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Complex analytic properties of certain uniform Fréchet–Schwartz algebras

by

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Abstract. For a class of uniform Fréchet–Schwartz algebras we prove some theorems which hold for Stein algebras; in particular, the principle of semicontinuity of fiber-dimensions and a hereditary maximum modulus principle. The proofs in complex analysis depend on the classical Weierstrass theorems. Since such a local theory is not available in our Fréchet–Schwartz setting, we had to develop new proofs. Besides their interest in themselves these theorems serve for a functional-analytic characterization of Stein algebras.

Let $\mathscr{A}$ be a uniform Fréchet algebra with locally compact spectrum X. Then the elements of $\mathscr{A}$ may be understood as continuous complex-valued functions on X. Thus the pair $(\mathscr{A}, X)$ constitutes a *natural system* (or *natural algebra*) in the sense of Rickart (cf. [7]). We shall always assume that $\mathscr{A}$ is, moreover, a *Schwartz space*; thereby we know that certain restriction maps are compact operators (see (1.3)). (Note that any Fréchet nuclear space is a Schwartz space.) A wide class of examples is provided by the algebras $\mathcal{O}(X)$ of all holomorphic functions on reduced complex analytic spaces $(X, \mathcal{O})$ having a countable basis for the topology of X. $\mathcal{O}(X)$ becomes a uniform Fréchet–Schwartz algebra when endowed with the compact open topology.

Now, assume that $(X, \mathcal{O})$ is a Stein space. Then the *Stein algebra* $\mathcal{O}(X)$ contains all information on the complex space $(X, \mathcal{O})$; in particular, the underlying topological space X is rediscovered by $\mathcal{O}(X)$ as a homeomorphic copy of its spectrum (cf. Forster [1]). Stein spaces enjoy a rich function theory (cf. [3]).

In this paper we shall prove some theorems, valid for Stein algebras, in the general uniform Fréchet–Schwartz setting. We summarize our results without listing up the hypotheses, for just now:

Recall that a hull in X is the zero set of some ideal in $\mathscr{A}$. There are *no compact hulls besides finite sets* (4.3). An important theorem in complex analysis asserts that the *fiber dimensions* of a holomorphic map vary *semicontinuously*; using the notion of Chevalley dimension, we prove this theorem in our setting (5.2). If a point of X, understood as a closed maximal ideal in $\mathscr{A}$, is topologically generated by a finite number of functions (on relatively compact neighbourhoods), then the *spectrum has finite dimension* in this point (5.3). As a byproduct we obtain the fact that the topology of X has a countable basis, and thus is *metrizable* (5.4). In Section 6 we shall establish a *maximum modulus principle* which holds for X as well as for all hulls in X.

Difficulties for the above theorems, in our setting, arise from a *lack of a local theory*. In complex analysis such a local theory is provided by the classical Weierstrass theorems. Therefore, the standard proofs in complex analysis are not transferable to our situation, when local problems are involved. So we had to develop independent proofs.

A hypothesis often assumed in this paper is *strong uniformity* (see (2.2)). $\mathscr{A}$ is called *strongly uniform* if for all kernel ideals $\mathscr{I} \subset \mathscr{A}$ the quotient algebras $\mathscr{A}/\mathscr{I}$, endowed with the natural quotient topology, are uniform Fréchet algebras, too. It might be of interest to know conditions for uniform Fréchet algebras which assure strong uniformity.

An application of the theory developed in this paper will be given in [6]. We shall characterize (reduced) Stein algebras by functional analytic conditions of the type we are using in this paper.

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1. Preliminaries

(1.1) A *Fréchet algebra* (= (F)-*algebra*) is a commutative, locally convex, complete algebra over the complex field $\boldsymbol{C}$ with unit whose topology is generated by a countable number of seminorms.

Now let $\mathscr{A}$ be a Fréchet-algebra. By $\sigma\mathscr{A}$ we denote the *spectrum of* $\mathscr{A}$, the set of all continuous $\boldsymbol{C}$-algebra homomorphisms $\varphi\colon \mathscr{A}\to\boldsymbol{C}$ with $\varphi\neq 0$; as usual it is endowed with the Gelfand topology (= weak-* topology $\sigma(\mathscr{A}',\mathscr{A})$). Let $\mathscr{C}(\sigma\mathscr{A})$ denote the algebra of all continuous functions on $\sigma\mathscr{A}$ endowed with the compact open topology.

Then the standard *Gelfand representation*

$$\Gamma\colon \mathscr{A}\to\mathscr{C}(\sigma\mathscr{A}),\qquad a\mapsto\hat{a},$$

given by setting $\hat{a}(\varphi)\colon=\varphi(a)$ for $a\in\mathscr{A}$, $\varphi\in\sigma\mathscr{A}$, is a continuous $\boldsymbol{C}$-algebra homomorphism.

