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An estimation of the Lebesgue functions of biorthogonal systems with an application to the non-existence of some bases in C and L^1

by

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Abstract. We prove the non-existence of a normalized basis in L^1 consisting of uniformly bounded functions and the dual fact for C. In the proofs we make use of Olevskiĭ's technique from [6], Chapter I. We show also, using methods of p-absolutely summing operators, some connections between integral and numerical inequalities, which together with considerations of Olevskiĭ's type give a new proof of the Bočkariev inequality from [1].

0. Introduction. In this paper we show, answering the question of Olevskiĭ ([6], p. 36, (vi)), that there is no normalized basis in $L^1(0,1)$ consisting of uniformly bounded functions. We prove also the "dual" fact for the space $C(0,1)$. These results generalize a theorem of Olevskiĭ (see [6], Chapter I, § 2, Theorems 2 and 9):

No uniformly bounded orthonormal system is a basis in L^1 or C. Our statements admit two methods of proof. The first one makes use of Olevskiĭ's technique, the second one starts from a certain inequality on averages of partial sums of numerical series proved by Bočkariev ([1]).

The paper consists of four sections. Section 1 has a preliminary character. In Section 2 we prove the equivalence of the approaches of Bočkariev and Olevskiĭ. As the common vocabulary for them we use the theory of absolutely summing operators. Section 3 contains the proofs of the non-existence of a normalized structurally bounded basis in L^1, the "dual" result for C and some further strengthenings. Section 4 contains in fact the new proof of the Bočkariev inequality, which is based on the results of Section 2 and the proof of Theorem 1 of Section 3.

To make the paper selfcontained we present a complete proof of Lemma B_1 (Section 3), which is essentially a special case (and consequently much easier to prove) of Theorem 1 (Chapter I, § 1) in [6] (see remarks on p. 35, [6], also Lemma 1 of [2]).

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1. Preliminaries. Our terminology and notation for classical Banach spaces is standard (see e.g. [5]). For normed spaces E, F we denote by $B(E, F)$ the space of bounded linear operators from E to F, considered as a normed space with the respective operator norm $\|\cdot\|_{B(E,F)}$. Instead of $B(E, E)$ we use $B(E)$.

Given Banach spaces E, F and an operator $T\colon E \to F$, we say that T is *p-absolutely summing* iff there exists a constant C such that $\forall n\ \forall x_1, x_2, \ldots, x_n \in E$

$$\sum_{i=1}^{n} \|Tx_i\|^p \leqslant C^p \sup_{y\in E^*, \|y\|\leqslant 1} \sum_{i=1}^{n} |y(x_i)|^p .$$

The infimum of such constants we denote by $\pi_p(T)$ (p-absolutely summing norm of T). If E, F are Hilbert spaces, we say that $T\colon E\to F$ is a *Hilbert–Schmidt operator* iff for a given (and then, in fact, for an arbitrary) orthonormal basis (e_j) of E $\sum_j \|Te_j\|^2 < \infty$. Then we write $\big(\sum_j \|Te_j\|^2\big)^{1/2} = \|T\|_{HS}$ (the Hilbert–Schmidt norm of T). It is a well-known fact that $\|T\|_{HS} = \pi_2(T)$.

Throughout the paper the capital C (possibly with some index) stands for universal constants.

In this paper we consider spaces over the real field, although all the results and their proofs are valid also in the complex case.

2. This section contains the proof of some "formal equivalences" of some facts proved by Bočkariev and Olevskiĭ and a statement in terms of 2-absolutely summing operators.

PROPOSITION. *Let C_0 be a positive constant. Then the following facts are equivalent:*

(A) *Given n, there exists a scalar matrix $[a_{ij}]_{i,j=1}^{n}$ such that*

$$\text{(i)}\qquad \sum_{i,j=1}^{n} |a_{ij}|^2 \geqslant C_0 \ln n,$$

$$\text{(ii)}\qquad \gamma(x) \overset{\mathrm{df}}{=} \left(\|x\|_\infty \cdot \frac{\sum_{k=1}^{n} \Big| \sum_{i=1}^{k} x_i \Big|}{n} \right)^{1/2} \geqslant \Big(\sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} x_j \Big|^2 \Big)^{1/2}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$.

(B) *Let $(S, \mathscr{B}, m)$ be a measure space. Then for every positive integer n and for every n measurable functions $h_1, h_2, \ldots, h_n$ on S such that*

$$\text{(j)}\qquad \|h_i\|_\infty \leqslant 1 \quad \textit{for} \quad i = 1, 2, \ldots, n,$$

$$\text{(jj)}\qquad \Big\| \sum_{i=1}^{n} a_i h_i \Big\|_2 \geqslant \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{1/2} \quad \textit{for} \quad a_1, \ldots, a_n \in \mathbf{R},$$

we have

$$\text{(jjj)}\qquad \frac{\sum_{k=1}^{n} \Big\| \sum_{i=1}^{k} h \Big\|_1}{n} \geqslant C_0 \ln n .$$

(C) *Given n, let $B_0 = \{x \in \mathbf{R}^n\colon \gamma(x) \leqslant 1\}$, $B = \operatorname{conv} B_0$ and $Y = (\mathbf{R}^n, \|\cdot\|_B)$, where $\|\cdot\|_B$ denote the Minkowski functional of B. Then*

$$\pi_2\big((i_{2,B})^*\big) \geqslant (C_0 \ln n)^{1/2},$$

where $i_{2,B}$ denotes the formal identity map regarded as an operator from l_n^2 into Y.

Remark I. (A) was proved by Bočkariev; a weaker version of (B) was established by Olevskiĭ ([1], [6]).

