[12] L. LeCam, *Remarque sur le théorème limite central dans les espaces localement convexes,* Les Problémes sur les Structures Algébriques. Colloq. de la Recherche Scientifique, Paris 1970, pp. 233–249.
[13] V. Mandrekar, *Central limit problem on $L_p$ ($p \geqslant 2$) I; Lévy-Khinchine representation,* Lecture Notes 526, pp. 159–166, Springer-Verlag, New York 1976.
[14] — *Lévy-Khinchine representation and Banach space type,* MSU-RM-374.
[15] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendentes et propriétés géométriques des espaces de Banach,* Studia Math. 58 (1976), pp. 45–90.
[16] K. R. Parthasarathy, *Probability measures on metric spaces,* Academic Press, New York 1967.
[17] A. Tortrat, *Sur la structure des lois infinitement divisibles dans les espaces vectoriels,* Z. Wahrscheinlichkeitstheorie 11 (1969), pp. 311–326.
[18] — *Structure de lois infinitement divisibles dans un espace vectoriel topologique,* Lecture Notes 31, pp. 299–328, Springer-Verlag, New York 1967.
[19] S. R. S. Varadhan, *Limit theorems for sums of independent random variables with values in a Hilbert space,* Sankhyā 24A (1962), pp. 213–238.

ARYA-MEHR UNIVERSITY OF TECHNOLOGY, TEHERAN, IRAN
and
MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN

*Received May 20, 1977*
*Revised version August 8, 1977* (1312)

---

STUDIA MATHEMATICA, T. LXVI. (1980)

# The stability radius of a bundle of closed linear operators

by

H. BART* (Amsterdam) and D. C. LAY (College Park, Md.)

**Abstract.** Given a bundle of linear operators $T-\lambda S$, where $T$ is closed and $S$ is bounded, a sequence $\{\gamma_m(T\colon S)\}$ of extended real numbers is defined. When $T$ is the identity operator, $\gamma_m(T\colon S)$ is equal to $\|S^{m-1}\|^{-1}$; when $S$ is the identity operator, $\gamma_m(T\colon S)$ is the reduced minimum modulus $\gamma(T^m)$ of $T^m$. It is shown that in several important cases (including the case when $T$ is a Fredholm operator and $S$ is arbitrary)

$$\lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}$$

exists and is equal to the supremum of all positive $r$ such that the ranges $R(T-\lambda S)$ are closed and $\dim N(T-\lambda S)$ and $\operatorname{codim} R(T-\lambda S)$ are constant on $0<|\lambda|<r$. This work generalizes the usual spectral radius formula, a recent theorem of K.-H. Förster and M. A. Kaashoek, and an earlier result of H. A. Gindler and A. E. Taylor.

**0. Introduction.** If $S$ is a bounded linear operator on a Banach space, the usual spectral radius formula implies that

$$\lim_{m\to\infty} \|S^m\|^{-1/m} \tag{0.1}$$

exists and is equal to the supremum of all $r>0$ such that $I-\lambda S$ is a bijective operator on $|\lambda|<r$. Recently, K.-H. Förster and M.A. Kaashoek [6] studied a similar limit, namely

$$\lim_{m\to\infty} \gamma(T^m)^{1/m}, \tag{0.2}$$

where $T$ is a (possibly unbounded) Fredholm operator and $\gamma(T^m)$ is the reduced minimum modulus of $T^m$. Förster and Kaashoek showed that the limit in (0.2) exists and equals the supremum of all $r>0$ such that the dimensions of the null spaces $N(T-\lambda I)$ and the codimensions of the ranges $R(T-\lambda I)$ are constant on $0<|\lambda|<r$.

In the present paper we describe a general setting which includes the results involving (0.1) and (0.2) as special cases. We consider an operator bundle $T-\lambda S$, where $S$ is a bounded linear operator between two

* The research for this paper was done while the first author was supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

Banach spaces $X$ and $Y$, and $T$ is a closed linear operator with domain in $X$ and range in $Y$. For such a bundle we define a sequence $\{\gamma_m(T\colon S)\}_{m=1}^{\infty}$ of non-negative extended real numbers. When $X = Y$ and $T = I$, one has $\gamma_m(T\colon S) = \|S^{m-1}\|^{-1}$; when $X = Y$ and $S = I$, one has $\gamma_m(T\colon S) = \gamma(T^m)$. We also introduce the notion of the stability radius $r(T\colon S)$ of $T$ and $S$. In most of the cases discussed in this paper $r(T\colon S)$ is equal to the supremum of all $r > 0$ such that $\dim N(T-\lambda S)$, $\operatorname{codim} R(T-\lambda S)$ and $\dim R(T-\lambda S)$ are constant on $0 < |\lambda| < r$. We prove that under rather general conditions

$$r(T\colon S) = \lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}. \tag{0.3}$$

Our main results apply to the following situations:

(i) $T$ is a Fredholm operator, $S$ is arbitrary;

(ii) $T$ is a semi-Fredholm operator with complemented range and null space, $S$ is compact;

(iii) $T$ has closed range, $S$ is degenerate (i.e., $S$ is of finite rank);

(iv) The resolvent $(T-\lambda S)^{-1}$ of $T$ and $S$ has a pole at the origin.