Call $\mathscr{A}$ a *uniform Fréchet algebra* (= (uF)-*algebra*) if the Gelfand representation Γ induces a topological isomorphism of $\mathscr{A}$ onto a closed subalgebra $\Gamma(\mathscr{A})\subset\mathscr{C}(\sigma\mathscr{A})$.

We shall identify $\mathscr{A}$ and $\Gamma(\mathscr{A})$; also, we shall identify the elements $f\in\mathscr{A}$ of the algebra and their Gelfand transforms $\hat{f}\in\Gamma(\mathscr{A})$.

For the most part we shall consider (uF)-algebras whose spectra are assumed to be locally compact.

Note that the pair $(\mathscr{A},\sigma\mathscr{A})$ is a *natural system* in the sense of Rickart (cf. [7], p. 357).

(1.2) Let X be a topological space. Then we call a *countable exhaustion* $\ldots\subset K_n\subset K_{n+1}\subset\ldots$ of X by compact subsets an *admissible* one if for every compact subset $K\subset X$ there exists an index $n\in\boldsymbol{N}$ such that $K\subset K_n$. If X is assumed to be locally compact and if there exists an admissible exhaustion of X, then one can even choose an admissible exhaustion satisfying $K_n\subset \mathring{K}_{n+1}$, $n\in\boldsymbol{N}$.

Now, let X be the spectrum of a (uF)-algebra $\mathscr{A}$. Then every admissible exhaustion $\ldots\subset K_n\subset K_{n+1}\subset\ldots$ of X describes the topology of $\mathscr{A}$ by means of the correspondent seminorms $\|\cdot\|_{K_n}$, $n\in\boldsymbol{N}$. Here for $f\in\mathscr{A}$ and a compact set $K\subset X$ the seminorm $\|\cdot\|_K$ is defined as usual

$$\|f\|_K\colon=\sup_{\varphi\in K}|f(\varphi)|.$$

Let $M\subset X$ be an arbitrary subset. By $\mathscr{A}_M$ we denote the separated completion of the restriction algebra $\{f_{|M}\colon f\in\mathscr{A}\}$ under the topology of uniform convergence on compact subsets of M. Obviously we have $\sigma\mathscr{A}_M=\hat{M}$, where $\hat{M}$ is the $\mathscr{A}$-convex hull of M in X; more precisely, $\hat{M}$ is the union of all sets

$$\hat{K}=\{\varphi\in X\colon |f(\varphi)|\leqslant\|f\|_K \text{ for all } f\in\mathscr{A}\}$$

with $K\subset M$ compact.

If M has been compact, then $\mathscr{A}_M$ is even a uniform Banach algebra with norm $\|\cdot\|_M$. But in general $\mathscr{A}_M$ need not even be a (uF)-algebra; namely, if M is not hemicompact, then $\mathscr{A}_M$ is a uniform locally m-convex complete algebra.

(1.3) Recall that a locally convex complete space $\mathscr{A}$ is a *Schwartz space* if for all Banach spaces $\mathscr{B}$, all continuous linear operators $\mathscr{A}\to\mathscr{B}$ are compact operators. For the theory of Schwartz spaces confer Horvath's book [4], p. 271 ff. For uniform Fréchet algebras this condition can be reformulated more conveniently:

Lemma. *Let $\mathscr{A}$ be a* (uF)*-algebra with spectrum X. Then $\mathscr{A}$ is a Schwartz algebra if and only if for every compact subset $K\subset X$ there exists a* (*larger*) *compact subset $L\subset X$ such that the restriction map $\mathscr{A}_L\to\mathscr{A}_K$ is a compact operator.*

We omit the simple proof.

(1.4) Let $\mathscr{A}$ be a (uF)-algebra with locally compact spectrum X.

We shall often assume the following condition:

(f) *For every $\varphi \in X$, the ideal* $\ker\varphi$ *is topologically finitely generated on (relatively) compact subsets $U \subset X$; more precisely: for every (relatively) compact neighbourhood $U \subset X$ of φ there exist $f_1, \ldots, f_n \in \mathscr{A}_U$ such that the ideal $(f_1, \ldots, f_n)$ is dense in* $(\ker\varphi)_U$.

In particular, if all ideals of the form $\ker\varphi$ with $\varphi \in X$ are topologically finitely generated in $\mathscr{A}$, then $\mathscr{A}$ satisfies (f). Examples will be given in (2.4) below.

2. Strongly uniform Fréchet algebras ((u^*F)-algebras)

(2.1) We shall use the terms "hull", "kernel" analogically as in the theory of uniform algebras (cf. [2]).

Again, let $\mathscr{A}$ be a (uF)-algebra with spectrum X, and let $\mathscr{F} \subset \mathscr{A}$ be a set of functions on X. (In most cases $\mathscr{F}$ will be an ideal.) Then the set

$$V(\mathscr{F}) := \{\varphi \in X : f(\varphi) = 0 \text{ for all } f \in \mathscr{F}\}$$

is called the *hull (in X) with respect to $\mathscr{F}$*. A set $M \subset X$ is called a *hull* if there exists a family $\mathscr{F} \subset \mathscr{A}$ of functions such that $M = V(\mathscr{F})$.