Remark II. In the proof of "formal equivalence" the quantity $C_0 \ln n$ may be replaced by an arbitrary one.

Proof. (A) $\Rightarrow$ (B). Let us consider a system of functions $h_1, h_2, \ldots, h_n$ satisfying conditions (j) and (jj) of (B). Then, by (A), there exists a matrix $[a_{ij}]_{i,j=1}^{n}$ satisfying (i) and such that (by (ii)) we have, for every $s \in S$,

$$\max_{1\leqslant i\leqslant n} |h_i(s)| \cdot \frac{\sum_{k=1}^{n} \Big| \sum_{i=1}^{k} h_i(s) \Big|}{n} \geqslant \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} a_{ij} h_j(s) \Big)^2 .$$

Integrating both sides of the above inequality and making use of (j) and (jj), we get

$$\frac{\sum_{k=1}^{n} \Big\| \sum_{i=1}^{k} h_i \Big\|_1}{n} \geqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2,$$

which combined with (i) yields (jjj).

(B) $\Rightarrow$ (C). Using (B), we estimate from below the quantity $\pi_2(i_{2,B}^*)$ ($i_{2,B}^*\colon Y^* \to l_n^2$ is the formal identity map). To attain this, consider the canonical isometrical embedding $j\colon Y^* \to K$, where K denotes the closure of the set of all the extreme points of the unit ball B of Y, i.e. j is defined by

$$[j(z)](k) = z(k) = \sum_{j=1}^{n} z_j k_j \qquad (z = (z_i) \in Y^*,\ K = (k_i) \in K \subset Y).$$

Clearly $K \subset \partial B_0$. Hence

$$\text{(1)}\qquad \gamma(k) = 1 \quad \text{for} \quad k \in K .$$

Now, by the Pietsch–Grothendieck theorem (cf. [8]), there exists a Borel measure on K, say m, such that

$$\int_K |j(z)(k)|^2 m(dk) \geqslant \|i_{2,B}^*(z)\|^2 \quad \text{for} \quad z \in Y^*$$

with $\|m\| = \pi_2(i_{2,B}^*)^2$. Hence we have

$$\sum_{j=1}^n |z_j|^2 = \|i_{2,B}^*(z)\|_2^2 \leqslant \int_K |j(z)(k)|^2 m(dk)$$
$$= \int_K \Big|\sum_{j=1}^n z_j k_j\Big|^2 m(dk) = \int_K \Big|\sum_{j=1}^n z_j \cdot \frac{k_j}{\|k\|_\infty}\Big|^2 \cdot \|k\|_\infty^2 m(dk).$$

Now it is easy to see that the measure space $(K, \|k\|_\infty^2 m(dk))$ and the functions $h_i = k_i/\|k\|_\infty$ $(i = 1, 2, \ldots, n)$ satisfy conditions (j) and (jj) of (B). Thus, by (jjj) and (1),

$$C_0 \ln n \leqslant \frac{\sum_{r=1}^n \big\|\sum_{i=1}^r h_i\big\|_1}{n} = \int_K \frac{1}{n}\sum_{r=1}^n \Big|\sum_{i=1}^r \frac{k_i}{\|k\|_\infty}\Big| \cdot \|k\|^2 \cdot m(dk)$$
$$= \int_K \gamma(k) m(dk) = m(K) \leqslant \|m\| = \pi_2(i_{2,B}^*)^2,$$

which yields the desired conclusion.

(C) $\Rightarrow$ (A). We recall first the following well-known and easy fact about 2-absolutely summing operators.

LEMMA 1. *Let H be a Hilbert space and let $S\colon X \to H$ be a linear operator. Then*

$$\pi_2(S) = \sup\{\|SA\|_{HS} \mid A\colon H \to X,\ \|A\| \leqslant 1\}.$$

If $\dim H < \infty$, then the supremum is attained (the proof follows immediately from the definitions).

Now assume that (C) holds. Thus, by Lemma 1, there exists an operator $A\colon l_n^2 \to Y^*$ such that

$$(2) \qquad \|A\| \leqslant 1,$$

$$(3) \qquad \|i_{2,B}^* A\|_{HS} \geqslant \sqrt{C_0 \ln n}.$$

Thus, remembering that $\|A\| = \|A^*\|$, $\|T\|_{HS} = \|T^*\|_{HS}$, we obtain

$$(2') \qquad \|A^*\| \leqslant 1,$$

$$(3') \qquad \|A^* i_{2,B}\|_{HS} \geqslant \sqrt{C_0 \ln n}.$$

Denote by $[a_{ij}]_{i,j=1}^n$ the matrix of the operator $i_{2,B}A^*$ in the natural basis of $\boldsymbol{R}^n$. Then

$$\|i_{2,B}A^*\|_{HS} = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$

Thus (3′) yields

$$(3'') \qquad \sum_{i,j=1}^n |a_{ij}|^2 \geqslant C_0 \ln n,$$

while (remembering that $i_{2,B}$ is a formal identity map) it follows from (2′) that

$$(2'') \qquad \sum_{i=1}^n \Big|\sum_{j=1}^n a_{ij} x_j\Big|^2 = \|A^* x\|_2^2 \leqslant \|x\|_B^2 \leqslant \gamma(x)^2$$
$$= \|x\|_\infty \frac{\sum_{k=1}^n \big|\sum_{i=1}^k x_i\big|}{n}.$$

This proves implication (C) $\Rightarrow$ (A) and completes the proof of the proposition.

