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Multiply self-decomposable measures in generalized convolution algebras

by

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Abstract. The aim of the present paper is to prove a representation theorem for multiply self-decomposable probability measures in generalized convolution algebras. Moreover, a one-to-one correspondence between the class of multiply self-decomposable measures in a generalized convolution algebra and the class of multiply monotone functions on the real line is established.

1. Introduction. Let Π denote the class of all probability measures supported by the half-line $[0, \infty)$, endowed with the weak convergence. Let δ_x denote the unit mass at the point $x \geqslant 0$ and let $T_x, x > 0$, denote the map given by $(T_x P)(E) = P(x^{-1}E)$ for $P \in \Pi$ and E a Borel subset of $[0, \infty)$. In the sequel we shall preserve the terminology of [3] and [5]. In particular, by $\circ$ we shall denote a generalized convolution defined on Π such that $(\Pi, \circ)$ stands for a regular generalized convolution algebra. Further, for $P \in \Pi$ we shall denote by Φ_P its characteristic function.

The concept of multiply self-decomposable measures for the ordinary convolution has been introduced in [2]. In a similar way one can define multiply self-decomposable measures in the algebra $(\Pi, \circ)$ as follows: Let Π_1 denote the class of all self-decomposable measures in $(\Pi, \circ)$, i.e. such measures p that for every number c in $(0, 1)$ there exists a measure Q_c in Π such that $p = T_c p \circ Q_c$ or, equivalently,

$$\Phi_p(t) = \Phi_p(ct)\Phi_{Q_c}(t) \qquad (t \in [0, \infty)). \tag{1.1}$$

Next for every integer $n > 1$ let Π_n denote the class of all measures in Π_1 such that for every number c in $(0, 1)$ the component Q_c belongs to Π_{n-1}. Every measure in Π_n $(n = 1, 2, \ldots)$ will be called *n-times self-decomposable* and every measure in $\Pi_\infty := \bigcap_{n=1}^{\infty} \Pi_n$ will be called *completely self-decomposable*. Since every stable measure in $(\Pi, \circ)$ is completely self-decomposable (Theorem 2, [5]) the set Π_∞ is non-empty. It is evident that $\Pi_\infty \subset \Pi_{n+1} \subset \Pi_n$ $(n = 1, 2, \ldots)$, which, however, according to Example 2.5 cannot be replaced by the equality.

2. n-times self-decomposable measures. Consider a measure $p \in \Pi_n$ $(n = 1, 2, \ldots)$. It is well known that for every number c in $(0, 1)$ the measures p and Q_c are both infinitely divisible. Further, for the characteristic function Φ_p of an infinitely divisible p in $(\Pi, 0)$ we have the formula

$$\Phi_p(t) = \exp \int_0^\infty \frac{\Omega(tx)-1}{W(x)}\, m(dx), \tag{2.1}$$

where m is a finite Borel measure on $[0, \infty)$, being the spectral measure for p, W is defined by

$$W(x) = \begin{cases} 1-\Omega(x), & 0 \leqslant x \leqslant x_0, \\ 1-\Omega(x_0), & x > x_0, \end{cases}$$

where x_0 is a positive number satisfying the condition $\Omega(x) < 1$ whenever $0 < x \leqslant x_0$ and Ω is the kernel corresponding to $(\Pi, 0)$.

We now proceed to establish some properties of spectral measures m corresponding to n-times self-decomposable measures.

