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A multiplier theorem for continuous measures

by

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Abstract. Let G be a non-discrete LCA group and X a norm-compact subset of continuous measures on G. Then there exists a singular continuous independent power Hermitian probability measure μ on G such that $\mu * \nu \in L^1(G)$ for all $\nu \in X$. Some applications of this are given, as well as a partial converse: If f is a trigonometric polynomial on the compact abelian group G and $\varepsilon > 0$, then there exist singular continuous measures $\mu, \nu \in M(G)$ such that $\mu * \nu = f$ and $\|\mu\| \|\nu\| \leqslant (1+\varepsilon)\|f\|$. Riesz products are used.

1. Introduction and statement of results. In this section we state our main results (Theorems 1 and 2) and two applications (Corollaries 3 and 4). In Section 2 we prove two lemmas. Theorems 1 and 2 are proved in Section 3. A converse to Theorem 1 appears in the last section.

Riesz products are described at the end of this first section.

THEOREM 1. *Let G be a compact abelian group with dual group Γ. Let X be a norm-compact subset of continuous measures on G. Then there exists a singular continuous independent power Hermitian probability measure μ on G such that $\mu * \nu$ has an absolutely convergent Fourier–Stieltjes series for all $\nu \in X$, and such that the map from X to $L^1(\Gamma)$ given by $\nu \to (\mu * \nu)^\wedge$ is continuous.*

THEOREM 2. *Let G be a non-compact, non-discrete, locally compact abelian group with dual group Γ. Let X be a norm-compact subset of continuous measures on G. Then there exists a singular continuous independent power Hermitian probability measure μ on G such that $\mu * \nu \in L^1(G)$ for all $\nu \in X$.*

Remarks. Doss [4] proved Theorem 1 when X is a singleton. The proofs of Theorems 1 and 2 are based to a large extent on his paper. Körner [9] shows that there exist probability measures μ, ν on Kronecker sets in T such that $\mu * \nu$ is a C^∞-function. Doss [4] obtains the following corollary for compact abelian groups; his proof carries over nearly *verbatim* to our more general context and will be omitted.

Corollary 3. *Let G be a σ-compact, locally compact abelian group. Let F be a perfect non-empty subset of G. Then there exists a Borel subset S of G of zero Haar measure such that $FS = G$.*

Remarks. (i) When G has a countable base, Corollary 3 has the following form: Let F be any uncountable closed subset of G; then there exists a Borel subset S of G of zero Haar measure such that $FS = G$.

(ii) Körner [9], Bernard and Varopoulos [1], and Varopoulos [13] have shown that if F is a perfect Kronecker set in a compact metrizable abelian I-group G, then there exists another Kronecker set S in G such that $FS = G$.

(iii) Talagrand [12] has shown that if F is a compact perfect subset of the locally compact abelian group G, then there exists a compact subset S of G such that FS has non-empty interior and S has zero Haar measure. We can obtain, by our methods, only the weaker version that FS has non-zero Haar measure.

(iv) After a version of these results had been prepared and circulated, we received a communication from S. Saeki pointing out that "norm compact" can be replaced by "pseudonorm compact" in Theorems 1 and 2, and that the conclusion of Theorem 2 holds in the (new) situation of Theorem 1; see [15], Theorem 7.5.1.

Corollary 4. *Let G be a non-discrete locally compact abelian group. Then*

(i) *$\Delta M_c(G)$ is not σ-compact;*

(ii) *$\partial M(G)$ is not a G_δ; and*

(iii) *$\Gamma^- \setminus \Gamma$ is not a G_δ.*

Proof. (i) If $\Delta M_c(G)$ were σ-compact, then there would exist a sequence of measures $\{\mu_1, \mu_2, \ldots\} \subset M_c(G)$ such that $\lim \mu_j = 0$, and such that for each $\chi \in \Delta M_c(G)$, there exists $1 \leqslant j < \infty$ such that $\hat{\mu}_j(\chi) \neq 0$. Let $X = \{\mu_1, \mu_2, \ldots\} \cup \{0\}$. Then X satisfies the hypotheses of Theorem 2. Therefore, there exists $\mu \in M_c(G)$ such that

$$(1) \qquad \mu * \mu_j \in L^1(G) \quad \text{for} \quad j = 1, 2, \ldots$$

and such that μ is a Hermitian, independent power probability measure.

This last sentence implies that there exists $\chi \in \Delta M_c(G) \setminus \Gamma$ such that $\hat{\mu}(\chi) = 1$. Indeed, $\mu^n \perp \operatorname{Rad} L^1(G)$ for $n = 1, 2, \ldots$, so the spectral radius of $\mu + \operatorname{Rad} L^1$ in $M_c(G)/\operatorname{Rad} L^1$ is one. If $\chi \in \Delta M_c(G)/\operatorname{Rad} L^1 = \Delta M_c(G) \setminus \Gamma$ has $|\hat{\mu}(\chi)| = 1$, then $\hat{\mu}(|\chi|) = 1$ and $\chi \notin \Gamma$.