Remark.

$$(4) \qquad \|i_{2,B}^*\| = \|i_{2,B}\| \leqslant C.$$

Proof. It is easy to see that it suffices to prove (4) for n of the form 2^k. Let $\chi^{(j)}$ $(1 \leqslant j \leqslant 2^k)$ be a basis of the Haar type in $\boldsymbol{R}^{2^k}$, normalized in l_n^2-norm, i.e.

$$\chi_i^{(1)} = 2^{-k/2}, \quad 1 \leqslant i \leqslant 2^k,$$

$$\chi_i^{(j)} = \begin{cases} 2^{\frac{r-k}{2}} & \text{for} \quad (l-1)2^{k-r} < i \leqslant (2l-1)2^{k-r-1}, \\ -2^{\frac{r-k}{2}} & \text{for} \quad (2l-1)2^{k-r-1} < i \leqslant l \cdot 2^{k-1}, \\ 0 & \text{otherwise} \end{cases}$$

for $j = 2^r + l$, $0 \leqslant r \leqslant k-1$, $0 < l \leqslant 2^r$.

Given $x \in \boldsymbol{R}^n$, we must show that $\|x\|_B \leqslant C\|x\|_2$. By the definition of $\|\cdot\|_B$ this is equivalent to

$$\inf_{\Sigma x_i = x} \sum \gamma(x_i) \leqslant C\|x\|_2.$$

Let x_m be the orthogonal projection of x on a subspace of $\boldsymbol{R}^n$ spanned by $\chi^{(j)}$ $(2^{m-1} < j \leqslant 2^m)$ for $m = 0, 1, \ldots, k$. Then, by easy computations, using the inequality

$$\frac{\|a\|_\infty \|a\|_1}{\|a\|_2^2} \leqslant \frac{\sqrt{p}+1}{2} \qquad (a \in \boldsymbol{R}^p),$$

we get

$$\gamma(x_m) \leqslant 2^{-m/4} \|x_m\|_2.$$

Hence, by the Schwartz inequality and the Pythagoras Theorem

$$\sum_{m=0}^{k} \gamma(x_m) \leqslant \sum_{m=0}^{k} 2^{-m/4} \|x_m\|_2 \leqslant \Big(\sum_{m=0}^{k} 2^{-m/2}\Big)^{1/2} \Big(\sum_{m=0}^{k} \|x_m\|_2^2\Big)^{1/2} \leqslant 2\,\|x\|_2,$$

and the proof is complete.

3. In the present section we show some applications of Lemma (B).

THEOREM 1. *There is no normalized structurally bounded basis in any space $L^1(S, \mathscr{B}, \mu)$ with $\dim L^1(S, \mathscr{B}, \mu) = \infty$. In particular, there is no normalized uniformly bounded basis in $L^1(0,1)$.*

THEOREM 2. *Let (f_n) be a normalized basis in $C(S)$ (S—compact, metric, infinite), and (μ_n)—its sequence of coefficient functionals. Then (μ_n) is not structurally bounded.*

Let us recall that $A \subset L$ (L—Banach lattice) is structurally bounded (bounded in order) iff it is contained in some interval $\langle -x, x\rangle$ ($x \in L$). Function spaces are considered as a lattice with respect to the pointwise order.

For the proofs of Theorems 1 and 2 we need the following weaker version of (B):

LEMMA B$_1$. *Let $(X, \mathscr{B}, m)$ be a measure space, and $h_1, h_2, \ldots, h_n$—measurable functions on X such that*

(j′) $$\|h_i\|_\infty \leqslant 1 \quad \textit{for} \quad 1 \leqslant i \leqslant n,$$

(jj′) $$\Big\|\sum_{i=k+1}^{k+l} h_i\Big\|_2^2 \geqslant l \quad \textit{for} \quad 0 \leqslant k \leqslant k+l \leqslant n.$$

Then

(jjj′) $$\sup_{1 \leqslant k \leqslant n} \Big\|\sum_{i=1}^{k} h_i\Big\|_1 \geqslant C \ln n.$$

Proof. Obviously it suffices to prove Lemma B_1 for $n = 5^r$ ($r = 1, 2, \ldots$). We shall show that there exist a positive integer q, a sequence of integers

$$r-1 = r_1 > r_2 > \ldots > r_{q+1} = -1$$

and a sequence (ε_k) with $\varepsilon_k = 0$ or $\varepsilon_k = 1$ for $k = 1, 2, \ldots, q$, such that if

$$n_k = \sum_{j=1}^{k} \varepsilon_j \cdot 5^{r_j}, \quad F_k = \sum_{i=1}^{n_k} h_i, \quad E_k = \{|F_k| > 5^{r_{k+1}+1}\}$$

for $0 \leqslant k \leqslant q$, then

(5) $$J_k \overset{\text{df}}{=} \int_{E_k} |F_k|\,dm \geqslant C_2(r_1 - r_{k+1}) \quad \text{for} \quad k = 0, 1, 2, \ldots, q,$$

which yields for $k = q$ the desired conclusion

$$\|F_q\|_1 \geqslant \int_{E_q} |F_q|\,dm \geqslant C_2 r = C \ln n.$$

For convenience we put

$$f_0 = 0, \quad f_k = F_{k-1}\chi_{X\setminus E_{k-1}} + \sum_{i=n_{k-1}+1}^{n_k} h_i \quad \text{for} \quad K = 1, 2, \ldots, q.$$

We define the sequences (r_k) and (ε_k) by induction:

1° We have $F_0 = 0$, $E_0 = \emptyset$, $n_0 = 0$. We put $r_1 = r-1$.

2° Suppose that r_j for $1 \leqslant j \leqslant k_0$ and ε_j for $1 \leqslant j \leqslant k_0 - 1$ have been chosen. We define ε_{k_0} to be either 0 or 1 in order to get

(6) $$\int_X |f_{k_0}|^2\,dm = \|f_{k_0}\|_2^2 \geqslant \tfrac{1}{4} 5^{r_{k_0}}.$$

More explicity, we put

$$\varepsilon_{k_0} = 0 \quad \text{if} \quad \int_{X\setminus E_{k_0-1}} |F_{k_0-1}|^2\,dm \geqslant \tfrac{1}{4} 5^{r_{k_0}},$$

$\varepsilon_{k_0} = 1$ otherwise.