We also present an example showing that the condition in (ii) that $T$ is a semi-Fredholm operator by itself is not enough, not even when $X = Y$ is a Hilbert space.

The idea of finding expressions (or estimates) similar to (0.3) for the stability radius originated in a paper of H. A. Gindler and A. E. Taylor [7]. They studied the case when $X = Y$ (possibly non-complete), $S = I$ and $T$ has a bounded inverse.

**1. Preliminaries.** Throughout this paper $X$ and $Y$ are complex Banach spaces, $T$ is a closed linear operator with domain $D(T)$ in $X$ and range $R(T)$ in $Y$, and $S$ is a bounded linear operator from (all of) $X$ into $Y$. Whenever we write $Tx$, it will be understood that $x \in D(T)$.

Define subspaces $N_m = N_m(T\colon S)$ and $R_m = R_m(T\colon S)$ of $X$ by

$$N_0 = (0), \quad R_0 = X,$$

$$N_{m+1} = T^{-1}SN_m, \quad R_{m+1} = S^{-1}TR_m, \quad m = 1, 2, \ldots$$

The sequence $\{N_m\}$ is increasing; the sequence $\{R_m\}$ is decreasing. Put

$$k(T\colon S) = \dim \frac{N(T)}{N(T)\cap R},$$

where $N(T)$ is the null space of $T$ and $R$ is the intersection of the subspaces $R_m$. The extended integer $k(T\colon S)$ will be called the *stability number* of $T$ and $S$. One can show that $k(T\colon S)$ is less than or equal to each of the extended integers

$$\dim N(T), \quad \operatorname{codim} R(T), \quad \dim SN(T).$$

Thus, for example, $k(T\colon S)$ is finite whenever $T$ is semi-Fredholm or $S$ is degenerate.



The *reduced minimum modulus* of $T$ is defined as the supremum $\gamma(T)$ of all $c \geq 0$ such that

$$\|Tx\| \geq c\cdot d(x, N(T)), \quad x \in D(T).$$

Here $d(x, N(T))$ denotes the distance from $x$ to $N(T)$. We recall from [9] that $T$ has closed range if and only if $\gamma(T) > 0$.

The following theorem is due to M. A. Kaashoek; it is a generalization of the well-known perturbation results of I. C. Gohberg, M. G. Krein and T. Kato for semi-Fredholm operators. (See [8], [10], [11]. Part (ii) of the theorem does not appear in [10] but may be deduced without difficulty from the statements (iii) and (iv).)

1.1. Theorem. *Suppose that $R(T)$ is closed and $k(T\colon S) = k < \infty$. Then there exists $r > 0$ such that, for $0 < |\lambda| < r$,*

(i) $\dim N(T-\lambda S) = \dim N(T) - k$,

(ii) $\dim R(T-\lambda S) = \dim R(T) + k$,

(iii) $\operatorname{codim} R(T-\lambda S) = \operatorname{codim} R(T) - k$,

(iv) $k(T-\lambda S\colon S) = 0$,

(v) $R(T-\lambda S)$ *is closed.*

*In the particular case when $k = 0$, the constant $r$ may be taken to be $\gamma(T)/\|S\|$.*

Motivated by this theorem we define the ***stability radius*** of $T$ and $S$ as the supremum $r(T\colon S)$ of all $r > 0$ such that $R(T-\lambda S)$ is closed and $k(T-\lambda S\colon S) = 0$ for $0 < |\lambda| < r$. It is clear that the functions

$$\lambda \mapsto \dim N(T-\lambda S), \quad \lambda \mapsto \dim R(T-\lambda S), \quad \lambda \mapsto \operatorname{codim} R(T-\lambda S), \tag{1.1}$$

are constant on $0 < |\lambda| < r(T\colon S)$. In most of the cases discussed in this paper $r(T\colon S)$ is actually the supremum of all $r > 0$ such that functions (1.1) are constant on $0 < |\lambda| < r$.

Given $m \geq 1$ and elements $x_1, \ldots, x_m$ in $D(T)$, we say that the tuple $(x_1, \ldots, x_m)$ is a *chain* for $T$ and $S$ if

$$Tx_i = Sx_{i-1}, \quad i = 2, \ldots, m.$$

It is easily verified that $x \in N_m$ if and only if there exists a chain $(x_1, \ldots, x_m)$ with $x_m = x$ and $Tx_1 = 0$. We let $\gamma_m = \gamma_m(T\colon S)$ denote the supremum of all $c \geq 0$ with the property that

$$\|Tx_1\| \geq c\cdot d(x_m, N_m)$$

for every chain $(x_1, \ldots, x_m)$. Observe that $\gamma_1$ is just the reduced minimum modulus $\gamma(T)$ of $T$.

To illustrate the definition of $\gamma_m$, we examine the two special cases mentioned in the introduction.