Now, if $\chi \in \Delta M_c(G) \setminus \Gamma$ and $\hat{\mu}(\chi) \neq 0$, then (1) implies that

$$\hat{\mu}_j(\chi) = 0 \quad \text{for} \quad j = 1, 2, \ldots$$

This proves that $\Delta M_c(G)$ is not σ-compact.

(ii)–(iii). Let X equal either $\partial M(G)$ or $\Gamma^- \setminus \Gamma$. Let $\{U_n\}$ be a sequence of open sets in $\Delta M(G)$ such that $X \subset \bigcap_{n=1}^{\infty} U_n$. Each set U_n is of the form

$$(2) \qquad \bigcup_{a} \bigcap_{k=1}^{K(n,a)} \{\chi \colon |\hat{\mu}_{k,n,a}(\chi) - \hat{\mu}_{k,n,a}(\varrho_{k,n,a})| < 1\}.$$

Since each U_n contains the compact set X, we may assume that the union over a in (2) is finite. We may also assume that each measure $\mu_{k,n,a}$ is either discrete or continuous. It is not hard to see that the discrete measures bring about no exclusion. This follows from the fact that $\Gamma^- \setminus \Gamma \subset X$, and the details are left to the reader. We may thus assume that the measures $\mu_{k,n,a}$, are all continuous. It is easy to see that for each $n = 1, 2, \ldots$, there exists an $a = a(n)$ such that $\hat{\mu}_{k,n,a(n)}(\varrho_{k,n,a(n)}) = 0$ for $k = 1, \ldots$ $\ldots, k(n, a)$. This follows from the fact that the zero functional is in the weak* closure of Γ in $L^\infty(\mu)$ for any continuous measure μ on G. Let

$$Y = \{\mu_{k,n,a(n)} \colon k = 1, \ldots, K(n, a(n)) \text{ and } n = 1, 2, \ldots\}.$$

Then there exists a measure $\omega \in M(G)$ such that $\omega * \nu \in L^1(G)$ for all $\nu \in Y$. (This follows from Theorem 2.) This measure is the extension of a Riesz product on a compact quotient of an open subgroup of G, and hence is "tame", by Brown [2]. Therefore, there exist (by the arguments of Brown [2]) elements χ of $\Delta M(G)$ such that $\hat{\omega}(\chi) = 0$, and $\chi \notin \partial M(G)$. In particular, $\chi \notin X$.

The corollary is proved.

Before proving Theorems 1 and 2 we present here a short description of Riesz products. Further discussion and references may be found in Zygmund [14], Hewitt and Zuckerman [8], Brown and Moran [3], and Brown [2].

Let G be a compact abelian group with dual group Γ. We shall use multiplicative notation for the group operation, except in R^n and Z^n. For $\gamma \in \Gamma$, let $O(\gamma)$ denote the order of γ. A subset $\Theta \subset \Gamma$ is said to be *dissociate* if it does not contain 1 and if every $\gamma \in \Gamma$ has at most one factorization (except for the order of the factors) of the form

$$(3) \qquad \gamma = \prod_{j=1}^{n} \gamma_j^{m_j},$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are distinct elements of Θ, $m_j \in \{\pm 1\}$ if $O(\gamma_j) > 2$, and $m_j = 1$ if $O(\gamma_j) = 2$. We let $\Omega(\Theta)$ denote the subset of Γ consisting of 1 and all characters of the form (3). Infinite dissociate sets exist in any infinite abelian group Γ.

Let Θ be dissociate and let $a \colon \Theta \to C$ be a function satisfying $|a(\gamma)| \leqslant \frac{1}{2}$

if $O(\gamma) > 2$, and $a(\gamma) \in (-1, 1)$ if $O(\gamma) = 2$. For $\gamma \in \Theta$, set

$$q_\gamma(x) = \begin{cases} 1+\overline{a(\gamma)\gamma(x)}+a(\gamma)\gamma(x) & \text{if} \quad O(\gamma) > 2, \\ 1+a(\gamma)\gamma(x) & \text{if} \quad O(\gamma) = 2, \end{cases}$$

and for each finite subset Φ of Θ define

$$p_\Phi(x) = \prod_{\gamma\in\Phi} q_\gamma(x).$$

We regard the trigonometric polynomials p_Φ as absolutely continuous probability measures on G. Then the net $\{p_\Phi: \Phi \subset \Theta, \Phi \text{ finite}\}$, directed by inclusion, converges weak-* to a probability measure μ. It is not difficult to see that $\hat{\mu}$ vanishes off $\Omega(\Theta)$, while

$$\hat{\mu}\Big(\prod_{j=1}^{n} \gamma_j^{m_j}\Big) = \prod_{j=1}^{n} a(\gamma_j)^{(m_j)}$$

for $\prod_{j=1}^{n} \gamma_j^{m_j} \in \Omega(\Theta)$, where $a(\gamma_j)^{(m_j)} = a(\gamma_j)$ if $m_j = 1$ and $a(\gamma_j)^{(m_j)} = \overline{a(\gamma_j)}$ if $m_j = -1$. The measure μ is called the *Riesz product* based on Θ and a. The phrase "a Riesz product μ" is used with the understanding that μ is the Riesz product based on some dissociate set Θ and function $a: \Theta \to C$ as above.