It follows from (j′) and the definitions of E_{k_0-1} and f_{k_0} that

(7) $$\|f_{k_0}\|_\infty \leqslant 6 \cdot 5^{r_{k_0}},$$

(8) $$|f_{k_0}| \leqslant 5^{r_{k_0}} \quad \text{on } E_{k_0-1}.$$

In the sequel we shall need the following

LEMMA 2. *Let $(X, \mathscr{B}, m)$ be a measure space. If a measurable function f on S satisfies the conditions*

(a) $$\|f\|_\infty \leqslant 10 A,$$

(b) $$\|f\|_2^2 \geqslant \tfrac{1}{4} A,$$

then there exists an integer $t \geqslant 1$ such that

(9) $$\int_{|f| > A \cdot 5^{-t+1}} |f|\,dm \geqslant 16 \cdot 10^{-3}\, t.$$

Proof.

$$\tfrac{1}{4}A \leqslant \int_X |f|^2 dm \leqslant \sum_{u=1}^{\infty} \int_{\frac{A}{5^{u-1}}<|f|\leqslant \frac{2A}{5^{u-2}}} |f|^2 dm$$

$$\leqslant \sum_{u=1}^{\infty} 2A\cdot 5^{-u+2}\cdot \int_{|f|>\frac{A}{5^{u-1}}} |f|\,dm \leqslant \Big(2A\sum_{u=1}^{\infty} u\cdot 5^{-u+2}\Big)\max_{u\geqslant 1}\frac{1}{u}\cdot \int_{|f|>\frac{A}{5^{u-1}}} |f|\,dm.$$

Now to get (9) it suffices to compare the first and the last terms. Thus Lemma 2 is proved.

To define r_{k_0+1} observe that $f = f_{k_0}$, $A = 5^{r_{k_0}}$ satisfy the assumptions of Lemma 2. Hence, for some positive integer t_{k_0}, we have

$$(10)\qquad I_{k_0} \overset{\text{df}}{=} \int_{V_{k_0}} |f_{k_0}|\,dm \geqslant 16\cdot 10^{-3} t_{k_0},$$

where $V_{k_0} = \{|f_{k_0}| > 5^{r_{k_0}-t_{k_0}+1}\}$.

If $t_{k_0} > r_{k_0}$, we put $q = k_0$ and $r_{k_0+1} = -1$, otherwise we define

$$(11)\qquad r_{k_0+1} = r_{k_0} - t_{k_0}.$$

It remains to prove that (5) holds for $k = k_0$. To do this, we make use of the inequality

$$(12)\qquad J_k \geqslant I_k + \sum_{i=1}^{k-1}\Big(1-\frac{2}{5}-\frac{2}{25}-\ldots-\frac{2}{5^{k-i}}\Big)I_i \qquad (k = 1, \ldots, q),$$

which combined with (10) and (11) gives (5).

We prove (12) by induction. To this end, we make some observations:

$$(13)\qquad E_k\setminus E_{k-1} = V_k\setminus E_{k-1}$$

because

$$(14)\qquad F_k = f_k \quad \text{on} \quad X\setminus E_{k-1}.$$

Also

$$(15)\qquad E_k \supset E_{k-1}$$

because

$$(16)\qquad F_k = F_{k-1}+f_k \quad \text{on} \quad E_{k-1};$$

so $|F_k| \geqslant |F_{k-1}|-|f_k| \geqslant 5^{r_k+1}-5^{r_k} > 5^{r_k} \geqslant 5^{r_{k+1}+1}$ on E_{k-1} (by the definitions, (8) and $r_k > r_{k+1}$), whence (15) follows.

Now we turn to the proof of (12).

Observe first that it is trivially satisfied for $k = 0$. Suppose that (12) has been shown for $k = k_0 < q$.

Then, remembering (13)–(16),

$$\begin{aligned} J_{k_0+1} &= \int_{E_{k_0+1}} |F_{k_0+1}|\,dm = \int_{E_{k_0+1}\setminus E_{k_0}} |F_{k_0+1}|\,dm + \int_{E_{k_0}} |F_{k_0+1}|\,dm \\ &= \int_{E_{k_0+1}\setminus E_{k_0}} |f_{k_0+1}|\,dm + \int_{E_{k_0}} |F_{k_0}+f_{k_0+1}|\,dm \\ &\geqslant \int_{E_{k_0+1}\setminus E_{k_0}} |f_{k_0+1}|\,dm + \int_{E_{k_0}} (|F_{k_0}|-|f_{k_0+1}|)\,dm \\ &= \int_{E_{k_0+1}} |f_{k_0+1}|\,dm + \int_{E_{k_0}} |F_{k_0}|\,dm - 2\int_{E_{k_0}} |f_{k_0+1}|\,dm \\ &= I_{k_0+1} + J_{k_0} - 2\int_{E_{k_0}} |f_{k_0+1}|\,dm; \end{aligned}$$

on the other hand, by (8) and the definition of E_j

$$\begin{aligned} \int_{E_{k_0}} |f_{k_0+1}|\,dm &\leqslant \int_{E_{k_0}} 5^{r_{k_0+1}}\,dm = \sum_{j=1}^{k_0} \int_{E_j\setminus E_{j-1}} 5^{r_{k_0+1}}\,dm \\ &\leqslant \sum_{j=1}^{k_0} \int_{E_j\setminus E_{j-1}} 5^{r_{k_0+1}} \frac{|F_j|}{5^{r_{j+1}+1}}\,dm \\ &= \sum_{j=1}^{k_0} 5^{r_{k_0+1}-r_{j+1}-1} \int_{E_j\setminus E_{j-1}} |f_j|\,dm \leqslant \sum_{j=1}^{k_0} \frac{1}{5^{k_0+1-j}} I_j, \end{aligned}$$

which combined with the previous estimation yields (12) for $k = k_0+1$. This completes the proof of (12) and Lemma B_1. Before passing to the proofs of Theorems 1 and 2 we prove the following