Let $[0, \infty)$ denote the compactified half-line. For a subset E of $[0, \infty]$ such that $\bar{E} \subset (0, \infty]$ and a Borel measure m on $[0, \infty]$ we put

$$I_m(E) = \int_E \frac{m(dx)}{W(x)}, \tag{2.2}$$

where the integrand is assumed to be $(1-\Omega(x_0))^{-1}$ if $x = \infty$. Denote by M_n the set of all finite Borel measures m on $[0, \infty]$ satisfying for every system of numbers $c_1, c_2, \ldots, c_n$ from the interval $(0,1)$ and all Borel subsets E with $\bar{E} \subset (0, \infty]$ the following condition:

$$I_m(E) + \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1, i_2, \ldots, i_k = 1 \\ \text{distinct}}}^{n} I_m(c_{i_1}^{-1} c_{i_2}^{-1} \ldots c_{i_k}^{-1} E) \geqslant 0. \tag{2.3}$$

By virtue of Lemma 8, [5], and by an easy induction we get the following proposition:

2.1. PROPOSITION. *A Borel measure m on $[0, \infty)$ is the spectral measure for a n-times self-decomposable measure in $(\Pi, 0)$ if and only if $m \in M_n$.*

Suppose that the measure m is concentrated on the open half-line $(0, \infty)$ and put

$$J_m(u) = \int_{e^{-u}}^{\infty} \frac{m(dt)}{W(t)} \quad (-\infty < u < \infty). \tag{2.4}$$

Obviously, for $a < b$ we have $I_m([e^{-b}, e^{-a})) = J_m(b) - J_m(a)$. Further, for arbitrary $t_1, t_2, \ldots, t_n > 0$ we put $c_i = e^{-t_i}$ $(i = 1, 2, \ldots, n)$. Consequently, by (2.3), $m \in M_n$ if and only if for every system $a, b, t_1, t_2, \ldots, t_n$ of real numbers with $a < b$ and $t_1, t_2, \ldots, t_n > 0$ we have the inequality

$$\mathop{\Delta}_{t_1 t_2 \ldots t_n} J_m(b) - \mathop{\Delta}_{t_1 t_2 \ldots t_n} J_m(a) \geqslant 0, \tag{2.5}$$

where $\mathop{\Delta}_{t_1 t_2 \ldots t_n}$ is the difference operator defined inductively on real functions g by

$$\mathop{\Delta}_{t_1 t_2 \ldots t_n} g(a) = \mathop{\Delta}_{t_1 t_2 \ldots t_{n-1}} (g(a) - g(a-t_n)) \quad (a \in R,\ t_1, t_2, \ldots, t_n > 0).$$

Moreover, the function J_m satisfies the condition $\lim_{u \to -\infty} J_m(u) = 0$, which together with (2.5) implies that the function J_m is n-times monotone (for the definition of multiply monotone functions, see [2]).

Hence we get the following criterion:

2.2. PROPOSITION. *A Borel measure m on the open half-line $(0, \infty)$ is the spectral measure for an n-times (resp. completely) self-decomposable probability measure in $(\Pi, 0)$ if and only if the corresponding function J_m defined by formula (2.4) is n-times (resp. completely) monotone.*

Let N_n be the subset of M_n consisting of probability measures on $[0, \infty]$. It is clear that the set N_n is convex and compact.

Given a measure m in N_n $(n = 1, 2, \ldots)$ concentrated on the open half-line $(0, \infty)$, we get the n-times monotone function J_m. By virtue of Proposition 4.1, [2], it follows that there exists a unique left-continuous monotone non-decreasing and non-negative function q_m on the real line such that

$$J_m(u) = \int_{-\infty}^{u} \int_{-\infty}^{u_{n-1}} \int_{-\infty}^{u_{n-2}} \ldots \int_{-\infty}^{u_1} q_m(t)\, dt\, du_1 \ldots du_{n-1} \tag{2.6}$$

$(-\infty < u < \infty)$. Hence and by virtue of (2.4) we get the formula

$$m(E) = \int_E w(t) \int_{-\infty}^{-\log t} \int_{-\infty}^{u_{n-2}} \int_{-\infty}^{u_{n-3}} \ldots \int_{-\infty}^{u_1} q_m(v)\, dv\, du_1\, du_2 \ldots du_{n-2} \frac{dt}{t} \tag{2.7}$$

