Big)^{1/2} \Big(\Big(\sum_{i=1}^n F_i F_i^*\Big)(\varphi_k), \varphi_k\Big)^{1/2}$$
$$= \sum_{k=1}^\infty (G^2(\varphi_k), \varphi_k)^{1/2} (F^2(\varphi_k), \varphi_k)^{1/2} = \sum_{k=1}^\infty g_k (F^2(\varphi_k), \varphi_k)^{1/2}$$
$$\leqslant \Big(\sum_{k=1}^\infty g_k^{p^*}\Big)^{1/p^*} \Big(\sum_{k=1}^\infty (F^2(\varphi_k), \varphi_k)^{p/2}\Big)^{1/p}$$
$$\leqslant \Big(\sum_{k=1}^\infty g_k^{p^*}\Big)^{1/p^*} \Big(\sum_{k=1}^\infty (F^p(\varphi_k), \varphi_k)\Big)^{1/p} = \|G\|_{p^*} \cdot \|F\|_p = n.$$

Therefore, in the estimations above, we have equalities in all places.

From these equalities we subsequently get the following conditions:

(22) *for every $k = 1, \ldots$ there is a $c_k \in C$ such that for every $i = 1, \ldots, n$*

$$G_i(\varphi_k) = c_k F_i^*(\varphi_k),$$

(23) *there is a $c \in C$ such that for every $k = 1, \ldots$*

$$(F^2(\varphi_k), \varphi_k)^{1/2} = c g_k^{p^*-1},$$

(24) *the vectors φ_k are eigenvectors of F^2 $(k = 1, \ldots)$, and*

$$F^2(\varphi_k) = f_k^2 \varphi_k.$$

(Condition (24) follows from Lemma 2.1.)

Conditions (23) and (24) give $f_k = c g_k^{p^*-1}$; hence $f_k^p = c^p g_k^{p^*}$ for $k = 1, \ldots$ By adding up these equalities we get

$$1 = (\|F\|_p)^p = \sum_{k=1}^\infty f_k^p = c^p \sum_{k=1}^\infty g_k^{p^*} = c^p (\|G\|_{p^*})^{p^*} = c^p n^{p^*};$$

hence $c = n^{-1/p-1}$. Thus $f_k = n^{-1/p-1} g_k^{p^*-1}$ for $k = 1, \ldots$ Since the operators F and G are both diagonal in the basis φ_k, we finally obtain

$$G = n F^{p-1}. \tag{25}$$

Combining conditions (22) and (25), one can verify that $c_k = n f_k^{p-2}$. Thus

$$G_i(\varphi_k) = n f_k^{p-2} F_i^*(\varphi_k) \quad \text{for} \quad i = 1, \ldots, n,\ k = 1, \ldots \tag{26}$$

One can prove, in a similar way, (25) and (26) also in the case $1 < p < 2$. Then in the proof by (φ_k) we mean the orthonormal basis of eigenvectors of the operator F.

We shall show that (for $1 < p < \infty$)

$$\delta_{ij} = \operatorname{trace} F_j F_i^* F^{p-2} \quad \text{for} \quad i, j = 1, \ldots, n. \tag{27}$$

Let (φ_k) denote the common orthonormal basis of eigenvectors of the operators F and G (it exists by (25)). Since $u^* \circ w^* = \mathrm{id}_{l_2^n}$, by the definition of F_i and G_i and (25) we have, for $i, j = 1, \ldots, n$,

$$\delta_{ij} = (e_i, e_j) = (u^* \circ w^*(e_i), e_j) = (G_i, F_j) = \operatorname{trace} F_j G_i$$
$$= \sum_{k=1}^\infty (F_j G_i(\varphi_k), \varphi_k) = \sum_{k=1}^\infty n f_k^{p-2} (F_j F_i^*(\varphi_k), \varphi_k)$$
$$= n \sum_{k=1}^\infty (F_j F_i^* F^{p-2}(\varphi_k), \varphi_k) = n \operatorname{trace} F_j F_i^* F^{p-2}.$$

Formula (27) obviously yields equality (17) and this completes the proof. ∎

To prove Theorem 2.2 we need some technical lemmas.