LEMMA 3. *Let $(S, \mathscr{B}, \nu)$ be a measure space and let the functions (f_k, g_k) $(k = 1, 2, \ldots, n)$ form a biorthogonal sequence with respect to ν, i.e.*

$$\int_S f_i g_j\,d\nu = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Then

$$\int_S \int_S \Big|\sum_{k=1}^{n} f_k(s) g_k(t)\Big|^2 \nu(ds)\,\nu(dt) \geqslant n.$$

Proof.

$$\int_S\int_S\Big|\sum_{k=1}^n f_k(s)g_k(t)\Big|^2\nu(ds)\nu(dt)$$
$$=\Big(\int_S\int_S\Big|\sum_{k=1}^n f_k(s)g_k(t)\Big|^2\nu(ds)\nu(dt)\Big)^{1/2}\cdot\Big(\int_S\int_S\Big|\sum_{k=1}^n f_k(t)g_k(s)\Big|^2\nu(ds)\nu(dt)\Big)^{1/2}$$
$$\geqslant\int_S\int_S\Big(\sum_{k=1}^n f_k(s)g_k(t)\Big)\Big(\sum_{l=1}^n f_l(t)g_l(s)\Big)\nu(ds)\nu(dt)$$
$$=\sum_{k,l=1}^n\int_S\int_S f_k(s)g_k(t)f_l(t)g_l(s)\nu(ds)\nu(dt)=n$$

by the biorthogonality of (f_j, g_j), the Schwartz inequality and the Fubini theorem.

Remark 1. Lemma 3 may also be proved as follows. The integral operator $\mathscr{K}f(s)=\int_S K(s,t)f(t)\nu(dt)$, where $K(s,t)=\sum_{j=1}^n f_j(s)g_j(t)$, coincides with the identity operator on the n-dimensional subspace of $L^2(\nu)$ spanned by $f_1,\ldots,f_n$. Hence the Hilbert–Schmidt norm of $\mathscr{K}$ is not less than $\sqrt{n}$; on the other hand, this norm is equal to $\big(\int_S\int_S|K(s,t)|^2\nu(ds)\nu(dt)\big)^{1/2}$.

Proof of Theorem 1. Assume, to the contrary, that there exists a normalized basis in $L^1(S,\mathscr{B},\mu)$ such that

$$|f_j(s)|\leqslant f(s)\quad \mu\text{-a.e. on } S \text{ for } j=1,2,\ldots$$

for some $f\in L^1(\mu)$. Since $(f_j)_{j=1}^\infty$ is a basis, $f>0$ μ-a.e. Let $(g_j)_{j=1}^\infty$ denote the sequence of coefficient functionals of the basis (f_j). Then

$$\|g_i\|_\infty\leqslant M\quad\text{for some } M \text{ and } i=1,2,\ldots$$

Put $\nu=M\cdot f\cdot\mu$ and let $T\colon L^1(S,\mu)\to L^1(S,\nu)$ be defined by

$$(Th)(s)=\frac{h(s)}{f(s)}.$$

Clearly T is an isomorphism. Hence, if we put $\tilde f_j=Tf_j$, then $(\tilde f_j)_{j=1}^\infty$ is a basis in $L^2(S,\nu)$ with coefficient functionals $\tilde g_j=g_j/M$ (in particular $(\tilde f_j,\tilde g_j)_{j=1}^\infty$ is biorthogonal). Moreover,

$$\|\tilde f_j\|_\infty\leqslant 1,\quad \|\tilde g_j\|_\infty\leqslant 1\quad\text{for}\quad j=1,2,3,\ldots$$

It now follows from Lemma 3 that the assumptions of Lemma B_1 are satisfied if we put

$$h_j=\tilde f_j\otimes\tilde g_j\quad(\text{i.e. } h_j(s,t)=\tilde f_j(s)\cdot\tilde g_j(t)),\quad m=\nu\otimes\nu,\quad X=S\times S,$$

for $j=1,2,\ldots,n$, where n is arbitrary.

Hence, by (jjj)′,

$$\sup_{1\leqslant k\leqslant n}\int_S\int_S\Big|\sum_{j=1}^k\tilde f_j(s)\tilde g_j(t)\Big|\nu(ds)\nu(dt)\geqslant C\ln n. \tag{17}$$

On the other hand, since $(\tilde f_j)$ is a basis, the norms of the operators of partial sums (with respect to $(\tilde f_j)$) $S_k\colon L^1\to L^1$ are uniformly bounded (say, by $C_1<\infty$). Since S_k is an integral operator with a kernel $\sum_{j=1}^k\tilde f_j(s)\tilde g_j(t)$, we have

$$C_1\geqslant\|S_k\|_{B(L^1)}=\operatorname*{sup\,ess}_{t\in S}\int_S\Big|\sum_{j=1}^k\tilde g_i(t)\tilde f_j(s)\Big|\nu(ds),$$

for $k=1,2,\ldots$, which, by the finiteness of ν, contradicts (17) for large n. Thus Theorem 1 is proved.