$(E \subset (0, \infty))$. In particular, for $E = (0, \infty)$, equation (2.7) becomes

$$1 = \int_0^\infty w(t) \int_{-\infty}^{-\log t} \int_{-\infty}^{u_{n-2}} \int_{-\infty}^{u_{n-3}} \ldots \int_{-\infty}^{u_1} q_m(v)\, dv\, du_1\, du_2 \ldots du_{n-2} \frac{dt}{t}. \tag{2.8}$$

Conversely, if q_m is a left-continuous, monotone non-decreasing and non-negative function normalized by condition (2.8), then the probability measure m defined by means of formula (2.7) is a spectral measure corresponding to an n-times self-decomposable measure in $(\Pi, 0)$. In such a way pe get a one-to-one correspondence between all measures in N_n concentrated on the open half-line $(0, \infty)$ and all left-continuous, monotone non-decreasing and non-negative functions normalized by condition (2.8). It hints at such a correspondence preserves convex combinations of elements. Consequently, extreme points are transformed into extreme points.

We now proceed to find all functions q being extreme points. First we note that the extreme points are functions which for some x are constant on both half-lines $(-\infty, s)$ and (s, ∞). Hence for an extreme point q there exists a unique number $s > 0$ such that $q(x) = (n-1)!\, C_s \chi_{(\log 1/s, \infty)}$, where C_s is a real constant (depending on s), χ_E is the indicator of a set E and the constant $(n-1)!$ is introduced to simplify further notations. Conversely, one can easily prove that such functions are extreme points. Let us denote by m_s $(s > 0)$ the extreme point of the set N_n, $n = 1, 2, \ldots$, corresponding to the function $q(x) = (n-1)!\, C_s \chi_{(\log 1/s, \infty)}$. By virtue of (2.7) we get the formula

$$m_s(E) = C_s \int_E w(t) \left(\log \frac{s}{t}\right)_+^{n-1} \frac{dt}{t}, \tag{2.9}$$

where $E \subset (0, \infty)$ and for a real number λ we write $\lambda_+ = \max(\lambda, 0)$. The constant C_s is determined by condition (2.8). Namely,

$$C_s^{-1} = \int_0^s w(t) \left(\log \frac{s}{t}\right)^{n-1} \frac{dt}{t}. \tag{2.10}$$

Putting, in addition, $m_s = \delta_s$ for $s = 0$ or ∞, respectively, we get the following proposition:

2.3. Proposition. *The set $\{m_s: s \in [0, \infty]\}$ coincides with the set of extreme points of N_n $(n = 1, 2, \ldots)$.*

Now, by Krein–Milman–Choquet Theorem [1], we get the following statement: $\mu \in N_n$ $(n = 1, 2, \ldots)$ if and only if there exists a probability measure ν on $[0, \infty]$ such that

$$\int_{[0,\infty]} f(x)\mu(dx) = \int_{[0,\infty]} \left(\int_{[0,\infty]} f(x)\, m_s(dx) \right) \nu(ds)$$

for all continuous bounded functions f on $[0, \infty]$. Moreover, if μ is concentrated on $[0, \infty)$, then ν does the same. Hence and by (2.1) we have the following theorem:

2.4. Theorem. *The class of characteristic functions of n-times $(n = 1, 2, \ldots)$ self-decomposable measures in $(\Pi, 0)$ coincides with the class of functions of the form*

$$\Phi_p(t) = \exp\left\{ \int_0^\infty \left(\int_0^s \frac{\Omega(tx)-1}{x} \left(\log \frac{s}{x}\right)^{n-1} dx \right) \times \left[\int_0^s w(u) \left(\log \frac{s}{u}\right)^{n-1} \frac{du}{u} \right]^{-1} \right\} \nu(ds), \tag{2.11}$$

where ν is a finite Borel measure on $[0, \infty)$.