Finally, let μ be the Riesz product based on Θ and a. Then μ is singular if $\sum_\Theta |a(\gamma)|^2 = \infty$, μ is absolutely continuous if $\sum_\Theta |a(\gamma)|^2 < \infty$, μ is continuous if $\sum_\Theta (1-|a(\gamma)|) = \infty$, and μ has independent powers if $\limsup \{|a(\gamma)|: \gamma \in \Theta\} > 0$. See [2], [3], [15] for more on these results.

2. Key lemmas.

LEMMA 5. *Let G be a compact abelian group. Let $X \subset M(G)$ be a norm compact subset of continuous measures. Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0, \ldots$ be a sequence of positive numbers. Then there exists a sequence $\gamma_1, \gamma_2, \ldots$ of elements of Γ such that for each $n = 1, 2, \ldots$*

(4) *if $m_1, \ldots, m_n \in \{-2, -1, 0, 1, 2\}$ and $\prod_{j=1}^{n} \gamma_j^{m_j} = 1$, then*

$$\gamma_1^{m_1} = \ldots = \gamma_n^{m_n} = 1;$$

and

(5) $\Big|\hat{\mu}\Big(\prod_{j=1}^{n} \gamma_j^{m_j}\Big)\Big| < \varepsilon_n$, *if $m_1, \ldots, m_n \in \{-1, 0, 1\}$, $m_n \neq 0$, and $\mu \in X$.*

Proof. We may assume that if $\mu \in X$, then $\bar{\mu} \in X$. We shall produce an infinite countable subgroup Λ of Γ and a sequence $\lambda_1, \lambda_2, \ldots$ of distinct elements of Λ such that for each finite set $F \subset \Lambda$,

(6) $$\lim_{k\to\infty} \hat{\mu}(\gamma\lambda_k) = 0 \quad \text{uniformly for } \mu \in X \text{ and } \gamma \in F.$$

We first claim that there exists a compact subgroup H of G such that G/H is metrizable and such that the map $\overline{\Pi}: M(G) \to M(G/H)$ (induced by the natural map $\Pi: G \to G/H$) sends each measure in X to a continuous measure on G/H. The proof is not difficult and is left to the reader.

Let H be such a subgroup and let Λ be the annihilator of H: $\Lambda = \{\gamma \in \Gamma: \langle y, \gamma\rangle = 1 \text{ for all } y \in H\}$. Then Λ is countable. Let $F \subset \Lambda$ be finite and let $\delta > 0$. Let $\mu_1, \ldots, \mu_s$ be $(\frac{1}{2}\delta)$-dense in X. Let ω be the continuous measure given by

$$\omega = \sum_{\gamma\in F} \sum_{j=1}^{s} (\bar{\gamma}\mu_j) * (\bar{\gamma}\mu_j)^{\sim}.$$

Let $a = [\delta/(2s \operatorname{Card} F)]^2$. Let V be any compact symmetric neighbourhood of H such that $\overline{\Pi}|\omega|[\Pi(VV)] < a$. Let f be the convolution square of the characteristic function of V, divided by the Haar measure of V. Then f is positive definite and $f(0) = 1$. If we take Fourier-(Stieltjes) transforms, then the estimate $\overline{\Pi}|\omega|[\Pi(VV)] < a$ becomes $\sum_{\lambda\in\Lambda} \hat{\omega}(\lambda)\hat{f}(\lambda) < a$. Since f is positive definite, $\sum \hat{f} = 1$, while $\hat{\omega} \geqslant 0$, we see that there exists $\lambda \in \Lambda$ such that $\hat{\omega}(\lambda) < a$. The Cauchy–Schwarz inequality then implies that

$$|\hat{\mu}_j(\gamma\lambda)| < \tfrac{1}{2}\delta \quad \text{for} \quad 1 \leqslant j \leqslant s \text{ and } \gamma \in F.$$

Since the set $\{\mu_j\}_{j=1}^{s}$ is $(\frac{1}{2}\delta)$-dense in X, we see that

$$|\hat{\mu}(\gamma\lambda)| < \delta \quad \text{for} \quad \mu \in X \text{ and } \gamma \in F.$$

The construction of the sequence $\lambda_1, \lambda_2, \ldots$ is now easy. We let $F_1, F_2, \ldots$ be a sequence of finite subsets of Λ such that $F_1 \subset F_2 \subset \ldots$ and $\bigcup_{j=1}^{\infty} F_j = \Lambda$. We choose distinct $\lambda_1, \lambda_2, \ldots \in \Lambda$ such that

$$|\hat{\mu}(\lambda_j\gamma)| < 2^{-j} \quad \text{for} \quad \mu \in X \text{ and } \gamma \in F_j, \text{ and } j = 1, 2, \ldots$$

This establishes the existence of Λ and λ_j such that (6) holds.