Proof of Theorem 2. Let us assume the converse. Let $(f_j)_{j=1}^\infty$ be a normalized basis in $C(S)$ (S compact metric) and let its sequence of coefficient functionals $(\mu_j)_{j=1}^\infty$ be contained in the interval $\langle-\nu,\nu\rangle$ $(\nu\in C(S^*))$ (i.e. $|\mu_j(A)|\leqslant\nu(A)$ for any Borel subset A of S and for $j=1,2,\ldots$). Then, by the Radon–Nikodym Theorem, $\mu_j=g_j\cdot\nu$ for $j=1,2,\ldots$ with some measurable g_j, $\|g_j\|_\infty\leqslant 1$. Hence, for the same reasons as in the proof of Theorem 1 we have simultaneously

$$\|S_k\|_{B(C)}=\sup_{s\in S}\int_S\Big|\sum_{j=1}^k g_j(s)f_j(t)\Big|\nu(dt)\leqslant C_1$$

for $k=1,2,\ldots$ and, by Lemma B_1,

$$\sup_{1\leqslant k\leqslant n}\int_S\int_S\Big|\sum_{j=1}^k f_j(s)g_j(t)\Big|\nu(ds)\nu(dt)\geqslant C\cdot\ln n,$$

a contradiction for large n; thus Theorem 2 is proved.

Remark 2. It is easy to show that Lemma B_1 remains true if we replace condition (j′) by (j″) $\|h_i\|_\infty\leqslant M$ and $C\ln n$ in (jjj′) by $(C/M)\ln n$ (using, for instance, the substitution $h_i=M\tilde h_i$, $m=\tilde m/M^2$. Thus Theorem 1 remains true after replacing the assumption of structural boundedness by $|f_n(t)|\leqslant a_nf(t)$ for all n and all t with some $f\in L^1$ and $a_n=o(\ln n)$.

Remark 3. It follows from Remark 2 by standard stability methods that if $m(t)\colon \boldsymbol{R}^+\to\boldsymbol{R}^+$ is such that $\lim_{t\to\infty}m(t)/t=\infty$, f_n is a normalized basis in L^1 and $\lambda_g(t)=\mu(\{x\colon |g(x)|\geqslant t\})$, then

$$\varlimsup_{n\to\infty}\int_0^\infty\lambda_{f_n}(t)e^{m(t)}dt=+\infty.$$

The above fact is slightly stronger than the non-equi-integrability of $e^{|f_n|}$.

Strengthenings of Theorem 2 analogous to Remark 2 and 3 also hold.

Remark 4. The following local version of Theorem 1 (resp. 2) follows from the local character of Lemma B_1.

THEOREM 1′ (resp. 2′). *Let $(f_i, g_i)_{1\leqslant i\leqslant n}$ be a uniformly bounded biorthogonal sequence (say, $\|f_i\|_\infty \leqslant M$, $\|g_i\|_\infty \leqslant M$). Then the basis constant (with respect to L^1 (resp. C) norm) of the sequence $(f_i)_{1\leqslant i\leqslant n}$ is not less than*

$$\frac{C}{M^2}\cdot \ln n.$$

As a consequence we have

THEOREM 1″. *Let $X \subset L^1$, $\dim X = \infty$ and let X be complemented in L^1. Then there is no normalized structurally bounded basis in X.*

CONJECTURE 1. *Let (f_n) be a normalized basis in L^1 (a sequence of coefficient functionals of a normalized basis in C). Then $\{f_n\}$ is not weakly conditionally compact (equivalently: (F_n) does not weakly converge to 0), i.e. $\{f_n\}$ are not equi-integrable.*

CONJECTURE 2. *There is no Hilbert (resp. Bessel) system which forms a basis in L^1 (resp. C) (see* [7]).

Recall that a sequence (e_n) in a Banach space E is said to be *Bessel* (resp. *Hilbert*) *system* iff

$$\Big\|\sum a_n e_n\Big\| \geqslant (\text{resp.} \leqslant)\, C\cdot\Big(\sum |a_n|^2\Big)^{1/2}$$

for some constant C and every choice of sequence of scalars (a_n).

Remark 5. Let us call a biorthogonal sequence (f_n, g_n) a pseudo-basic sequence (in C or L^1) iff

$$(*)\qquad \sup_n \int_S\int_S \Big|\sum_{k=1}^{n} f_k(s)g_k(t)\Big|\, m(ds)m(dt) \leqslant \infty$$

and each function f_n (and g_n) belongs to a proper class of functions. In particular, any basis is a pseudobasic sequence, but the converse is not true, even if we add some density and totality assumptions. There exists a suitable block permutation of the Haar system such that it satisfies $(*)$, partial sums of any continuous function converge everywhere, but there is no uniform convergence. Clearly our proofs of Theorems 1 and 2 hold for a pseudobasis.

4. In the present section we prove (B), improving Lemma B_1. We show that an average L_1-norm of partial sums is large (and even most of them — we know that at least one of them is large). Precisely, we prove

LEMMA B_2. *Let $n = 5^{4k}$, let $I = [1, p^{4k}]$ be a segment of positive integers and let $A \subset I$ be such that $\operatorname{card} A \geqslant \frac{1}{2}\operatorname{card} I$. Then, under the assumptions of Lemma B_1 (formally weaker than those of* (B)),

$$\sup_{j_1, j_2 \in A}\Big\|\sum_{i=j_1+1}^{j_2} h_i\Big\|_1 \geqslant C_3 k = C_3' \ln n.$$

Suppose that we have made this; put

$$A = \Big\{1\leqslant j\leqslant n:\ \Big\|\sum_{i=1}^{j} h_i\Big\|_1 < \tfrac{1}{2}C_3 k\Big\}.$$