1.2. Example. Assume that $T$ is bijective and let $L$ be the inverse of $T$. Then $L$ is a bounded linear operator from all of $Y$ into $X$. It is clear that $N_m = (0)$ for all $m$. Also, the chains for $T$ and $S$ are of the form

$$(Ly, LSLy, \ldots, L(SL)^{m-1}y), \qquad y \in Y.$$

It follows that, for $m = 1, 2, \ldots,$

$$\gamma_m = \|L(SL)^{m-1}\|^{-1}. \tag{1.2}$$

But then, if $r_\sigma(SL)$ denotes the spectral radius of $SL$, we have

$$r_\sigma(SL)^{-1} = \lim_{m\to\infty} \gamma_m^{1/m}. \tag{1.3}$$

Observe that $r_\sigma(SL)^{-1}$ is also equal to the supremum of all $r > 0$ such that $T - \lambda S$ is bijective for $|\lambda| < r$. When $X = Y$ and $T = I$, the right-hand side of (1.2) becomes $\|S^{m-1}\|^{-1}$ and the left-hand side of (1.3) is equal to $r_\sigma(S)^{-1}$.

1.3. Example. Suppose that $X = Y$ and $S = I$. Then

$$\gamma_m = \gamma(T^m), \qquad m = 1, 2, \ldots$$

The proof of this is based on the observation that the chains are now of the form

$$(T^{m-1}x, \ldots, Tx, x), \qquad x \in D(T^m),$$

and $N_m = N(T^m)$.

We conclude this section with two lemmas, the second of which will be used in Sections 3 and 4.

1.4. Lemma. *Let $T$ be surjective and suppose that*

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

*Then $R(T-\mu S)$ is dense in $Y$.*

Proof. Take $0 \neq y \in Y$, let $\varepsilon > 0$ and choose $r$ such that

$$|\mu| < r < \limsup_{m\to\infty} \gamma_m^{1/m}.$$

Select $m$ so that

$$r^m < \gamma_m, \qquad \frac{|\mu|^m \|S\| \|y\|}{r^m} < \varepsilon.$$



Since $R(T) = Y$, there exists a chain $(x_1, \ldots, x_m)$ with $Tx_1 = y$. Observe that $d(x_m, N_m) < r^{-m}\|y\|$. Hence there is a chain $(u_1, \ldots, u_m)$ with $Tu_1 = 0$ and $\|x_m - u_m\| < r^{-m}\|y\|$. Now put

$$x = \sum_{i=1}^{m} \mu^{i-1}(x_i - u_i), \qquad z = (T - \mu S)x.$$

Then $z \in R(T - \mu S)$ and a straightforward calculation shows that $\|y - z\| < \varepsilon$. This proves the lemma.

The Banach space of all bounded linear operators from $X$ into $Y$ will be denoted by $\mathscr{L}(X, Y)$.

1.5. Lemma. *Let $R(T)$ be closed and $k(T\colon S) = 0$. If*

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}$$

*and $R(T - \mu S)$ is closed, then $k(T - \mu S\colon S) = 0$.*

Proof. Define

$$X_\infty = \bigcap_m R_m, \qquad Y_\infty = \bigcap_m TR_m.$$

From [10], Lemma 2.3 we know that

$$TX_\infty = Y_\infty, \qquad S^{-1}Y_\infty = X_\infty. \tag{1.4}$$

Furthermore, since $\mu \neq 0$,

$$N(T - \mu S) \subset X_\infty. \tag{1.5}$$

The condition $k(T\colon S) = 0$ means that $N(T) \subset R_m$ for all $m$. Using this one easily shows, as in the proof of Theorem 3.1 in [10], that $R_m$ and $TR_m$ are closed. Hence $X_\infty$ and $Y_\infty$ are closed.

Let $T_\infty$ and $S_\infty$ be the restrictions of $T$ and $S$ to $X_\infty$ considered as operators into $Y_\infty$. Then $T_\infty$ is closed and $S_\infty \in \mathscr{L}(X_\infty, Y_\infty)$. Observe that $R(T_\infty) = Y_\infty$. By [11], Lemma 5.11, the hypothesis $k(T\colon S) = 0$ implies that $N_m(T\colon S) \subset X_\infty$ for all $m$. Hence $N_m(T_\infty\colon S_\infty) = N_m(T\colon S)$ and consequently

$$\gamma_m(T\colon S) \leq \gamma_m(T_\infty\colon S_\infty), \qquad m = 1, 2, \ldots$$

But then we have

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T_\infty\colon S_\infty)^{1/m}.$$

Applying Lemma 1.4 we find that $(T - \mu S)X_\infty = R(T_\infty - \mu S_\infty)$ is dense in $Y_\infty$. However, in view of (1.5), Lemma IV.2.9 in [9] implies that $(T - \mu S)X_\infty$ is closed. So $(T_\infty - \mu S_\infty)X_\infty = Y_\infty$. This, together with (1.5) and the second part of (1.4), gives

$$N(T-\mu S) \subset \bigcap_m R_m(T-\mu S:\ S),$$

and the proof is complete.

