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Multiply self-decomposable probability measures on Banach spaces

by

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Abstract. In the present paper we define multiply self-decomposable probability measures on a Banach space and give a general form of their characteristic functionals.

1. Introduction. This paper is concerned with probability measures defined on Borel subsets of a real separable Banach space X. For a probability measure μ on X, the characteristic functional $\hat{\mu}$ is defined on the dual space X^* by the formula

$$\hat{\mu}(y) = \int_X e^{i\langle y,x\rangle}\,\mu(dx) \qquad (y \in X^*),$$

where $\langle\cdot,\cdot\rangle$ denotes the dual pairing between X and X^*.

Recall that a probability measure μ on X is self-decomposable if for every number c in $(0,1)$ there exists a probability measure μ_c on X such that

$$\hat{\mu}(y) = \hat{\mu}(cy)\hat{\mu}_c(y) \qquad (y \in X^*). \tag{1.1}$$

The problem of describing the class of characteristic functionals of self-decomposable probability measures has been completely solved by Urbanik [8]. In the same paper the author has obtained a general form of characteristic functionals even for a larger class of probability measures, namely, for Levy's measures on X.

We now introduce a concept of multiply self-decomposable probability measures on Banach spaces. Let $L_1(X)$ denote the class of all self-decomposable probability measures on X. For every integer $n>1$, let $L_n(X)$ denote the class of all measures μ in $L_1(X)$ such that for every number c in $(0,1)$ the component μ_c in (1.1) belongs to $L_{n-1}(X)$. Every measure in $L_n(X)$ will be called *n-times self-decomposable*. Further, every measure in $L_\infty(X) := \bigcap_{n=1}^{\infty} L_n(X)$ will be called *completely self-decomposable*. Since every stable measure on X is completely self-decomposable (Proposition 1.9, [3]), the set $L_\infty(X)$ is non-empty.

2. A characterization of multiply self-decomposable probability measures on X. It is well known ([2], [6]) that every infinitely divisible probability measure μ on X has a unique representation

$$\mu = \varrho * \tilde{e}(M), \tag{2.1}$$

where ϱ is a symmetric Gaussian measure and $\tilde{e}(M)$ a generalized Poisson measure on X. In terms of characteristic functionals we have the formulas

$$\hat{\varrho}(y) = \exp(-\tfrac{1}{2}\langle y, Ry\rangle) \quad (y \in X^*), \tag{2.2}$$

where R is a covariance operator and

$$\hat{\tilde{e}}(M)(y) = \exp\Big(i\langle y, x_0\rangle + \int_X k(x, y)\, M(dx)\Big) \tag{2.3}$$

for certain $x_0 \in X$. The kernel k is defined by the formula

$$k(y, x) = e^{i\langle y, x\rangle} - 1 - i\langle y, x\rangle 1_W(x),$$

where 1_W denotes the indicator of a compact subset W of X. Moreover, the measure M, being a generalized Poisson exponent, has a finite mass outside every neighbourhood of 0 in X. Let $L(M)$ denote the set of all generalized Poisson exponents on X. Recall that $M(X)$ is a cone and if $M \in M(X)$ and $M \geqslant N \geqslant 0$, then N, $M - N \in M(X)$.

Given a measure $\mu \in L_n(X)$ $(n = 1, 2, \ldots)$ and a system of numbers $c_1, c_2, \ldots, c_n$ in $(0, 1)$, there exist, by the definition of the set $L_n(X)$, probability measures $\mu_{c_1 c_2 \ldots c_k}$ $(k = 1, 2, \ldots, n)$ such that for any $y \in X^*$

$$\hat{\mu}_{c_1 c_2 \ldots c_{k-1}}(y) = \hat{\mu}_{c_1 c_2 \ldots c_{k-1}}(c_k y)\, \hat{\mu}_{c_1 c_2 \ldots c_k}(y), \tag{2.4}$$

where, for $k = 1$, $\mu_{c_1 c_2 \ldots c_{k-1}}$ is defined simply as μ. By virtue of Theorem 2.6, [3], it follows that for any $c_1, c_2, \ldots, c_n$ in $(0, 1)$ all measures $\mu_{c_1 c_2 \ldots c_k}$ $(k = 1, 2, \ldots, n)$ are infinitely divisible. Let M and $M_{c_1 c_2 \ldots c_k}$ $(k = 1, 2, \ldots, n)$ denote the generalized Poisson exponents corresponding to the measures μ and $\mu_{c_1 c_2 \ldots c_k}$, respectively. From (2.4) we get the formula

$$M_{c_1}(E) = M(E) - M(c_1^{-1} E)$$

for every Borel subset E of X such that $0 \notin \bar{E}$. By easy induction we have the equation

$$M_{c_1 c_2 \ldots c_n}(E) = M(E) + \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1, i_2, \ldots, i_k = 1 \\ \text{distinct}}}^{n} M(c_{i_1}^{-1} c_{i_2}^{-1} \ldots c_{i_k}^{-1} E) \tag{2.5}$$

for every Borel subset E of X such that $0 \notin \bar{E}$. Consequently, the right-hand side of (2.5) is non-negative.