For a given subset $M \subset X$ we consider the ideal

$$k(M) := \{f \in \mathscr{A} : f_{|M} = 0\}.$$

It is called the *kernel (in $\mathscr{A}$) with respect to M*. An ideal $\mathscr{I} \subset \mathscr{A}$ is said to be a *kernel ideal* (or a *kernel*, for short), if it is the kernel with respect to $V(\mathscr{I})$.

(2.2) Let $\mathscr{I} \subset \mathscr{A}$ be a closed ideal and $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ an admissible exhaustion. Then the quotient algebra $\mathscr{A}/\mathscr{I}$ carries the natural quotient topology given by the sequence of seminorms

$$\|f+\mathscr{I}\|_n := \inf_{g \in \mathscr{I}} \|f+g\|_{K_n}, \qquad n \in N.$$

$\mathscr{A}/\mathscr{I}$ is an (F)-algebra under this topology. Now assume, moreover, that $\mathscr{I}$ is a kernel in $\mathscr{A}$. Then $\mathscr{A}/\mathscr{I}$ is a semisimple (F)-algebra whose seminorms satisfy

$$\|f\|_{K_n \cap V(\mathscr{I})} \leqslant \|f+\mathscr{I}\|_{K_n} \quad \text{for all } f \in \mathscr{A}.$$

But the natural quotient topology and the coarser uniform quotient topology seem not to be equivalent, in general. Since we must consider such quotient algebras without leaving the category of (uF)-algebras, we are induced to define as follows:

DEFINITION. A (uF)-algebra $\mathscr{A}$ is called *strongly uniform* ($=(u^*)$, for short) if for any kernel ideal $\mathscr{I} \subset \mathscr{A}$, the quotient algebra $\mathscr{A}/\mathscr{I}$ endowed with the natural quotient topology is again a (uF)-algebra.

(2.3) Let $\mathscr{A}$ be a (uF)-algebra and $\mathscr{I} \subset \mathscr{A}$ a kernel ideal. Then we have the following stability properties:

(i) if $\mathscr{A}$ is a (u^*F)-algebra, then $\mathscr{A}/\mathscr{I}$ is one as well;

(ii) if $\mathscr{A}$ satisfies condition (f), then $\mathscr{A}/\mathscr{I}$ does so, too;

(iii) if $\mathscr{A}$ is a Schwartz-algebra, then $\mathscr{A}/\mathscr{I}$ is one as well.

The proofs of (i) and (ii) are trivial; for (iii) confer [4].

(2.4) There are two classes of standard examples for (u^*F)-algebras.

(i) The algebras $\mathscr{C}(X)$ of all continuous C-valued functions on a locally compact hemicompact space X.

$\mathscr{C}(X)$ never is a Schwartz-algebra, except for trivial cases of X. If X is a subspace of C^n, then $\mathscr{C}(X)$ satisfies condition (f). We omit the easy proofs.

(ii) The algebras $\mathscr{O}(X)$ of all holomorphic functions on a (reduced) Stein space $(X, \mathscr{O})$.

$\mathscr{O}(X)$ is a Schwartz-algebra (cf. Gunning and Rossi [3], p. 236) and satisfies condition (f) (cf. Forster [1]).

3. On the topology of spectra

(3.1) The spectrum of a (uF)-algebra is hemicompact. This seems to be the only obvious topological property of such a spectrum. We shall give conditions for the algebra which imply good topological properties of the spectrum, except for local compactness which we shall always assume, explicitly. First we show that the rather weak condition (f), see (1.4), assures first countability of the spectrum.

THEOREM. *Let $\mathscr{A}$ be a (uF)-algebra with locally compact spectrum X. If $\mathscr{A}$ satisfies condition (f), then every point $\varphi \in X$ possesses a countable neighbourhood basis consisting of $\mathscr{A}$-convex sets.*

Proof. Let $\varphi \in X$ be given. Since X is locally compact, we can choose a compact and $\mathscr{A}$-convex neighbourhood U_0 of φ. By assumption (f), there exist functions $f_1, \ldots, f_n \in \mathscr{A}_{U_0}$ such that the ideal $(f_1, \ldots, f_n)$ is a dense subspace of $(\ker\varphi)_{U_0}$. For any $\varepsilon > 0$ we consider the neighbourhood of φ

$$U_\varepsilon := f_1^{-1}(\Delta_\varepsilon) \cap \ldots \cap f_n^{-1}(\Delta_\varepsilon),$$

where $\Delta_\varepsilon := \{z \in C : |z| < \varepsilon\}$. We shall show that for any neighbourhood V of φ there exists an $\varepsilon > 0$ such that $U_\varepsilon \cap U_0 \subset V$.