**2. An upper bound for the stability radius.** The proof of Proposition 2.1 will be based on the existence of global relative inverses of certain holomorphic operator functions [2]. A similar argument was used in [6].

An operator $L \in \mathscr{L}(Y, X)$ is called a *relative inverse* of $T$ if $R(L) \subset D(T)$ and

$$Tx = TLTx, \qquad Ly = LTLy$$

for $x \in D(T)$ and $y \in Y$. If $D(T) = X$, then $LT$ is a projection of $X$ along $N(T)$ and $TL$ is a projection of $Y$ onto $R(T)$.

Let $\hat{X}$ be $D(T)$ endowed with the graph norm $|||x||| = \|x\|+\|Tx\|$. Then $\hat{X}$ is a Banach space.

2.1. Proposition. *Suppose $N(T-\lambda S)$ is complemented in $\hat{X}$ and $R(T-\lambda S)$ is complemented in $Y$ for $|\lambda| < r(T: S)$, and $k(T: s) < \infty$. Then*

$$r(T:\ S) \leqslant \liminf_{m\to\infty} \gamma_m(T:\ S)^{1/m}. \tag{2.1}$$

Proof. Let $\hat{T}$ and $\hat{S}$ be the operators $T$ and $S$ considered as maps from $\hat{X}$ into $Y$. Then $\hat{T}, \hat{S} \in \mathscr{L}(\hat{X}, Y)$ and, for $|\lambda| < r(T: S)$, the operator $\hat{T}-\lambda\hat{S}$ has complemented range and null space. Also $k(\hat{T}-\lambda\hat{S}:\ \hat{S}) = k(T-\lambda S:\ S) = 0$ for $0<|\lambda|<r(T: S)$. But then [2], Theorem 2.2 ensures the existence of an $\mathscr{L}(Y, \hat{X})$-valued function $L$ with the following properties:

(i) $L$ is holomorphic on $0<|\lambda|<r(T: S)$ and meromorphic on $|\lambda|<r(T: S)$;

(ii) $L(\lambda)$ is a relative inverse of $\hat{T}-\lambda\hat{S}$ for $0<|\lambda|<r(T: S)$;

(iii) The projection functions $\lambda \mapsto L(\lambda)(\hat{T}-\lambda\hat{S})$ and $\lambda \mapsto (\hat{T}-\lambda\hat{S})L(\lambda)$ have holomorphic extensions to all of $|\lambda|<r(T: S)$.

Now let $p$ be a positive integer and let

$$L(\lambda) = \sum_{n=-p}^{\infty} \lambda^n L_n$$

be the Laurent expansion of $L$ at the origin. From (i) we have

$$r(T:\ S) \leqslant \liminf_{m\to\infty} \|L_{m-1}\|^{-1/m}.$$

So it suffices to show that $\gamma_m(T:\ S) \geqslant \|L_{m-1}\|^{-1}$ for $m \geqslant p$.



Take $m \geqslant p$. From (iii) we have

$$\hat{T}L_{-p} = 0, \qquad \hat{T}L_{-p+i} = \hat{S}L_{-p+i-1}, \qquad i = 1, \ldots, p-1.$$

Hence $R(L_{-1}) \subset N_p$. Since $m \geqslant p$, it follows that

$$R(L_{-1}) \subset N_m \tag{2.2}$$

(notice that $N_m(T:\ S) = N_m(\hat{T}:\ \hat{S})$).

Now let $(x_1, \ldots, x_m)$ be a chain for $\hat{T}$ and $\hat{S}$. Define, for $0<|\lambda| < r(T:\ S)$,

$$\psi(\lambda) = [I_X - L(\lambda)(\hat{T}-\lambda\hat{S})]\Big(\sum_{i=1}^{m} \lambda^{i-1}x_i\Big).$$

By (iii), the function $\psi$ has a holomorphic extension, also denoted by $\psi$, to all of $|\lambda|<r(T: S)$. On account of (ii), we have

$$(\hat{T}-\lambda\hat{S})\psi(\lambda) = 0, \qquad |\lambda|<r(T:\ S).$$

It follows that the $(m-1)$-th coefficient $\psi_{m-1}$ in the Taylor expansion of $\psi$ at the origin belongs to $N_m$. A routine computation, based on the fact that $(x_1, \ldots, x_m)$ is a chain, shows that

$$\psi_{m-1} = x_m - L_{m-1}Tx_1 + L_{-1}Sx_m.$$

But then we see from (2.2) that $x_m - L_{m-1}Tx_1 \in N_m$, and so

$$d(x_m, N_m) \leqslant \|L_{m-1}Tx_1\| \leqslant |||L_{m-1}Tx_1||| \leqslant \|L_{m-1}\|\,\|Tx_1\|.$$

Hence $\gamma_m(T:\ S) \geqslant \|L_{m-1}\|^{-1}$, and the proof is complete.

In the following remark we show how to apply Proposition 2.1 in three important cases.