LEMMA 2.7. *Let $A_1, \ldots, A_m$ be a sequence of bounded operators in H, and let $\alpha_1, \ldots, \alpha_m$ be a sequence of scalars. Then*

$$\Big(\sum_{i=1}^{m}\alpha_i A_i\Big)\Big(\sum_{i=1}^{m}\alpha_i A_i\Big)^* \leqslant \Big(\sum_{i=1}^{m}|\alpha_i|^2\Big)\Big(\sum_{i=1}^{m}A_iA_i^*\Big).$$

Proof. The argument is elementary. We can develop

$$\Big(\sum_{i=1}^{m}\alpha_i A_i\Big)\Big(\sum_{i=1}^{m}\alpha_i A_i\Big)^* = \sum_{i=1}^{m}|\alpha_i|^2 A_iA_i^* + \sum_{i<j}(\alpha_i\bar\alpha_j A_iA_j^* + \bar\alpha_i\alpha_j A_jA_i^*)$$
$$= \sum_{i=1}^{m}|\alpha_i|^2 A_iA_i^* + \sum_{i<j}[(\bar\alpha_j A_i)(\bar\alpha_i A_j)^* + (\bar\alpha_i A_j)(\bar\alpha_j A_i)^*].$$

But for any operators A, B one has

$$AB^* + BA^* \leqslant AA^* + BB^*,$$

(since $AA^* - AB^* - BA^* + BB^* = (A-B)(A-B)^* \geqslant 0$). Then we can estimate

$$\Big(\sum_{i=1}^{m}\alpha_i A_i\Big)\Big(\sum_{i=1}^{m}\alpha_i A_i\Big)^* \leqslant \sum_{i=1}^{m}|\alpha_i|^2 A_iA_i^* + \sum_{i>j}[|\alpha_j|^2 A_iA_i^* + |\alpha_i|^2 A_jA_j^*]$$
$$= \sum_{i=1}^{m}|\alpha_i|^2 A_iA_i^* + \sum_{i=1}^{m}\sum_{j>i}|\alpha_j|^2 A_iA_i^* + \sum_{j=1}^{m}\sum_{i<j}|\alpha_i|^2 A_jA_j^*.$$

By changing the indices in the last sum we see that this quantity is equal to

$$\sum_{i=1}^{m}|\alpha_i|^2 A_iA_i^* + \sum_{i=1}^{m}\sum_{j>i}|\alpha_j|^2 A_iA_i^* + \sum_{i=1}^{m}\sum_{j<i}|\alpha_j|^2 A_iA_i^* = \sum_{j=1}^{m}|\alpha_j|^2\Big(\sum_{i=1}^{m}A_iA_i^*\Big).$$

This completes the proof. ■

LEMMA 2.8. *Let A, B, C be operators in H, $0 \leqslant A \leqslant B$. Let $0 < r \leqslant 1$ be a real number. Then*

$$C^*A^rC \leqslant C^*B^rC.$$

Proof. Our assumptions imply that $0 \leqslant A^r \leqslant B^r$ ([12]). On the other hand, for any operators C and D, $C^*DC \geqslant 0$ whenever $D \geqslant 0$. Thus $C^*A^rC \leqslant C^*B^rC$. ■

LEMMA 2.9. *Let A, B be operators in H with $0 \leqslant A \leqslant B$, and let $r > 0$ be a real number. Then*

$$\operatorname{trace} A^{r+2} \leqslant \operatorname{trace} AB^rA.$$

Proof. If $0 < r \leqslant 1$, then our statement follows from Lemma 2.8 and the monotonicity of the trace, i.e. the property that $\operatorname{trace} D \geqslant 0$ whenever $D \geqslant 0$. In the case $r > 1$ we denote the orthonormal basis of eigenvectors of the operator A by (φ_k); then $A(\varphi_k) = a_k\varphi_k$ with $a_k \geqslant 0$ for $k = 1, \ldots$ Since $A \leqslant B$, we have $a_k \leqslant \langle B(\varphi_k), \varphi_k\rangle$ for $k = 1, \ldots$ On the other hand, Lemma 2.1 implies that $\langle B(\varphi_k), \varphi_k\rangle^r \leqslant \langle B^r(\varphi_k), \varphi_k\rangle$. Combining these inequalities, we get

$$\operatorname{trace} A^{r+2} = \sum_{k=1}^{\infty} a_k^{r+2} \leqslant \sum_{k=1}^{\infty} a_k^2 \langle B(\varphi_k), \varphi_k\rangle^r$$
$$\leqslant \sum_{k=1}^{\infty} a_k^2 \langle B^r(\varphi_k)\cdot\varphi_k\rangle = \sum_{k=1}^{\infty}\langle AB^rA(\varphi_k), \varphi_k\rangle = \operatorname{trace} AB^rA.$$