According to the definition of the Gelfand topology for X we may assume without loss of generality, that V has the shape

$$V = h_1^{-1}(\Delta_1) \cap \ldots \cap h_m^{-1}(\Delta_1)$$

with appropriate $h_1, \ldots, h_m \in \ker\varphi$.

Now choose an arbitrary δ with $0 < \delta < 1$ and fix it for the sequel. Since the ideal $(f_1, \ldots, f_n)$ is a dense subspace of $(\ker\varphi)_{U_0}$, there exist functions $g_j, g_{ji} \in \mathscr{A}$ such that, when restricted to U_0,

$$g_j = \sum_{i=1}^{n} g_{ji} f_i \in (\ker\varphi)_{U_0}, \quad 1 \leqslant j \leqslant m,$$

satisfy

$$\|h_j - g_j\|_{U_0} < \delta.$$

Setting $V' := U_0 \cap g_1^{-1}(\Delta_{1-\delta}) \cap \ldots \cap g_m^{-1}(\Delta_{1-\delta})$, we have $V' \subset V$. Choose an $\varepsilon > 0$ satisfying

$$\varepsilon \cdot \sum_{i,j} \|g_{ji}\|_{U_0} < 1 - \delta.$$

Then, for all $\varphi \in U_0 \cap U_\varepsilon$ and $1 \leqslant j \leqslant m$, we obtain the estimation

$$|g_j(\varphi)| = \Big|\sum_i g_{ji}(\varphi) f_i(\varphi)\Big| < \varepsilon \cdot \sum_i |g_{ji}(\varphi)|$$
$$< (1-\delta) \cdot \sum_j |g_{ji}(\varphi)| \cdot \Big(\sum_{i,j} \|g_{ji}\|_{U_0}\Big)^{-1} \leqslant 1 - \delta,$$

hence $\varphi \in V'$ and $U_0 \cap U_\varepsilon \subset V$, as desired.

(3.2) In order to obtain further topological qualities for X we must add two more hypotheses. Not until (5.4), we shall be able to show that the locally compact spectrum X of a (u*F)-Schwartz algebra $\mathscr{A}$ satisfying (f) has a *countable basis for its topology*. From this it follows that any open or closed subset $U \subset X$ is hemicompact and hence the correspondent algebras $\mathscr{A}_U$ (see (1.2)) are also (uF)-algebras.

In particular, such spectra are *metrizable*. Using the stability properties (2.3), we obtain that all hulls in X enjoy these properties, too.

In our paper [5] we have intensified the Schwartz property locally, in a natural way. Theorem 6 in [5] asserts that X and all hulls in X are under this assumption locally connected (even without satisfying condition (f)).

4. Chevalley dimension for (*uF*)-algebras and a main lemma

(4.1) We introduce the notion of a complex dimension for (uF)-algebras, which is motivated by the Chevalley dimension in complex analysis.

Let $\mathscr{A}$ be a (uF)-algebra with spectrum X. For any $\varphi \in X$ we consider the integer $d(\varphi)$, defined as the minimum of all $n \in \boldsymbol{N}$ such that there exist $f_1, \ldots, f_n \in \ker\varphi$ and there exists a neighbourhood U of φ such that the fibers of the mapping $(f_1, \ldots, f_n)\colon U \to \boldsymbol{C}^n$ are finite sets.

If this minimum does not exist, we set $d(\varphi) = \infty$.

The *dimension of* φ *in* X is defined by

$$\dim_\varphi X := \begin{cases} 0, & \text{if } \varphi \text{ is an isolated point in } X, \\ d(\varphi), & \text{otherwise.} \end{cases}$$

Let Y be a hull in X with $\varphi \in Y$. Then we canonically define the *dimension of* φ *with respect to the hull* Y

$$\dim_\varphi Y := \dim_\varphi \sigma\big(\widehat{\mathscr{A}/k(Y)}\big).$$

It is well known that for example for Stein algebras the above dimension equals the topological Krull dimension (cf. [1]).

(4.2) Definition (4.1) immediately yields the

PROPOSITION. *Let* $\mathscr{A}$, $Y \subset X$ *be as above. Then*

(i) *the mapping* $Y \mapsto \boldsymbol{N} \cup \{0, \infty\}$,

$$\varphi \mapsto \dim_\varphi Y$$

is semicontinuous, more precisely, for every $\varphi \in Y$ *there exists a neighbourhood* $U \subset Y$ *of* φ *such that* $\dim_\varphi Y \geqslant \dim_\psi Y$, *for all* $\psi \in U$.

If $\dim_\varphi Y < \infty$ *for all* $\varphi \in Y$, *then this mapping is bounded on any relatively compact subset of* Y.

(ii) *The set* $\{\varphi \in Y\colon \dim_\varphi Y \leqslant n\}$ *is open, the set* $\{\varphi \in Y\colon \dim_\varphi Y \geqslant n\}$ *is closed, for all* $n \in \boldsymbol{N} \cup \{0\}$.