We now proceed to construct the sequence $\{\gamma_j\}$. We first suppose that $\{\lambda_j^2\}_{j=1}^{\infty}$ is finite, where $\{\lambda_j\}$ and Λ are as above. Then we may pass to a subsequence and assume that $\lambda_j^2 = \lambda_k^2$ for $1 \leqslant j, k < \infty$. If we replace λ_j by $\lambda_j' = \lambda_j\lambda_1^{-1}$, then we may assume that $\lambda_j^2 = 1$ for all j. These changes do not affect the validity of (6).

Now, since the λ_j all have order two, there is an infinite subsequence $\{\lambda_{j(k)}\}_{k=1}^{\infty}$ which forms an infinite independent set. The independence ensures that (4) will hold if $\{\lambda_j\}_{j=1}^{\infty}$ is a subset of $\{\lambda_{j(k)}\}_{k=1}^{\infty}$ and an easy induction using (6) shows that there exists an infinite sequence $\gamma_1 = \lambda_{j(k(1))}$, $\gamma_2 = \lambda_{j(k(2))}, \ldots$ such that (5) holds.

The proof of the lemma is almost finished. We only need to consider the case in which $\{\lambda_j^2\}_{j=1}^{\infty}$ is infinite. By passing to subsequence, we may assume that $\lambda_j^2 \neq \lambda_k^2$ if $1 \leqslant j \neq k < \infty$. Then for every finite set $F = F^{-1} \subset \Lambda$, there exists $J = J(F)$ such that $\lambda_j \notin F$ and $\lambda_j^2 \notin F$, if $j \geqslant J$.

Choose (using (6)) $\gamma_1 = \lambda_{j(1)}$ such that

$$|\hat{\mu}(\gamma_1)| < \varepsilon_1 \quad \text{for all } \mu \in X.$$

Suppose that $\gamma_1 = \lambda_{j(1)}, \ldots, \gamma_k = \lambda_{j(k)}$ have been found such that (4) and (5) hold for $n = 1, \ldots, k$.

Let $F = \left\{\prod_{j=1}^{k} \gamma_j^{m_j}\colon m_1, \ldots, m_k \in \{-2, -1, 0, 1, 2\}\right\}$. Let $J = J(F)$ be so large that $\lambda_j \notin F$ and $\lambda_j^2 \notin F$ if $j \geqslant J$. Since $F = F^{-1}$, we see also that λ_j^{-1} and $\lambda_j^{-2} \notin F$ if $j \geqslant J$.

Let $j(k+1) \geqslant J$ be so large that

$$(7) \qquad |\hat{\mu}(\gamma\lambda_{j(k+1)})| < \varepsilon_{n+1}, \quad \text{if} \quad \mu \in X \text{ and } \gamma \in F.$$

This is possible by (6). Since $\bar{\mu} \in X$ if $\mu \in X$, we also see that

$$(8) \qquad |\hat{\mu}(\gamma\lambda_{j(k+1)}^{-1})| < \varepsilon_{n+1}, \quad \text{if} \quad \mu \in X \text{ and } \gamma \in F.$$

Set $\gamma_{k+1} = \lambda_{j(k+1)}$. Then, because $\gamma_{k+1}^{\pm 1} \notin F$ and $\gamma_{k+1}^{\pm 2} \notin F$, formula (4) holds for $n = k+1$. Also, by (7)–(8) and the fact that $F = F^{-1}$, we see that (5) holds for $n = k+1$.

This completes the proof of Lemma 5.

The following Lemma is an elaboration of an old result for R; see Goldberg [5]. Our proof differs from that found in [5].

Lemma 6. *Let G be a compact abelian group and let $\nu \in M^+(T^n \times G)$. Then there exists $\omega \in M^+(R^n \times G)$ such that $\hat{\omega}(z, \gamma) = \hat{\nu}(z, \gamma)$ for all $(z, \gamma) \in Z^n \times \hat{G}$. Moreover, if ν is continuous, absolutely continuous, singular, Hermitian, or has independent powers, then ω possesses the corresponding properties. Finally, $\omega \in M_0^+(R^n \times G)$, if $\nu \in M_0^+(T^m \times G)$.*