2.2. Remark. (i) Suppose that $T$ is a Fredholm operator. Then it follows from Theorem 1.1 that $T-\lambda S$ is a Fredholm operator for $|\lambda| < r(T:\ S)$. In particular the conditions of Proposition 2.1 are satisfied. We conclude that formula (2.1) holds.

(ii) Assume that $S$ is compact and that $T$ is a semi-Fredholm operator with complemented range and null space. Let $\hat{T}$ and $\hat{S}$ be as in the proof of Proposition 2.1. Then $\hat{S}$ is compact and $\hat{T}$ is a bounded semi-Fredholm operator with complemented range and null space. This implies that, for all $\lambda \in C$, the operator $\hat{T}-\lambda\hat{S}$ is semi-Fredholm with complemented range and null space (cf. [4]). In particular the hypotheses of Proposition 2.1 are satisfied, and so we obtain the inequality (2.1).

(iii) The set of all $\lambda \in C$ such that $T-\lambda S$ is bijective is called the *resolvent set* of $T$ and $S$ and denoted by $\varrho(T:\ S)$. For $\lambda$ in $\varrho(T:\ S)$, let

$R(\lambda)$ be the *resolvent* of $T$ and $S$ given by

$$R(\lambda)y = (T-\lambda S)^{-1}y, \quad y \in Y.$$

Then $\varrho(T\colon S)$ is an open (possibly empty) subset of $C$ and $R$ is a holomorphic $\mathscr{L}(Y, X)$-valued function (cf. [3], Section 5). Suppose now that $R$ has a pole (or is holomorphic) at the origin. Then it follows from Theorem 1.1 that $\lambda \in \varrho(T\colon S)$ for $0 < |\lambda| < r(T\colon S)$. Moreover, if $L(\lambda)$ is $R(\lambda)$ viewed as an operator from $Y$ into $\hat{X}$, then obviously $L$ has the properties mentioned in the proof of Proposition 2.1. As a result, formula (2.1) is valid.

**3. The case when $T$ is a Fredholm operator.** When $T$ is a Fredholm operator, $r(T\colon S)$ is equal to the supremum of all $r > 0$ such that $\dim N(T-\lambda S)$ and $\operatorname{codim} R(T-\lambda S)$ are constant on $0 < |\lambda| < r$. This is a consequence of Theorem 1.1.

3.1. THEOREM. *Let $T$ be a Fredholm operator. Then*

$$r(T\colon S) = \lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

This theorem is a generalization of [6], Theorem 5 and [7], Theorem 3.5. The proof requires the following lemma.

3.2. LEMMA. *Suppose that $T$ is a Fredholm operator and $k(T\colon S) = 0$. Let*

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}. \tag{3.1}$$

*Then $T-\mu S$ is a Fredholm operator and $k(T-\mu S\colon S) = 0$.*

**Proof.** In order to show that $T-\mu S$ is a Fredholm operator, we shall relate the right-hand side of (3.1) to the spectral radius of a certain element in the Calkin algebra over $X$. Let $m$ be a positive integer. As $k(T\colon S) = 0$, we have $N(T) \subset R_{m-1}$. Since $T$ is Fredholm, the subspaces $R_n$ are closed and have finite codimension. Using these facts one can construct a relative inverse $L_m$ of $T$ such that $L_m T R_n \subset R_n$ for $n = 0, \ldots, m-1$. (First construct a projection of $X$ along $N(T)$ which maps $R_n$ into $R_n$ for $n = 0, \ldots, m-1$.)

Since $T$ is a Fredholm operator, we have $TR_{m-1}$ closed and

$$\dim N_m \leqslant mN, \quad \operatorname{codim} TR_{m-1} \leqslant mN,$$

where $N = \max\{1, \dim N(T), \operatorname{codim} R(T)\}$. Hence there exist a projection $P$ of $X$ along $N_m$ and a projection $Q$ of $Y$ onto $TR_{m-1}$ such that

$$\|P\| \leqslant 2(mN)^{1/2}, \quad \|Q\| \leqslant 3(mN)^{1/2}$$

(cf. [5], and the references given there).



Take $y \in Y$ and put

$$x_i = (L_m S)^{i-1} L_m Q y, \quad i = 1, \ldots, m.$$

Using the special properties of $L_m$, one can show that $(x_1, \ldots, x_m)$ is a chain with $Tx_1 = Qy$. But then

$$d(x_m, N_m) \leqslant \frac{\|Qy\|}{\gamma_m} \leqslant \frac{\|Q\|\,\|y\|}{\gamma_m}.$$

Since $P$ is a projection of $X$ along $N_m$, we have

$$\|Px_m\| \leqslant \|P\|\cdot d(x_m, N_m) \leqslant \frac{\|P\|\,\|Q\|\,\|y\|}{\gamma_m}.$$

It follows that

$$\|P(L_m S)^{m-1} L_m Q\| \leqslant \frac{\|P\|\,\|Q\|}{\gamma_m} \leqslant \frac{6mN}{\gamma_m}.$$

Since $I_X - P$ and $I_Y - Q$ are degenerate, we have

$$\|(L_m S)^m - F_m\| \leqslant \frac{6mN\,\|S\|}{\gamma_m}, \tag{3.2}$$

for some degenerate operator $F_m$ in $\mathscr{L}(X, X)$.