2.5. Example. For a fixed number $s > 0$ let us form a characteristic function φ as follows:

$$\varphi(t) = \exp\left\{ \int_0^s \frac{\Omega(tx)-1}{x} \left(\log \frac{s}{x}\right)^{n-1} dx \right\}. \tag{2.12}$$

By virtue of Theorem 2.4, φ is n-times self-decomposable. Further, by the uniqueness of representation (2.1) the spectral measure m corresponding to the function φ is given by the formula

$$m(E) = \int_E \left(\log \frac{s}{x}\right)_+^{n-1} \frac{\omega(x)}{x}\, dx. \tag{2.13}$$

Hence the function J_m defined by (2.4) is of the form

$$J_m(u) = \int_{e^{-u}}^\infty \left(\log \frac{s}{x}\right)_+^{n-1} \frac{dx}{x} \tag{2.14}$$

which, of course, is n-times monotone. On the other hand, J_m is not $(n+1)$-times monotone. Consequently, by Proposition 2.2, φ is not $(n+1)$-times self-decomposable, which shows that for every $n = 1, 2, \ldots$

$$\Pi_\infty \subsetneq \Pi_{n+1} \subsetneq \Pi_n.$$

3. Completely self-decomposable measures. Given a finite Borel measure m on the open half-line $(0, \infty)$ we define function J_m by means of formula (2.4). By Proposition 2.2, m is a spectral measure for a completely self-decomposable measure in $(\Pi, 0)$ if and only if the function J_m is completely monotone. Suppose that J_m is completely monotone; then there exists a unique completely monotone function P_m such that

$$J_m(t) = \int_{-\infty}^t P_m(u)\, du \quad (-\infty < t < \infty). \tag{3.1}$$

Hence and by (2.4) the measure m is uniquely determined by P_m:

$$m(E) = \int_E w(t) p_m(-\log t)\frac{dt}{t}. \tag{3.2}$$

In particular, for a probability measure m on $(0, \infty)$ we get the formula

$$1 = \int_0^\infty w(t) p_m(-\log t)\frac{dt}{t}. \tag{3.3}$$

Conversely for every completely monotone function p_m on the real-line normalized by condition (3.3) the probability measure m defined by (3.2) is completely self-decomposable. Of course the correspondence $m \leftrightarrow p_m$ is one-to-one and preserves the convex combinations of elements.

Denote by $\mathscr{K}$ the class of all completely monotone functions p on the real line normalized by condition (3.3). Given $t > 0$ and an extreme point p of $\mathscr{K}$, define two functions p_1 and p_2

$$p_1(u) = \frac{p(u)+p(u-t)}{1+c},$$

$$p_2(u) = \frac{p(u)-p(u-t)}{1-c} \quad (-\infty < u < \infty),$$

where $c = \int_0^\infty w(u) p(-\log u - t)\frac{du}{u}$. It is evident that for sufficiently large t we have $0 < c < 1$ and then the functions p_1 and p_2 are both completely monotone. Moreover, p_1 and p_2 are normalized by condition (3.3). On the other hand, we have, for every $u \in (-\infty, \infty)$,

$$p(u) = \tfrac{1}{2}(1+c)p_1(u) + \tfrac{1}{2}(1-c)p_2(u)$$

which, by the assumption that p is an extreme point, implies that for all $u \in (-\infty, \infty)$ and for sufficiently large $t > 0$

$$p(u-t) = p(u)\int_0^\infty w(v) p(-\log v - t)\frac{dv}{v}.$$

Consequently, the function p is of the form

$$p(u) = ae^{su} \quad (a, s > 0;\ -\infty < u < \infty).$$

Since, by the proof of Theorem 2, [5], the integral

$$\int_0^\infty w(t) p(-\log t)\frac{dt}{t} = a\int_0^\infty \frac{w(t)}{t^{1+s}}\,dt$$

is finite if and only if $0 < s < \varkappa$, where $\varkappa$ is the characteristic exponent of the algebra in question, and by condition (3.3) the constant a is given by

$$a = \left(\int_0^\infty \frac{w(t)\,dt}{t^{1+s}}\right)^{-1} \tag{3.4}$$

the function p being an extreme point of the set $\mathscr{K}$ is of the form

$$p_s = \left(\int_0^\infty \frac{w(t)\,dt}{t^{1+s}}\right) e^{su} \quad (-\infty < u < \infty;\ 0 < s < \varkappa). \tag{3.5}$$