Proof. For this proof $\sum_z$ denotes n-fold summation $\sum_{k_1=-\infty}^{\infty} \ldots \sum_{k_n=-\infty}^{\infty}$ over Z^n; $x \leqslant y$ $(x, y \in R^n)$ means that this relation holds coordinatewise; and for $z = (z_1, \ldots, z_n) \in Z^n$ we write

$$z+1 = (z_1+1, \ldots, z_n+1) \quad \text{and}$$
$$I_z = [2z_1\pi, 2(z_1+1)\pi) \times \ldots \times [2z_n\pi, 2(z_n+1)\pi).$$

Define Δ and δ on R^n at $x = (x_1, \ldots, x_n)$ by

$$\delta(x) = 2^n \prod_{j=1}^{n} x_j^{-2}(1-\cos x_j);$$

$$\Delta(x) = \begin{cases} \prod_{j=1}^{n}(1-|x_j|), & |x_j| \leqslant 1;\ j = 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\hat{\delta} = (2\pi)^n \Delta$. Let k be the function defined on Z^n by

$$k(z) = \sup\{\delta(x)\colon x \in I_z\},$$

and note that

$$\sum_z k(z) \leqslant \Big[\sum_{j=-\infty}^{\infty} \sup\{2x^{-2}(1-\cos x)\colon 2j\pi \leqslant x < 2(j+1)\pi\}\Big]^n < \infty.$$

Fix $x \in R^n$ and let $p \in Z^n$ be such that $p \leqslant x \leqslant p+1$. Then we claim that

$$(9) \qquad \sum_z e^{-ix\cdot(t+2z\pi)} \delta(t+2z\pi) = \sum_{p \leqslant z \leqslant p+1} \Delta(x-z) e^{-iz\cdot t}$$

for all $t \in [0, 2\pi)^n$. This is seen as follows. Define ψ_1 and ψ_2 on R^n by

$$\psi_1(t) = e^{-ix\cdot t}\delta(t) \quad (t \in R^n);$$

$$\psi_2(t) = \begin{cases} (2\pi)^{-n} \sum_{p \leqslant z \leqslant p+1} \Delta(x-z) e^{-iz\cdot t}, & t \in [0, 2\pi)^n, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $q \in Z^n$

$$(10) \qquad \hat{\psi}_1(q) = (2\pi)^n \Delta(x+q) = (2\pi)^n \hat{\psi}_2(q).$$

Since $\psi_j \in L^1(R^n)$, the functions $g_j(t) = (2\pi)^n \sum_z \psi_j(t+2z\pi)$ $(t \in [0, 2\pi)^n)$ are in $L^1[0, 2\pi)^n$, and, as is easily seen, $\hat{\psi}_j(q) = \hat{g}_j(q)$ for all $q \in Z^n$. Thus, in view of (10), $g_1 = (2\pi)^n g_2$ a.e. on $[0, 2\pi)^n$, i.e., (9) holds for almost all $t \in [0, 2\pi)^n$. In fact, it holds everywhere on $[0, 2\pi)^n$ by continuity.

Now, let δ_1 and Δ_1 be defined on $R^n \times G$ and $R^n \times \hat{G}$ by

$$\delta_1(x, s) = \delta(x) \quad \text{and} \quad \Delta_1(x, \gamma) = \begin{cases} \Delta(x), & \gamma = 1, \\ 0, & \gamma \neq 1, \end{cases}$$

and let $f\colon R^n \times \hat{G} \to C$ be defined by

$$f(x, \gamma) = \sum_{(z,\chi)} \hat{\nu}(z, \chi) \Delta_1(x-z, \gamma\chi^{-1}),$$

where the sum is over all $(z, \chi) \in Z^n \times \hat{G}$. This sum actually has only finitely many non-zero terms for a given (x, γ). Indeed, if $p \leqslant x \leqslant p+1$ $(p \in Z^n)$, then $\Delta_1(x-z, \gamma\chi^{-1}) = 0$ unless $p \leqslant z \leqslant p+1$ and $\gamma = \chi$. Hence,

$$(11) \qquad f(x, \gamma) = \sum_{p \leqslant z \leqslant p+1} \hat{\nu}(z, \gamma) \Delta(x-z).$$

In particular, it is easy to check that when $x = p$

$$(12) \qquad f(p, \gamma) = \hat{\nu}(p, \gamma) \quad ((p, \gamma) \in Z^n \times \hat{G}).$$

Let μ be the periodic extension of ν to $R^n \times G$ (for E in $R^n \times G$, $\mu(E)$

$= \sum_z \nu(E \cap I_z \times G \cdot (2z\pi, 1)^{-1})$ and set $\omega = \delta_1 \mu$. Then $\omega \in M^+(R^n \times G)$ since

$$(13) \qquad \int_{R^n \times G} \delta_1(t, s)\, d\mu(t, s) = \sum_z \int_{I_z \times G} \delta_1(t, s)\, d\mu(t, s)$$
$$\leqslant \int_{[0,2\pi)^n \times G} d\nu(t, s) \cdot \sum_z k(z) < \infty.$$