Let $\varkappa$ denote the canonical mapping from $\mathscr{L}(X, X)$ onto the Calkin algebra over $X$ (see for instance [4]). Since for each $m$ the operators $L_m$ and $L_1$ are relative inverses of the Fredholm operator $T$, we have that $L_m - L_1$ is degenerate. Hence $\varkappa L_m S = \varkappa L_1 S$, and it follows from formula (3.2) that

$$\|(\varkappa L_1 S)^m\| \leqslant \frac{6mN\,\|S\|}{\gamma_m}. \tag{3.3}$$

The spectral radius $r_\sigma(\varkappa L_1 S)$ of $\varkappa L_1 S$ is given by

$$r_\sigma(\varkappa L_1 S) = \lim_{m\to\infty} \|(\varkappa L_1 S)^m\|^{1/m}.$$

Formula (3.3) now implies that

$$\limsup_{m\to\infty} \gamma_m^{1/m} \leqslant r(\varkappa L_1 S)^{-1}.$$

Thus $\mu^{-1}\varkappa I_X - \varkappa L_1 S$ is invertible. As is well known, this means that $I_X - \mu L_1 S$ is Fredholm, and hence so is $T - \mu T L_1 S$. But $TL_1 = I_Y - Q_1$, where $Q_1$ is a degenerate projection. It follows that $T-\mu S$ is a Fredholm operator. In particular, $R(T-\mu S)$ is closed. But then we know from Lemma 1.5 that $k(T-\mu S\colon S) = 0$, and the proof is complete.

We are now able to prove Theorem 3.1. The proof uses a decomposition theorem due to T. Kato [11]. In this connection we note that the hypothesis $k(T\colon S) = 0$ is equivalent to the condition $\nu(T\colon S) = \infty$ appearing in [11].

Proof of Theorem 3.1. In view of Remark 2.2(i), it suffices to show that

$$r(T\colon S) \geqslant \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

So take $\mu$ satisfying

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

We need to show that $R(T-\mu S)$ is closed and $k(T-\mu S\colon S) = 0$.

Consider topological decompositions $X = X_0 \oplus X_1$ and $Y = Y_0 \oplus Y_1$ as given in [11], Theorem 4. These decompositions completely reduce $T$ and $S$. For $i = 0, 1$, let $T_i$ and $S_i$ denote the restrictions of $T$ and $S$ to $X_i$ viewed as operators into $Y_i$. From [11] we know that $T_1, S_1 \in \mathscr{L}(X_1, Y_1)$, that $S_1$ is bijective and that $S_1^{-1}T_1$ is nilpotent. Since $\mu \neq 0$, it follows that $T_1 - \mu S_1$ is bijective.

Next we consider $T_0$ and $S_0$. Clearly $S_0 \in \mathscr{L}(X_0, Y_0)$. Recall from [11], that $T_0$ is Fredholm and $k(T_0\colon S_0) = 0$. Let $P$ be the projection of $X$ onto $X_0$ along $X_1$. A routine argument shows that $N_m(T_0\colon S_0) = PN_m(T\colon S)$. Hence

$$\gamma_m(T_0\colon S_0) \geqslant \|P\|^{-1}\cdot\gamma_m(T\colon S), \qquad m = 1, 2, \ldots,$$

and consequently

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T_0\colon S_0)^{1/m}.$$

But then Lemma 3.2 yields that $T_0 - \mu S_0$ is Fredholm and $k(T-\mu S_0\colon S_0) = 0$. It is now clear that $T-\mu S$ is Fredholm. In particular $R(T-\mu S)$ is closed. Moreover, $k(T-\mu S\colon S) = k(T_0-\mu S_0\colon S_0) + k(T_1 - \mu S_1\colon S_1) = 0$, and the proof is complete.

We now apply Theorem 3.1 to the case when $T$ has closed range and $S$ is degenerate.

3.3. Theorem. *Suppose $T$ has closed range and $S$ is degenerate. Then*

$$r(T\colon S) = \lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

Proof. Put $Y_1 = R(T) + R(S)$. Then $Y_1$ is a closed subspace of $Y$. Further, let $\hat{X} = X/W$, where $W = N(T)\cap N(S)$, and let $\hat{T}$ and $\hat{S}$ be the induced operators from $\hat{X}$ into $\hat{Y}$. The hypothesis that $S$ is degenerate ensures that

$$\text{(3.4)} \qquad \dim\frac{N(T)}{N(T)\cap N(S)} < \infty, \qquad \dim\frac{R(T)+R(S)}{R(T)} < \infty.$$



Hence $\hat{T}$ is a Fredholm operator, and so, by Theorem 3.1,

$$r(\hat{T}\colon \hat{S}) = \lim_{m\to\infty} \gamma_m(\hat{T}\colon \hat{S})^{1/m}.$$

It remains to prove that $\gamma_m(\hat{T}\colon \hat{S}) = \gamma_m(T\colon S)$ and $r(\hat{T}\colon \hat{S}) = r(T\colon S)$.