In order to prove our theorem on the semicontinuity of fiber dimensions, we need Theorem (4.3) and Lemma (4.4). Theorem (4.3) is an important theorem in Complex analysis of Stein spaces (cf. [3], p. 241), whereas Lemma (4.4) has a more technical character.

(4.3) THEOREM. *Let* $\mathscr{A}$ *be a (u*F)-Schwartz algebra and* Y *a compact hull in* $X = \sigma\mathscr{A}$. *Then* Y *is a finite set.*

Proof. Let $\mathscr{I} \subset \mathscr{A}$ be the kernel with respect to Y (see (2.1)). Then the spectrum of $\mathscr{A}/\mathscr{I}$ obviously equals Y. By the strong uniformity of $\mathscr{A}$ the quotient algebra $\mathscr{A}/\mathscr{I}$ is also a (uF)-algebra; it is even a uniform Banach algebra with norm $\|\cdot\|_Y$ since Y has been assumed to be compact. On the other hand $\mathscr{A}/\mathscr{I}$ is a Schwartz space since $\mathscr{I}$ is a closed ideal. But a Banach–Schwartz space is a finite dimensional vector space (cf. [4]). Hence the spectrum Y of $\mathscr{A}/\mathscr{I}$ consists of a finite number of points.

(4.4) Lemma. *Let $\mathscr{A}$ be a (u*F)-Schwartz algebra with locally compact spectrum X. Let $Y \subset X$ be a hull and $f_1, \ldots, f_n \in \mathscr{A}$ functions such that $Y \cap V(f_1, \ldots, f_n) \cap \mathring{K}$ is a finite set for a compact subset $K \subset X$.*

Then there exists a $\delta > 0$ such that

$$Y \cap V(f_1 - g_1, \ldots, f_n - g_n) \cap K$$

is a finite set for all $g_1, \ldots, g_n \in \mathscr{A}$ with $\|g_i\|_K < \delta$.

Proof. Abbreviate $f := (f_1, \ldots, f_n)$, $g := (g_1, \ldots, g_n)$. Let $Y \cap V(f) \cap \cap \mathring{K}$ be the finite set $\{\varphi_1, \ldots, \varphi_r\}$.

We want to prove our lemma by contradiction. Suppose there is no $\delta > 0$ with the asserted property. Then there exist a sequence $\delta_m > 0$, $m \in N$, tending to zero, and n-tuples $g^{(m)} \in \mathscr{A}^n$ satisfying $\|g_i^{(m)}\|_K < \delta_m$, $1 \leqslant i \leqslant n$, such that $K \cap Y_m$ is an infinite set, where $Y_m := Y \cap V(f - g^{(m)})$. Y_m is obviously a hull as well as its relatively open connected components, since hulls are $\mathscr{A}$-convex. Now choose open neighbourhoods U_ϱ of φ_ϱ satisfying $\overline{U}_\varrho \subset \mathring{K}$, $1 \leqslant \varrho \leqslant r$. For all $m \in N$, we then have

$$Y_m \cap \Big(K - \bigcup_{\varrho=1}^{r} U_\varrho\Big) \neq \emptyset;$$

for, if the contrary were the case, then $Y_m \cap K$ would be a compact hull and hence a finite set by (4.3), in contradiction to the construction of Y_m.

Select arbitrary points $\psi_m \in Y_m \cap (K - \bigcup_{\varrho=1}^{r} U_\varrho)$. By the compactness of K there exists a convergent subsequence ψ_{m_k} with limit $\psi_0 \in K$. Clearly $\psi_0 \notin \bigcup_{\varrho=1}^{r} U_\varrho$, thus there is an index i such that $f_i(\psi_0) \neq 0$.

But on the other hand we have

$$f(\psi_{m_k}) - g^{(m_k)}(\psi_{m_k}) = 0,$$

since $\psi_{m_k} \in Y_{m_k}$; thus $|f(\psi_{m_k})| < \delta_{m_k}$ and $\lim f(\psi_{m_k}) = f(\psi_0) = 0$. Contradiction.

5. Semicontinuity of fiber dimensions and further applications of Section 4

(5.1) The semicontinuity of fiber dimensions is an important principle for complex analytic mappings (for a proof cf. [3], p. 114). We do not have a local theory for our function algebraic setting like in complex analysis, where such a theory is provided by the Weierstrass theorems. So we must give an independent proof for Theorem (5.2); it will be based on (4.3) and (4.4).

As further applications of Section 4 we shall obtain the finite dimensionality (Theorem (5.3)) and the second countability (Theorem (5.4)) for spectra satisfying condition (f).