We will show that $\hat{\omega} = f$ on $R^n \times \hat{G}$. This will establish, via (12), the first assertion of the lemma. Fix (x, γ) and let $p \in Z^n$ be such that $p \leqslant x \leqslant p+1$. Then, using (9) and (11),

$$\hat{\omega}(x, \gamma) = \int_{R^n \times G} e^{-ix\cdot t} \bar{\gamma}(s)\, \delta_1(t, s)\, d\mu(t, s)$$
$$= \sum_z \int_{[0,2\pi)^n \times G} e^{-ix\cdot(t+2z\pi)} \bar{\gamma}(s)\, \delta(t+2z\pi)\, d\nu(t, s)$$
$$= \int_{[0,2\pi)^n \times G} \sum_{p \leqslant z \leqslant p+1} \Delta(x-z) e^{-iz\cdot t} \bar{\gamma}(s)\, d\nu(t, s)$$
$$= \sum_{p \leqslant z \leqslant p+1} \hat{\nu}(z, \gamma) \Delta(x-z) = f(x, \gamma),$$

where the interchange of sum and integral is justified by absolute convergence as in (13).

It is clear from the definition of f that ω is Hermitian if ν is. Now, the map $R^n \times G \to R^n/2\pi Z^n \times G = T^n \times G$ induces a Banach algebra homomorphism $\boldsymbol{p}\colon M(R^n \times G) \to M(T^n \times G)$ given by

$$(\boldsymbol{p}\tau)(E) = \tau(E')$$

for Borel sets $E \subseteq T^n \times G$, where

$$E' = \{(x, s) \in R^n \times G\colon \text{there exists } z \in Z^n \text{ with } x+z \in [0, 2\pi)^n \text{ and } (x+z, s) \in E\}.$$

Then if ω, which is $\geqslant 0$, is not continuous (or absolutely continuous, or singular), $\nu = \boldsymbol{p}\omega$ is not continuous (or absolutely continuous, or singular). If ω does not have independent powers, ν does not have independent powers. Finally, the claim that $\omega \notin M_0(R^n \times G)$ implies $\nu \notin M_0(T^m \times \times G)$ is established as in Graham [6].

3. Proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $X \subset M(G)$ be a norm compact subset of continuous measures on the compact abelian group G. Let $Y = (X \cup \cup \{0\}) - (X \cup \{0\})$. Suppose that we can show that there exists a singular continuous independent power Hermitian probability measure ν such that every element of $\nu * Y$ has an absolutely convergent Fourier series and such that the map $\mu \to (\nu * \mu)\hat{}$ is continuous, at 0, from Y to $L^1(\Gamma)$. Then the map $\mu \to (\nu * \mu)\hat{}$ is a continuous map from X to $L^1(\Gamma)$.

Thus, to prove Theorem 1 it will be sufficient to show that there exists for each norm compact set $X \subset M_c(G)$, containing 0, a singular continuous independent power Hermitian probability measure ν such that $(\nu * X)\hat{} \subset L^1(\Gamma)$ and such that $\mu \to (\nu * \mu)\hat{}$ is continuous at zero.

Let X_0 be defined by

$$(14) \qquad X_0 = X \cup \bigcup_{n=1}^{\infty} \{2^n \mu\colon \mu \in X \text{ and } 4^{-n-1} \leqslant \|\mu\| \leqslant 4^{-n}\}.$$

Then X_0 is norm compact, since $0 \in X$.

We apply Lemma 5 to X_0 and $\varepsilon_n = 6^{-n}$, $n = 1, 2, \ldots$

We let $\Theta = \{\gamma_n\}_{n=1}^{\infty}$, where $\{\gamma_n\}_{n=1}^{\infty}$ satisfy (4)–(5) for all $n = 1, 2, \ldots$ and $\mu \in X_0$. Then, by (4), Θ is dissociate. Let ν be the Riesz product generated by Θ and $a(\gamma) \equiv \frac{1}{2}$. Then ν is a singular continuous independent power Hermitian probability measure. Also, if $\mu \in X_0$, then (5) implies that

$$(15) \qquad \sum_{\Gamma} |\hat{\nu}(\gamma)\hat{\mu}(\gamma)| \leqslant |\hat{\mu}(1)| + \sum_{n=1}^{\infty} \tfrac{1}{2} 3^n \varepsilon_n \leqslant |\hat{\mu}(1)| + \sum_{n=1}^{\infty} 3^{-n}/2.$$

Now (14)–(15) show that if $4^{-n-1} \leqslant \|\mu\| \leqslant 4^{-n}$, then

$$\|(\nu * \mu)\hat{}\|_{L^1(\Gamma)} \leqslant |\hat{\mu}(1)| + 2^{-n-1} \sum_{m=1}^{\infty} 3^{-m} \leqslant \|\mu\| + \tfrac{1}{2}\|\mu\|^{1/2},$$

that is, $\mu \to (\nu * \mu)\hat{}$ is continuous at 0 as a function from X to $L^1(\Gamma)$.