Let $\varphi$ be the canonical mapping from $X$ onto $\hat{X}$. Observe that $(x_1, \ldots, x_m)$ is a chain for $T$ and $S$ if and only if $(\varphi(x_1), \ldots, \varphi(x_m))$ is a chain for $\hat{T}$ and $\hat{S}$. Hence $N_m(\hat{T}\colon \hat{S}) = \varphi N_m(T\colon S)$. Using the fact that $W \subset N_m(T\colon S)$ one now easily deduces that

$$d\big(x, N_m(T\colon S)\big) = d\big(\varphi(x), N_m(\hat{T}\colon \hat{S})\big), \qquad x \in X.$$

It follows that $\gamma_m(\hat{T}\colon \hat{S}) = \gamma_m(T\colon S)$.

Let $\lambda \in C$. It is clear that $R(T-\lambda S)$ is closed if and only if $R(\hat{T}-\lambda\hat{S})$ is closed. Observe that $N(\hat{T}-\lambda\hat{S}) = \varphi N(T-\lambda S)$ and $R_m(\hat{T}-\lambda\hat{S}\colon \hat{S}) = \varphi R_m(T-\lambda S\colon S)$. Put

$$R_\lambda = \bigcap_m R_m(T-\lambda S\colon S), \qquad \hat{R}_\lambda = \bigcap_m R_m(\hat{T}-\lambda\hat{S}\colon \hat{S}).$$

Since $W \subset N(T-\lambda S)\cap R_\lambda$, we have

$$\begin{aligned}\dim\frac{N(T-\lambda S)}{N(T-\lambda S)\cap R_\lambda} &= \dim\frac{\varphi N(T-\lambda S)}{\varphi[N(T-\lambda S)\cap R_\lambda]}\\ &= \dim\frac{\varphi N(T-\lambda S)}{\varphi N(T-\lambda S)\cap\varphi R_\lambda}\\ &= \dim\frac{N(\hat{T}-\lambda\hat{S})}{N(\hat{T}-\lambda\hat{S})\cap\hat{R}_\lambda}.\end{aligned}$$

Thus $k(\hat{T}-\lambda\hat{S}\colon \hat{S}) = k(T-\lambda S\colon S)$, and the proof is complete.

Theorem 3.3 remains valid if the condition that $S$ is degenerate is replaced by the weaker assumption that (3.4) holds. Also, Theorem 3.3 applies when $T$ and $S$ are both degenerate. In this latter case, Theorem 1.1 implies that $r(T\colon S)$ is equal to the supremum of all $r > 0$ such that $\dim R(T-\lambda S)$ is constant on $0 < |\lambda| < r$.

**4. The case when $T$ is a semi-Fredholm operator.** Theorem 3.1 does not remain true if the condition that $T$ is Fredholm is replaced by the weaker assumption that $T$ is semi-Fredholm.

4.1. Example. Let $X = Y$ be the Hilbert space $l_2$ and define $T, S \in \mathscr{L}(l_2, l_2)$ by

$$T(x_0, x_1, x_2, \ldots) = (x_0, x_1, x_3, x_5, x_7, \ldots),$$
$$S(x_0, x_1, x_2, \ldots) = (\alpha x_0, x_1 + x_2, x_3 + \tfrac{1}{2}x_4, x_5 + \tfrac{1}{3}x_6, \ldots).$$

Here $0 \leqslant \alpha < 1$. One can show that

$$\gamma_m(T\colon S) = \alpha^{-(m-1)}, \quad m = 2, 3, \ldots,$$

where, as usual, $0^{-1} = \infty$. Therefore

$$\lim_{m\to\infty} \gamma_m(T\colon S)^{1/m} = \alpha^{-1} > 1.$$

On the other hand, the range of $T-S$ is not closed, and so $r(T\colon S) \leqslant 1$. In fact, $r(T\colon S) = 1$. Observe that $T$ is surjective. Hence this example also shows that the hypotheses of Lemma 1.4 do not imply that $T-\mu S$ is surjective.

Suppose that $T$ is a semi-Fredholm operator and that $S$ is compact. Then $T-\lambda S$ is semi-Fredholm for all $\lambda \in C$. It follows from Theorem 1.1 that $r(T\colon S)$ is equal to the supremum of all $r > 0$ such that $\dim N(T-\lambda S)$ and $\operatorname{codim} R(T-\lambda S)$ are constant on $0 < |\lambda| < r$.