(5.2) Theorem (Semicontinuity of fiber dimensions). *Let $\mathscr{A}$ be a (u*F)-Schwartz algebra with locally compact spectrum X, and let $f = (f_1, \ldots \ldots, f_n)$: $X \to C^n$ be a map with $f_i \in \mathscr{A}$, $1 \leqslant i \leqslant n$.*

Then the map

$$X \to N \cup \{0, \infty\}, \quad \varphi \mapsto \dim_\varphi f^{-1}(f(\varphi))$$

is semicontinuous, i.e. for every $\varphi \in X$ there exists a neighbourhood U of φ such that

$$\dim_\varphi f^{-1}(f(\varphi)) \geqslant \dim_\psi f^{-1}(f(\psi)) \quad \textit{for all } \psi \in U.$$

Proof. Let $\varphi \in X$ be given. We may assume: $\dim_\varphi f^{-1}(f(\varphi)) := r < \infty$ and $f(\varphi) = 0$. Thus there exist functions $h_1, \ldots, h_r$, lying in the (uF)-algebra $\mathscr{A}/k(f^{-1}(f(\varphi)))$, such that the fibers of the map

$$(h_1, \ldots, h_r): f^{-1}(f(\varphi)) \to C^r$$

are finite sets in a neighbourhood of φ. Select representatives $g_\varrho \in \mathscr{A}$ of h_ϱ, that is,

$$g_\varrho | f^{-1}(f(\varphi)) = h_\varrho, \quad 1 \leqslant \varrho \leqslant r.$$

Then the set

$$V(f_1, \ldots, f_n) \cap V(g_1, \ldots, g_r) = V(f_1, \ldots, f_n, g_1, \ldots, g_r)$$

is a finite set, when intersected with an appropriate relatively compact neighbourhood $U \subset X$ of φ.

Now apply Lemma (4.4), setting $Y = X$. Thus there exists a $\delta > 0$ such that

$$V(f_1 - \delta_1, \ldots, f_n - \delta_n, g_1 - \delta_{n+1}, \ldots, g_r - \delta_{n+r}) \cap U$$

is a finite set for all $\delta_i \in C$ satisfying $|\delta_i| < \delta$, $1 \leqslant i \leqslant n+r$. This means that the map $(g_1, \ldots, g_r)$ restricted to $f^{-1}(\Delta_\delta^n) \cap U$, has finite fibers ($\Delta_\delta^n$ denotes the open n-polycylinder centered at 0 with radius δ). Thus for all $\psi \in f^{-1}(\Delta_\delta^n) \cap U$ we have obtained $\dim_\psi f^{-1}(f(\psi)) \leqslant r$, as desired.

(5.3) Theorem. *Let $\mathscr{A}$ be a (u*F)-Schwartz algebra with locally compact spectrum X. If $\mathscr{A}$ satisfies condition* (f), *then we have* $\dim_\varphi X < \infty$ *for all* $\varphi \in X$.

Proof. Let $\varphi \in X$ and a compact neighbourhood U of φ be given. Then condition (f) yields functions $g_1, \ldots, g_n \in \mathscr{A}_U$ which generate an ideal that is dense in $(\ker\varphi)_U$. Obviously we have $V(g_1, \ldots, g_n) = \{\varphi\}$.

In order to obtain $\dim_\varphi X \leqslant n$ we wish to apply Lemma (4.4). For that purpose, however, we need *global* functions $h_1, \ldots, h_n \in \mathscr{A}$ such that $V(h_1, \ldots, h_n) = \{\varphi\}$, locally around φ. We proceed as follows. According to the proof of Theorem (3.1) we can choose an $\varepsilon > 0$ such that $g_1^{-1}(\Delta_\varepsilon) \cap \ldots \cap g_n^{-1}(\Delta_\varepsilon) \cap U$ is an open neighbourhood of φ. Since $\mathscr{A}$ has dense restriction image in $\mathscr{A}_U$, there exist $h_1, \ldots, h_n \in \ker\varphi$ such that $\|h_i - g_i\|_U < \varepsilon$, $1 \leqslant i \leqslant n$.

By the above choice of ε we obtain that $V(h_1, \ldots, h_n) \cap U$ is a compact hull and hence a finite set, by Theorem (4.3). Thus there is a small neighbourhood $W \subset U$ of φ such that

$$V(h_1, \ldots, h_n) \cap W = \{\varphi\}.$$

Now we are ready to apply Lemma (4.4). It asserts for our situation the existence of a $\delta > 0$ such that

$$V(h_1 - \delta_1, \ldots, h_n - \delta_n) \cap W$$

is a finite set for all $\delta_i \in \Delta_\delta$.

Thus the mapping $(h_1, \ldots, h_n)$ restricted to the set $\big((h_1, \ldots, h_n)^{-1}(\Delta_\delta^n)\big) \cap W$ has finite fibers, and hence

$$\dim_\varphi X \leqslant n < \infty.$$

Remark. The proof yields even the result that the dimension in a point $\varphi \in X$ is bounded by the minimal number of topological generators of the ideals $(\ker\varphi)_U$, the minimum taken from all relatively compact neighbourhoods U of φ.

In complex analysis this number is interpreted as the *embedding dimension of φ in X*.