This completes the proof. ■

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Let $2 \leqslant p < \infty$ and let $E \subset S_p$ be an n-dimensional subspace. We define $F_1, \ldots, F_n$ and F as in Theorem 2.3. On the space $\mathscr{F}(H, H)$ of all finite rank operators we define the scalar product by $\langle A, B\rangle = (\operatorname{trace} AB^*F^{p-2})^{1/2}$, the completion of $\mathscr{F}(H, H)$ under the norm $|||\cdot|||$ determined by this scalar product will be denoted by $\bar S_2$. Then $\bar S_2$ is a Hilbert space. We define $P = i_2 \circ Q \circ i_1$, where i_1: $S_p \to \bar S_2$ is the identity map, Q: $\bar S_2 \to i_1(E)$ is the orthogonal projection, i_2: $i_1(E) \to E$ is the identity map. Obviously, P is a projection onto E and $\gamma_2(P) \leqslant \|i_1\|\cdot\|i_2\|$.

To estimate the norm of i_1 let us recall that for any $1 < q < \infty$ and operators $B \in S_q$, $C \in S_{q^*}$ ($q^* = q(q-1)$), the composition CB is nuclear and $|\operatorname{trace} CB| \leqslant \|B\|_q \cdot \|C\|_{q^*}$ (cf. [6], III.7.2). So, let $q > 1$ be such real number that $2/p - 1/q = 1$. We have

$$|||A||| = (\operatorname{trace} AA^*F^{p-2})^{1/2} \leqslant \|AA^*\|_{p/2}^{1/2}\cdot\|F^{p-2}\|_q^{1/2}$$
$$= \|A\|_p \|F\|_p^{1/2-1/p}$$

for every $A \in S_p$. Thus $\|i_1\| \leqslant \|F\|_p^{1/2-1/p} = 1$.

To estimate the norm of i_2 let us consider any $T \in E$, $T = \sum_{i=1}^{n}\alpha_i F_i$. By the properties of the trace we have

$$\|i_2(T)\|^p = \|T\|_p^p = \operatorname{trace}(TT^*)^{p/2}$$
$$= \operatorname{trace} TT^*(TT^*)^{p/2-1} = \operatorname{trace} T^*(TT^*)^{p/2-1}T.$$

Applying Lemma 2.7 to the operator TT^* and then Lemma 2.9 with the exponent $r = p/2-1 > 0$, we can estimate

$$\|i_2(T)\|^p \leqslant \Big(\sum_{i=1}^{n} |a_i|^2\Big)^{p/2-1} \cdot \operatorname{trace} T^*(FF^*)^{p/2-1}T$$
$$= \Big(\sum_{i=1}^{n} |a_i|^2\Big)^{p/2-1} \cdot \operatorname{trace} TT^*F^{p-2},$$

since $F > 0$. Now we can use equality (17) and find the last line is equal to

$$n^{p/2-1}(\operatorname{trace} TT^*F^{p-2})^{p/2} = n^{p/2-1}|||T|||^p.$$

Thus $\|i_2\| \leqslant n^{1/2-1/p}$. So $\gamma_2(P) \leqslant n^{1/2-1/p}$ and this completes the proof. ■

COROLLARY 2.10. *Let G be a n-dimensional Banach space isometric to a quotient of a subspace of S_p, $1 < p < \infty$. Then $d(G, l_2^n) \leqslant n^{|1/p-1/2|}$.*

Proof. As in the proof of Corollary 1.11, we can assume that $p \geqslant 2$. Let $E \subset S_p$ be a subspace and $Q: E \to G$ the quotient map. By Theorem 2.2 and Proposition 0.2 there is an extension $\tilde{Q}: S_p \to G$ with $\gamma_2(\tilde{Q}) \leqslant n^{1/2-1/p}$. Thus for an isometric embedding $Q^*: G^* \to E^*$ we have $\gamma_2(Q^*) = \gamma_2(Q) \leqslant \gamma_2(\tilde{Q}) \leqslant n^{1/2-1/p}$. Hence $d(G, l_2^n) = d(G^*, l_2^n) \leqslant n^{|1/2-1/p|}$. ■

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