Then, by Lemma B_2, $\operatorname{card} A < \frac{1}{2}n$; hence

$$\operatorname{card}\Big\{j:\ \Big\|\sum_{i=1}^{j} h_i\Big\|_1 \geqslant \tfrac{1}{2}C_3 k\Big\} > \tfrac{1}{2}n,$$

whence assertion (jjj) *of* (B) *follows.*

Thus it suffices to prove Lemma B_2. To this end, we need the following combinatorial result:

LEMMA 4. *Let $J = [0, 2^k - 1]$ be a segment of positive integers and let I, A be the same as in Lemma B_2. Denote by J_m^s the segment of positive integers $[m\cdot 2^s, (m+1)\cdot 2^s - 1]$ for $0 \leqslant s \leqslant k$, $0 \leqslant m \leqslant 2^{k-s}-1$. Then there exists a map $\lambda\colon J \to I$ such that*

(a) $\lambda(J) \subset A$,

(b) *λ is strictly increasing,*

(c) $\max\{\operatorname{diam}\lambda(J_{2j}^r), \operatorname{diam}\lambda(J_{2j+1}^r)\} \leqslant \frac{1}{5}\operatorname{dist}(\lambda(J_{2j}^r), \lambda(J_{2j+1}^r))$ *for* $0 \leqslant r < k$, $0 \leqslant j \leqslant 2^{k-s-1}-1$.

Suppose we have proved Lemma 4. We show that

$$(**)\qquad \sup_{j\in J}\Big\|\sum_{i=\lambda(0)+1}^{\lambda(j)} h_i\Big\|_1 \geqslant C_3 k,$$

which immediately, by (a), yields Lemma B_2.

Our argument differs from the proof of Lemma B_1 only in technical details, and so we only give a sketch of it.

We define two finite sequences:

(r_j) such that $k-1 = r_1 > r_2 > \ldots > r_{q+1} = -1$,

(ε_j) such that $\varepsilon_j = 0$ or $\varepsilon_j = 1$ for $1 \leqslant j \leqslant q$, such that if

$$n_j = \sum_{i=1}^{j}\varepsilon_i\cdot 2^{r_i},\quad F_j = \sum_{i=\lambda(0)+1}^{\lambda(n_j)} h_i,\quad E_j = \{|F_j| > 5\cdot[\lambda(n_j + 2^{r_j+1}) - \lambda(n_j)]\}$$

for $0 \leqslant j \leqslant q$,

then

$$J_k = \int_{E_k} |F_k|\,dm > C_3(r_1 - r_{k+1}) \quad \text{for } k = 0, \quad 1, \ldots, q, \tag{5'}$$

which for $k = q$ yields (**).

We also put $f_0 = 0$, $f_j = F_{j-1}\cdot \chi_{X\setminus E_{j-1}} + \sum_{i=\lambda(n_{j-1}+1)}^{\lambda(n_j)} h_i$.

We define (r_j) and (ε_j) by induction. Put $r_1 = k-1$ and suppose that we have defined r_i for $1 \leqslant i \leqslant j$ and ε_i for $1 \leqslant i \leqslant j-1$. We let ε_j be 0 or 1 in order to get

$$\int_X |f_j|^2\,dm \geqslant \tfrac{1}{4}[\lambda(n_{j-1}+2^{r_j}) - \lambda(n_{j-1})]. \tag{6'}$$

Then $f = f_j$ satisfies the assumptions of Lemma 2 with $A = \lambda(n_{j-1}+2^{r_j}) - \lambda(n_{j-1})$. Hence there exists a $t_j \geqslant 1$ such that

$$\int_{|f_j|>5^{-t_j+1}\cdot A} |f_j|\,dm \geqslant 16\cdot 10^{-3} t_j.$$

If $t_j > r_j$, we put $q = j$ and $r_{j+1} = -1$; otherwise we define

$$r_{j+1} = r_j - t_j. \tag{11'}$$

In any case we have

$$\begin{aligned}
5\cdot[\lambda(n_j+2^{r_{j+1}}) - \lambda(n_j)] &\leqslant 5\,\mathrm{diam}\,\lambda[n_j, n_j + 2^{r_{j+1}+1} - 1] \\
&\leqslant 5\cdot 5^{-t_j}\,\mathrm{dist}\big(\lambda([n_{j-1}, n_{j-1}+2^{r_j}-1]), \lambda([n_{j-1}+2^{r_j}], n_{j-1}+2^{r_j+1}-1])\big) \\
&\leqslant 5^{-t_j+1}\big(\lambda(n_{j-1}+2^{r_j}) - \lambda(n_{j-1})\big)
\end{aligned}$$

by (c) applied t_j times. Hence we have

$$\int_{V_j} |f_j|\,dm \geqslant 16\cdot 10^{-3}\cdot t_j \geqslant 16\cdot 10^{-3}\cdot (r_j - r_{j+1}), \tag{10'}$$

where $V_j = \{|f_j| > 5\cdot[\lambda(n_j+2^{r_{j+1}}) - \lambda(n_j)]\}$.

Analogues of (13)–(16) also hold; we have on $E_i \setminus E_{i-1}$ for $i < j$

$$\begin{aligned}
|f_j| &\leqslant \lambda(n_{j-1}+2^{r_j}) - \lambda(n_{j-1}) \leqslant 5^{-(r_i - r_j)}\big(\lambda(n_i + 2^{r_{i+1}}) - \lambda(n_i)\big) \\
&\leqslant 5^{i-j}\big(\lambda(n_i+2^{r_{i+1}}) - \lambda(n_i)\big) \leqslant 5^{i-j-1}|F_i| = 5^{i-j-1}|f_i|.
\end{aligned}$$

We use here (c) (applied $r_i - r_j$ times), the fact that (r_j) strictly decreases and the definitions of E_s and f_s. Thus we are now able to prove (12′), which, combined with (10′), yields Lemma B_2.