(5.4) Theorem. *Let $\mathscr{A}$ be a (u*F)-Schwartz algebra with locally compact spectrum X. If $\mathscr{A}$ satisfies condition* (f), *then the topology of X has a countable basis.*

For further consequences see (3.2).

Proof. First we shall show that any compact subset $K \subset X$ has a countable basis for its relative topology. By Theorem (5.3) we know that $\dim_\varphi X < \infty$ for all $\varphi \in K$. Since K is compact, we can choose an open covering $U_1, \ldots, U_n$ of K and functions $f_1^{(1)}, \ldots, f_{r_1}^{(1)}; f_1^{(2)}, \ldots, f_{r_2}^{(2)}; \ldots \ldots, f_1^{(n)}, \ldots, f_{r_n}^{n} \in \mathscr{A}$ such that the mapping

$$(f_1^{(1)}, \ldots, f_{r_1}^{(1)};\ f_1^{(2)}, \ldots, f_{r_n}^{(n)}): \bigcup_{\nu=1}^{n} U_\nu \to \boldsymbol{C}^{r_1+\ldots+r_n}$$

has finite fibers. It can easily be seen that, by adding further appropriate functions $g_1, \ldots, g_{r_0} \in \mathscr{A}$, one can even achieve the injectivity of the mapping

$$(f_1^{(1)}, \ldots, f_{r_n}^{(n)}, g_1, \ldots, g_{r_0}): K \to \boldsymbol{C}^{r_0+\ldots+r_n}.$$

Moreover, this map embeds K homeomorphically into $\boldsymbol{C}^{r_0+\ldots+r_n}$—since K is compact—and thus there is a countable basis for the topology of K.

Choose an admissible exhaustion $\ldots \subset K_m \subset K_{m+1} \subset \ldots$ of X. By the above considerations there is a countable basis $\mathfrak{B}_m'$ for the topology of K_m, $m \in \boldsymbol{N}$. Using the elements of $\mathfrak{B}_m'$ we shall construct families $\mathfrak{B}_m$ of open subsets of X, as follows: let all the elements $B_m \in \mathfrak{B}_m'$ satisfying $B_m \subset \mathring{K}_m$ belong to $\mathfrak{B}_m$ as well; for any $B_m \in \mathfrak{B}_m'$ with $B_m \not\subset \mathring{K}_m$, choose an X-open U_m such that $U_m \cap B_m = K_m$, and send this element U_m to the family $\mathfrak{B}_m$. From the local compactness of X it follows easily that the union $\bigcup_{m\in N} \mathfrak{B}_m$ provides a countable basis for the topology of X.

(5.5) Remark. It turns out that all theorems of this paper remain valid, if one replaces assumption (f) by $\dim_\varphi X < \infty$, for all $\varphi \in X$. In particular: (3.1), (3.2), (5.3), (5.4), (6.3). (The proofs are slightly more complicated.)

6. A hereditary maximum modulus theorem

(6.1) Let $\mathscr{A}$ be a (uF)-algebra with locally compact spectrum X.

Definition. (i) $\mathscr{A}$ is called a *local maximum modulus algebra with respect to $\varphi \in X$* if there is a relatively compact neighbourhood U of φ such that $\|f\|_U = \|f\|_{\partial U}$, for all $f \in \mathscr{A}$. (∂U denotes the topological boundary of U.)

(ii) $\mathscr{A}$ is called a *maximum modulus algebra* if it is a local maximum modulus algebra with respect to φ, for all non-isolated $\varphi \in X$.

(iii) $\mathscr{A}$ is called a *hereditary maximum modulus algebra* if $\mathscr{A}/\mathscr{I}$ is a maximum modulus algebra for all kernel ideals $\mathscr{I} \subset \mathscr{A}$ (see (2.1)).

(6.2) Lemma. *Let $\mathscr{A}$, X be as above. $\mathscr{A}$ is a maximum modulus algebra if and only if the equation*

$$\|f\|_K = \|f\|_{\partial K}$$

holds, for all compact subsets $K \subset X$ without isolated points and for all $f \in \mathscr{A}$.

Proof. Only necessity needs to be shown. Without loss of generality, let a compact $K \subset X$ be given such that $\mathring{K} \neq \emptyset$. By hypothesis, for any $\varphi \in \mathring{K}$ there is a compact neighbourhood $L(\varphi)$ such that $\|f\|_{L(\varphi)} = \|f\|_{\partial L(\varphi)}$ for all $f \in \mathscr{A}$. Choose a compact neighbourhood $K(\varphi) \subset L(\varphi) \cap \mathring{K}$. Again, we have $\|f\|_{K(\varphi)} = \|f\|_{\partial K(\varphi)}$ for all $f \in \mathscr{A}$, by Rossi's local maximum modulus principle (cf. [2], p. 92). Since φ lies in the $\mathscr{A}$-convex hull of $\partial K(\varphi)$, there exists a (positive) representing measure μ_φ for φ which is supported on a subset of $\partial K(\varphi)$ (cf. [2], p. 33). Clearly, $\mu_\varphi \neq \delta_\varphi$ (δ_φ—unit point mass at φ). Therefore, φ cannot belong to the Choquet boundary $\chi\mathscr{A}_K$ with respect to $\mathscr{A}_K$ (cf. [8]). As this is true for all $\varphi \in \mathring{K}$, we have $\mathring{K} \cap \chi\mathscr{A}_K = \emptyset$; consequently even $\mathring{K} \cap \overline{\chi\mathscr{A}_K} = \emptyset$. Since the closure of the Choquet boundary equals the Shilov boundary $\gamma\mathscr{A}_K$, we obtain the desired result: $\gamma\mathscr{A}_K \subset \partial K$.