Putting $N_\infty = \bigcap_{n=1}^\infty N_n$,

$$m_s = \left(\int_0^\infty \frac{w(t)}{t^{1+s}}\,dt\right)^{-1} \cdot \int_E \frac{w(t)}{t^{1+s}}\,dt \quad (0 < s < \varkappa) \tag{3.6}$$

and, in addition, $m_s = \delta_s$ for $s = 0$ or ∞, we get the following proposition:

3.1. Proposition. *The set* $\{m_s: s \in [0, \varkappa) \cup \{\infty\}\}$ *coincides with the set of extreme points of* N_∞.

Now, by Krein–Milman–Choquet Theorem [1], we get the following statement: $\mu \in N_\infty$ if and only if there exists a probability measure ν on $[0, \varkappa) \cup \{\infty\}$ such that

$$\int_{[0,\infty]} f(x)\mu(dx) = \int_{[0,\varkappa)\cup\{\infty\}} \left(\int_{[0,\infty]} f(x) m_s(dx)\right)\nu(ds)$$

for all continuous bounded functions f on $[0, \infty]$. Moreover, if μ is concentrated on $[0, \infty)$, then ν is concentrated on $[0, \varkappa)$. Consequently, by (2.1), it follows the following theorem:

3.2. Theorem. *The class of characteristic functions of completely self-decomposable measures in* $(\Pi, 0)$ *coincides with the class of functions of the form*

$$\Phi_p(t) = \exp\left\{\int_0^\varkappa \int_0^\infty \frac{\Omega(tx)-1}{x^{1+s}}\,dx \left(\int_0^\infty \frac{w(u)}{u^{1+s}}\,du\right)^{-1} \nu(ds)\right\}, \tag{3.7}$$

where $\varkappa$ *is the characteristic exponent of the algebra in question and* ν *is a finite Borel measure on the interval* $[0, \varkappa)$.

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An estimation of the Lebesgue functions of biorthogonal systems with an application to the non-existence of some bases in C and L^1

by

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Abstract. We prove the non-existence of a normalized basis in L^1 consisting of uniformly bounded functions and the dual fact for C. In the proofs we make use of Olevskiĭ's technique from [6], Chapter I. We show also, using methods of p-absolutely summing operators, some connections between integral and numerical inequalities, which together with considerations of Olevskiĭ's type give a new proof of the Bočkariev inequality from [1].

0. Introduction. In this paper we show, answering the question of Olevskiĭ ([6], p. 36, (vi)), that there is no normalized basis in $L^1(0,1)$ consisting of uniformly bounded functions. We prove also the "dual" fact for the space $C(0,1)$. These results generalize a theorem of Olevskiĭ (see [6], Chapter I, § 2, Theorems 2 and 9):

No uniformly bounded orthonormal system is a basis in L^1 or C. Our statements admit two methods of proof. The first one makes use of Olevskiĭ's technique, the second one starts from a certain inequality on averages of partial sums of numerical series proved by Bočkariev ([1]).

The paper consists of four sections. Section 1 has a preliminary character. In Section 2 we prove the equivalence of the approaches of Bočkariev and Olevskiĭ. As the common vocabulary for them we use the theory of absolutely summing operators. Section 3 contains the proofs of the non-existence of a normalized structurally bounded basis in L^1, the "dual" result for C and some further strengthenings. Section 4 contains in fact the new proof of the Bočkariev inequality, which is based on the results of Section 2 and the proof of Theorem 1 of Section 3.

To make the paper selfcontained we present a complete proof of Lemma B_1 (Section 3), which is essentially a special case (and consequently much easier to prove) of Theorem 1 (Chapter I, § 1) in [6] (see remarks on p. 35, [6], also Lemma 1 of [2]).

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