It remains to prove Lemma 4. To this end, define $\mathscr{K}_J = \{J_m^s \cap J\}$, $\mathscr{K}_I = \{I_l^r \cap I\}$, where $I_l^r = [(l-1)\cdot 5^{2r}+1, l\cdot 5^{2r}]$. We define by induction (with respect to the inclusion order) a map

$$\Lambda\colon \mathscr{K}_J \to \mathscr{K}_I$$

satisfying the following conditions:

$$\Lambda(J_m^s) \cap A \neq \emptyset \quad \text{for all } J_m^s, \tag{a'}$$

$$\sup J_{m_1}^{s_1} < \inf J_{m_2}^{s_2} \Rightarrow \sup \Lambda(J_{m_1}^{s_1}) < \inf \Lambda(J_{m_2}^{s_2}), \tag{b'}$$

$$\max\{\mathrm{diam}\,\Lambda(J_{2j}^r), \mathrm{diam}\,\Lambda(J_{2j+1}^r)\} \leqslant \tfrac{1}{5}\mathrm{dist}\big(\Lambda(J_{2j}^r), \Lambda(J_{2j+1}^r)\big), \tag{c'}$$

$$J_{m_1}^{s_1} \subset J_{m_2}^{s_2} \Rightarrow \Lambda(J_{m_1}^{s_1}) \subset \Lambda(J_{m_2}^{s_2}). \tag{d'}$$

Obviously, having such a Λ, it suffices to put for $m \in J$ as $\lambda(m)$ any element of $\Lambda(J_m^0) \cap A \neq \emptyset$ (by (a′)) (remember that $J_m^0 = \{m\}$).

Construction of Λ. Let us introduce some notations. Let

$$\varrho(I_m^t) = \frac{\mathrm{card}(I_m^t A)}{\mathrm{card}\, I_m^t}.$$

We shall say that $I_{m'}^{t-1}$ is a *subsegment of* I_m^t (respectively, that the inclusion $I_{m'}^{t-1} \subset I_m^t$ is):

of type I iff $\varrho(I_{m'}^{t-1}) \geqslant \frac{16}{19}\varrho(I_m^t)$,

of type II iff $\varrho(I_m^{t-1}) \geqslant \frac{3}{2}\varrho(I_m^t)$.

Note that I_m^t with $t > 0$ contains either a subsegment of type II or 7 different subsegments of type I. Otherwise we have

$$\begin{aligned}
\varrho(I_m^t) &= \tfrac{1}{25}\sum_{u=1}^{25} \varrho(I_{(m-1)\cdot 25+u}^{t-1}) \\
&< \tfrac{1}{25}\cdot[6\cdot\tfrac{3}{2}\varrho(I_m^t) + 19\cdot\tfrac{16}{19}\cdot\varrho(I_m^t)] = \varrho(I_m^t),
\end{aligned}$$

a contradiction.

Now we turn to the inductive construction of Λ. Put $\Lambda(J_0^k) = I_1^{2k}$ (i.e. $\Lambda(J) = I$) and suppose we have defined $\Lambda(J_j^{r+1}) = I_l^s$, $r \geqslant 0$.

We shall define $\Lambda(J_{2j}^r)$ and $\Lambda(J_{2j+1}^r)$. To this end, consider any maximal sequence

$$I_l^s = I_{m_s}^s \supset I_{m_{s-1}}^{s-1} \supset \ldots \supset I_{m_t}^t$$

with all inclusions of type II. Suppose we have shown that $t > 0$. Then I_m^t contains at least 7 subsegments of type I. We define $\Lambda(J_{2j}^r)$ and $\Lambda(J_{2j+1}^r)$ to be the first and the seventh of them, respectively. Then (a′) follows inductively from the fact that by inclusion of type either I or II we pass from segments with non-empty intersections with A to those possessing the same property; (c′) follows from the fact that between each pair $\Lambda(J_{2j}^r)$, $\Lambda(J_{2j+1}^r)$ we have 5 other segments of the same length; (b′) and (d′) follow immediately by induction.

Thus it remains to show that $t > 0$. Suppose that this is not true, then, taking into account the preceding $k-r-1$ steps of induction, we get a descending sequence of segments:

$$I = I_1^{2k} = I_{m_{2k}}^{2k} \supset I_{m_{2k-1}}^{2k-1} \supset \ldots \supset I_l^s = I_{m_s}^s \supset \ldots \supset I_{m_l}^l = I_{m_0}^0,$$

where each inclusion is of type either I or II and inclusion of type I takes place at most $k-r-1 \leqslant k-1$ times; hence

$$1 \geqslant \varrho(I_{m_0}^0) \geqslant (\tfrac{3}{2})^{k+1} \cdot (\tfrac{16}{19})^{k-1} \cdot \varrho(I) \geqslant (\tfrac{3}{2})^2 \cdot (\tfrac{24}{19})^{k-1} \cdot \tfrac{1}{2} > 1,$$

a contradiction.

This completes the proof of Lemmas 4, B_1 and B_2.

Added in proof. After this paper has been submitted for publication the second named author proved that every normalized basis in an $\mathscr{L}_1$-space contains a subbasis equivalent to the unit vector basis of l^1. This establishes conjecture I, cf. S. J. Szarek, *Bases and biorthogonal systems in the spaces C and L^1*, Ark. Mat., to appear.

For a simple proof of the Bočkariov inequality cf. B. S. Kašin, *Remarks on estimation of Lebesgue functions of orthonormal systems*, Mat. Sb. 106 (148) (1978), pp. 380–385 (Russian).

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