The proof of Theorem 1 is finished.

Proof of Theorem 2. We first observe that it suffices to prove Theorem 2 for the special case $X = \{\nu_j\} \cup \{0\}$, where $\sum \|\nu_j\| < \infty$. The theorem in general is then established as follows.

For each $n = 1, 2, \ldots$, let $\{\nu(n, j)\colon 1 \leqslant j \leqslant J(n)\} \subset X$ be 2^{-n}-dense in X. Choose numbers $\alpha(n, j) > 0$ such that

$$\sum_{n=1}^{\infty} \sum_{j=1}^{J(n)} \alpha(n, j) \|\nu(n, j)\| < \infty,$$

and set $X' = \{\alpha(n, j)\nu(n, j)\colon n = 1, 2, \ldots, 1 \leqslant j \leqslant J(n)\} \cup \{0\}$. Then the special case above implies the existence of a singular continuous independent power Hermitian probability measure μ such that $\mu * \nu(n, j) \in L^1(G)$ for all n, j. But if $\nu(n_k, j_k) \to \nu \in X$, then $\mu * \nu(n_k, j_k) \to \mu * \nu$ so $\mu * \nu \in L^1(G)$ for all $\nu \in X$.

It remains to prove Theorem 2 for $X = \{\nu_j\} \cup \{0\}$ with $\sum \|\nu_j\| < \infty$.

By the structure theorem (Rudin [11], Hewitt and Ross [7]), G has an open subgroup of the form $R^n \times D$, where D is compact. Let

$F = [0, 2\pi)^n \times D$. Then, because X is countable and each element of X has σ-compact support, there exists a sequence $\{y_k\}_{k=1}^{\infty}$ of elements of G such that

(16) $$|\mu|(G \setminus \bigcup_{k=1}^{\infty} y_k F) = 0 \quad \text{for} \quad \mu \in X;$$

and such that

(17) $$|\mu|(\bigcap_{j=1}^{n} y_{k(j)} F) = 0 \quad \text{for} \quad \mu \in X \text{ and } 1 \leqslant k(1) < \ldots < k(n) < \infty.$$

(This follows by choosing $y_k = z_k x_k$, where the z_k are elements of $2\pi Z^n \times \{1\}$ and the x_k are in cosets of $R^n \times D$. An appropriate choice of the z_k and x_k will give a covering of the σ-compact "support" of X by disjoint sets. Then (16) and (17) follow.)

For each $k = 1, 2, \ldots$ let X_k denote the set of restrictions of elements of X to $y_k F$ and let $X_0 = \bigcup_{k=1}^{\infty} \delta_{y_k^{-1}} * X_k \cup \{0\}$. Then $\sum \{\|\nu\|: \nu \in X_0\} < \infty$ (by (16)–(17)) and $X_0 \subset M_c(F)$. We now let Π be the natural projection of $R^n \times D$ onto $T^n \times D$ given by $R^n \times D \to (R^n/2\pi Z^n) \times D = T^n \times D$, and let $\overline{\Pi}$ be the induced map of measures. Let $Y_0 = \overline{\Pi} X_0$. Let B denote the set of Borel functions f on $T^n \times D$ of the form

$$f(x_1, \ldots, x_n, d) = \exp\big(i(x_1 t_1 + \ldots + x_n t_n)\big) \quad \text{for} \quad 0 \leqslant t_1, \ldots, t_n < 2\pi.$$

Now $Y_0 = \{\mu_j\}$, where $\sum_{j=1}^{\infty} \|\mu_j\| < \infty$, since this property is inherited from X_0. Let $\{b_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $\lim b_j = 0$, $b_j \leqslant 1$, and such that $\sum_{j=1}^{\infty} \|\mu_j\|/b_j < \infty$. Let Y be defined by

$$Y = \bigcup_{j=1}^{\infty} \{(b_j/\|\mu_j\|) f\mu : f \in B,\ \mu \in Y_0 \text{ and } \|\mu\| \leqslant \|\mu_j\|\} \cup \{0\}.$$

Then Y is a norm compact subset of the unit ball. Thus, by Lemma 5, there exists a sequence $\{\gamma_k\} \subset Z^n \times \hat{D}$ such that (4)–(5) hold for all $\mu \in Y$, with $\varepsilon_n = 6^{-n}$, for $n = 1, 2, \ldots$

Let ν be the Riesz product on $T^n \times D$ generated by $\{\gamma_k\}$ and $a(\gamma_k) \equiv \frac{1}{2}$. Then, for all $\mu \in Y$, we see that

$$\sum |(\nu * \mu)\hat{\ }(\gamma)| \leqslant |\hat{\mu}(1)| + \sum_{n=1}^{\infty} 3^n 6^{-n} = |\hat{\mu}(1)| + 1 \leqslant 2.$$