(6.3) THEOREM. *Let $\mathscr{A}$ be a* (u*F)*-Schwartz algebra, satisfying condition* (f), *with locally compact spectrum X. Then $\mathscr{A}$ is a hereditary maximum modulus algebra.*

Proof. By the stability properties as listed up in (2.3), we know about all kernel ideals $\mathscr{I} \subset \mathscr{A}$ that the quotient algebras $\mathscr{A}/\mathscr{I}$ also fulfill the assumptions of our theorem. Thus it suffices to show that $\mathscr{A}$ is a local maximum modulus algebra with respect to φ, for all $\varphi \in X$.

We shall proceed indirectly. Let $\varphi_0 \in X$ be non-isolated and a compact $\mathscr{A}$-convex neighbourhood K of φ_0 be given. Suppose that $\mathscr{A}$ is not a local maximum modulus algebra with respect to φ_0. For any compact neighbourhood $L \subset K$ of φ_0 we then have $\gamma\mathscr{A}_L \not\subset \partial L$. Rossi's local maximum modulus principle asserts on the Shilov boundary for $\mathscr{A}_L$

$$\gamma\mathscr{A}_L \subset \partial L \cup \big(L \cap (\gamma\mathscr{A}_K)\big).$$

Hence $\gamma\mathscr{A}_K \cap \mathring{L} \neq \emptyset$. Thus φ_0 is a cluster point for $\gamma\mathscr{A}_K$ and, since Shilov boundaries are compact, φ_0 is an element of $\gamma\mathscr{A}_K$.

Now, choose an admissible exhaustion $\ldots \subset K_n \subset K_{n+1} \subset \ldots$ of X and a neighbourhood basis $\ldots \supset U_n \supset U_{n+1} \supset \ldots$ of φ_0 (which exists by (3.1)), satisfying $U_1 \subset \mathring{K}_1$. By the above considerations we know that $\varphi_0 \in \gamma\mathscr{A}_{K_n}$, $n \in \boldsymbol{N}$.

We use a standard argument from the theory of Shilov boundaries (cf. [8], p. 62) and obtain the following. For any $n \in \boldsymbol{N}$, there exists a function $h_n \in \mathscr{A}_{K_n}$ such that

$$\|h_n\|_{U_n} = 1 \quad \text{and} \quad \|h_n\|_{K_n - U_n} < 1/n.$$

Approximate the h_n's on K_n by global functions $g_n \in \mathscr{A}$:

$$\|g_n - h_n\|_{K_n} < 1/n.$$

Setting $C_n := \max_{m \leqslant n} \|g_m\|_{K_n}$, we obtain the estimation

$$\|g_m\|_{K_n} \leqslant \begin{Bmatrix} C_n & \text{for } m \leqslant n \\ 1+1/n & \text{for } m > n \end{Bmatrix} \leqslant \max(2, C_n) < \infty,$$

for all $m, n \in \boldsymbol{N}$. As the right-hand side is independent on m, the sequence $(g_m)_{m\in\boldsymbol{N}}$ is bounded in $\mathscr{A}$. Every Fréchet-Schwartz space is Montel, in

particular (cf. [4], p. 277); hence there exists a convergent subsequence $(g_{m_k})_{k\in\boldsymbol{N}}$ with limit $g \in \mathscr{A}$. Now observe that g is the zero function, because (g_{m_k}) is converging pointwise to zero on $X - \{\varphi_0\}$ and g is a continuous function.

But contradicting this fact, we know that

$$1 \leqslant \|g_{m_k}\|_{K_l} = \|g - g_{m_k}\|_{K_l} \quad \text{for all } k, l \in \boldsymbol{N}.$$

Hence there exists a neighbourhood $L \subset K$ of φ_0 satisfying $\gamma\mathscr{A}_L \subset \partial L$, which establishes our theorem.

(6.4) If one relinquishes the heredity of the maximum modulus principle, then the proof of Theorem (6.3) yields the maximum modulus principle under weaker hypotheses:

COROLLARY. *Let $\mathscr{A}$ be a (uF) Montel algebra with locally compact and connected spectrum. Then $\mathscr{A}$ is a maximum modulus algebra.*

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