4.2. Theorem. *Suppose that $S$ is compact and that $T$ is a semi-Fredholm operator with complemented range and null space. Then*

$$r(T\colon S) = \lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

Proof. In view of Remark 2.2(ii), it suffices to show that

$$r(T\colon S) \geqslant \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

So take $\mu$ satisfying

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

We know already that $R(T-\mu S)$ is closed. In order to prove that $k(T-\mu S\colon S) = 0$, we once more apply the decomposition result of T. Kato ([11], Theorem 4). Using the same notation as in the proof of Theorem 3.1, we have that $T_0$ is a semi-Fredholm operator, $k(T_0\colon S_0) = 0$ and

$$0 < |\mu| < \limsup_{m\to\infty} \gamma_m(T_0\colon S_0)^{1/m}.$$

But then it follows from Lemma 1.5 that $k(T_0-\mu S_0\colon S_0) = 0$. This, together with the fact that $T_1-\mu S_1$ is bijective, gives $k(T-\mu S\colon S) = 0$, and the proof is complete.

The conclusion of Theorem 4.2 also holds when $X = Y$ is a Hilbert space, $T$ is a semi-Fredholm operator on $X$ with a non-empty resolvent set and $S = I$ is the identity operator on $X$. The proof is basically the same as that of [6], Theorem 5, with the reference to [1], Theorem 5.2 replaced by a reference to [2], Theorem 2. The hypothesis that $X$ is a Hilbert space serves only to ensure that $T-\lambda I$ has complemented range and null space for $|\lambda| < r(T\colon I)$.



**5. The case when the resolvent of $T$ and $S$ has a pole at the origin.** Let $\sigma(T\colon S)$ be the set of all $\lambda \in C$ such that $T-\lambda S$ is not bijective. If the resolvent $(T-\lambda S)^{-1}$ has a pole at the origin, then $r(T\colon S)$ is equal to the distance from 0 to $\sigma(T\colon S)\setminus\{0\}$.

5.1. Theorem. *Suppose that the resolvent of $T$ and $S$ has a pole at the origin. Then*

$$r(T\colon S) = \lim_{m\to\infty} \gamma_m(T\colon S)^{1/m}.$$

The proof follows the same pattern as that of Theorem 3.1. The main differences are: the reference to Remark 2.2(i) should be replaced by a reference to Remark 2.2(iii), the reference to [11], Theorem 4 should be replaced by a reference to a decomposition result obtained in Sections 4 and 5 of [3] and, finally, Example 1.2 should be used instead of Lemma 3.2.

In the following corollary $T$ is a closed linear operator with domain and range in $X$. The spectrum of $T$ is denoted by $\sigma(T)$ and the identity operator on $X$ by $I$.

5.2. Corollary. *Suppose that the resolvent $(T-\lambda I)^{-1}$ has a pole at the origin. Then*

$$\lim_{m\to\infty} \gamma(T^m)^{1/m}$$

*exists and is equal to the distance from 0 to $\sigma(T)\setminus\{0\}$.*

The special case when $(T-\lambda I)^{-1}$ is holomorphic at the origin was treated by H. A. Gindler and A. E. Taylor ([7], Theorem 3.5). The result of K.-H. Förster and M. A. Kaashoek ([6], Theorem 5) covers the situation where $(T-\lambda I)^{-1}$ has a pole of finite rank.

The conclusion of Corollary 5.2 need not be true if the origin is merely an isolated point of the spectrum of $T$. To see this, let $T$ be a quasi-nilpotent operator such that for all $n$ the range of $T^n$ is not closed.

## References

[1] H. Bart, *Holomorphic relative inverses of operator valued functions*, Math. Ann. 208 (1974), pp. 179–194.

[2] —, M. A. Kaashoek and D. C. Lay, *Relative inverses of meromorphic operator functions and associated holomorphic projection functions*, Math. Ann. 218 (1975), pp. 199–210.

[3] — and D. C. Lay, *Poles of a generalised resolvent operator*, Proc. Roy. Irish Acad. 74A (1974), pp. 147–168.

[4] S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces*, Marcel Dekker, Inc., New York 1974.

[5] W. J. Davis, *Remarks on finite rank projections*, J. Approx. Theory 9 (1973), pp. 205–211.

[6] K.-H. Förster and M. A. Kaashoek, *The asymptotic behaviour of the reduced*

*minimum modulus of a Fredholm operator*, Proc. Ann. Math. Soc. 49 (1975), pp. 123–131.

[7] H. A. Gindler and A. E. Taylor, *The minimum modulus of a linear operator and its use in spectral theory*, Studia Math. 22 (1962–1963), pp. 15–41.

[8] I. C. Gohberg and M. G. Krein, *The basic propositions on defect numbers, root numbers and indices of linear operators*, Uspekhi Mat. Nauk 12, 2 (74) (1957), pp. 43–118 (Russian); Amer. Math. Soc. Transl. (2) 13 (1960), pp. 185–265.

[9] S. Goldberg, *Unbounded linear operators*, McGraw-Hill, New York 1966.

[10] M. A. Kaashoek, *Stability theorems for closed linear operators*, Proc. Acad. Sci. Amsterdam A 68 (1965), pp. 452–466.

[11] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. 6 (1958), pp. 261–322.

WISKUNDIG SEMINARIUM, VRIJE UNIVERSITEIT, AMSTERDAM
and
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK

*Received July 26, 1977* (1333)