Thus, if $\mu \in Y_0$ and $\|\mu\| \leqslant \|\mu_j\|$, then

$$\|(\nu * f\mu)\hat{\ }\|_{L^1(Z^n \times \hat{D})} \leqslant 2\|\mu_j\|/b_j \quad \text{for} \quad f \in B.$$

Let ω be the lifting of ν to $R^n \times D$ which is given by Lemma 6. Then ω is a singular continuous independent power Hermitian probability measure since ν is. Furthermore, $\hat{\omega}$ is supported on the set of elements of the form

(18) $$(t, 1) \prod_{j=1}^{k} \gamma_j^{m_j}, \quad \text{where } t \in [0, 2\pi)^n \text{ and } m_j \in \{0, -1, 1\}$$
$$\text{for } 1 \leqslant j \leqslant k < \infty.$$

This follows from (11).

Let t denotes an element of $[0, 2\pi)^n$, and let f_t denote the corresponding element of B. Then, if $z \in Z^n$, $\varrho \in \hat{D}$, and $\mu \in X_0$, then we see that

$$(\omega * \mu)\hat{\ }(t + z, \varrho) = (f_t \nu * f_t \overline{\Pi} \mu)\hat{\ }(z, \varrho).$$

Thus, if $(t + z, \varrho)$ has the form (18), then

$$|(\omega * \mu)\hat{\ }(t + z, \varrho)| \leqslant 6^{-k},$$

provided that not all m_j in (18) are zero. Therefore, for all elements $\mu_j \in X_0$, we see that

(19) $$\|(\omega * \mu_j)\hat{\ }\|_{L^1(R^n \times \hat{D})} \leqslant 2(2\pi)^n \|\mu_j\|/b_j.$$

Of course, translating μ_j by an element $y \in G$ does not change the estimate (19). Therefore, since each $\mu \in X$ is a sum,

$$\mu = \sum_{k=1}^{\infty} \delta_{y(k)} * \mu_{j(k)},$$

of translates of elements of X_0, we have

$$\|(\omega * \mu)\hat{\ }\|_{L^1(\Gamma)} \leqslant \sum_{k=1}^{\infty} (\|\mu_{j(k)}\|/b_{j(k)}) < \infty.$$

The proof of Theorem 2 is finished.

4. A converse to Theorem 1.

THEOREM 7. *Let G be an infinite compact abelian group and let f be a trigonometric polynomial on G. Let $\varepsilon > 0$. Then there exist singular continuous measures μ, ν on G such that $f = \mu * \nu$ and $\|\mu\|\,\|\nu\| \leqslant (1+\varepsilon)\|f\|$.*

Proof. By Rudin ([11], 2.6.8) there exists a trigonometric polynomial g on G such that $g * f = f$ and $\|g\| < (1+\varepsilon)^{1/2}$.

Let Θ be any infinite dissociate subset of the dual Γ of G. Let E denote the support of $\hat{g}$ and F the support of $\hat{f}$. It is easy to see that there exists a finite subset Φ of Θ such that

(20) $$\Omega(\Theta \setminus \Phi) \cap E E^{-1} = \{1\}.$$

($\Omega(\cdot)$ is defined at the end of Section 1.)

Let Θ_1 and Θ_2 be two disjoint infinite subsets of $\Theta\setminus\Phi$ and let μ_1 and μ_2 be singular continuous Riesz products based on Θ_1, a_1 and Θ_2, a_2, respectively. For each finite subset Ψ of Θ_1 let $(\mu_1)_\Psi$ be the Riesz product based on $\Theta_1\setminus\Psi$ obtained by restricting a_1 to $\Theta_1\setminus\Psi$. Then the net $\{(\mu_1)_\Psi: \Psi\subset\Theta_1, \Psi \text{ finite}\}$ tends weak-* in $M(G)$ to Haar measure. Hence, the continuity of f implies that

$$\inf\{\|f(\mu_1)_\Psi\|\} = \|f\|,$$

where the infimum is taken over all finite subsets Ψ of Θ_1. Thus, we may assume that $\mu = f(\mu_1)_\Psi$ has norm at most $(1+\varepsilon)^{1/3}\|f\|$. Similarly, there exists a finite subset Λ of Θ_2 such that $\nu = g(\mu_2)_\Lambda$ has norm at most $(1+\varepsilon)^{2/3}$. The measures μ and ν are singular and continuous since both $(\mu_1)_\Psi$ and $(\mu_2)_\Lambda$ are. Comparing transforms using (20) yields $(\mu*\nu)\hat{} = \hat{f}$ on F, while $(\mu*\nu)\hat{}$ vanishes off F. Thus, $\mu*\nu = f$ and the inequality $\|\mu\|\,\|\nu\|\leqslant(1+\varepsilon)\|f\|$ follows from our estimates for $\|\mu\|$ and $\|\nu\|$.

This result is due to MacLean [10]. Whether a similar factorization holds for all elements of $L^1(G)$ seems to be an open question.

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