Conversely, if $M \in M(X)$ and for any $c_1, c_2, \ldots, c_n$ in $(0, 1)$ the set function $M_{c_1 c_2 \ldots c_n}$ defined by formula (2.5) is a non-negative measure, then all measures $M_{c_1 c_2 \ldots c_k}$ $(k = 1, 2, \ldots, n)$ belong to $M(X)$. Consequently, the generalized Poisson measure $\tilde{e}(M)$ is n-times self-decomposable. Since every symmetric Gaussian measure ϱ on X is completely self-decomposable, we get the following

2.1. THEOREM. (i) $\mu \in L_n(X)$ $(n = 1, 2, \ldots, \infty)$ *if and only if its generalized Poisson component* $\tilde{e}(M)$ *in* (2.1) *belongs to* $L_n(X)$ *respectively.*

(ii) $\tilde{e}(M) \in L_n(X)$ $(n = 1, 2, \ldots)$ (*resp.* $\in L_\infty(X)$) *if and only if for any numbers* $c_1, c_2, \ldots, c_n$ *in* $(0, 1)$ (*resp. for any numbers* $c_1, c_2, \ldots, c_n$ *in* $(0, 1)$ *and for every* $n = 1, 2, \ldots$) *the set function* $M_{c_1 c_2 \ldots c_n}$ *defined by formula* (2.5) *is a non-negative measure on* X.

3. A reduction of the problem. In [8] Urbanik has introduced a concept of weight functions on a real separable Banach space X. Roughly speaking, a weight function on X is every real-valued continuous function Φ on X such that

(a) $\Phi(0) = 0$ and $\Phi(x) > 0$ for all $x \neq 0$,

(b) $\Phi(x)$ is convergent to a positive limit as $\|x\| \to \infty$,

(c) $\Phi(x) \leqslant a\|x\|^2$ for a certain positive constant a and for all $x \in X$,

(d) $\int_X \Phi(x) M(dx) < \infty$ for every $M \in M(X)$,

(e) if $M_n \in M(X)$, $\tilde{e}(M_n) \to \mu$ and $\int_X \Phi(x) M_n(dx) \to 0$, then $\mu = \delta_x$

for a certain $x \in X$, where δ_x denotes the unit mass at x.

In the sequel a weight function Φ on X is said to *satisfy condition* $(*)$ if for every $x \in X$ there exist positive numbers ε_x and a_x such that for every number r in $(0, \varepsilon_x)$

$$\Phi(rx) \geqslant a_x r^2. \tag{$*$}$$

It is well known that if X is a Hilbert space, then as a weight function we can take $\Phi(x) = \dfrac{\|x\|^2}{1 + \|x\|^2}$ ([5], Chapter VI, Theorem 4.10) which, of course, satisfies condition $(*)$. Moreover, from Urbanik's construction for a weight function on X ([8], Proposition 5.2) we have

3.1. PROPOSITION. *For every* X *there exists a weight function on* X *satisfying condition* $(*)$.

Now by the fact that $\int_0^\infty e^{at}\, dt < \infty$ if and only if $a < 0$ and by conditions (c) and $(*)$, one can easily prove the following proposition:

3.2. PROPOSITION. *Let Φ be a weight function on X satisfying condition (*). Then for every $x \in X$ the integral $\int_{-\infty}^{\infty} \Phi(e^{-t}x)e^{at}dt$ is finite if and only if $0 < a < 2$.*

Given a subset E of X, we put $\tau(E) = \{tx: x \in E, t > 0\}$. It is clear that the set $\tau(E)$ is invariant in the sense that $\tau(\tau(E)) = \tau(E)$.

3.3. LEMMA ([8], Lemma 5.4). *For every $M \in M(X)$ there exists a sequence $\{E_k\}$ of compact subsets of X such that $0 \notin E_k$ $(k = 1, 2, \ldots)$, $\tau(E_k) \cap \tau(E_j) = \emptyset$ if $k \neq j$ $(k, j = 1, 2, \ldots)$ and $M = \sum_{k=1}^{\infty} M_k$, where M_k is the restriction of M to $\tau(E_k)$.*

Let U be an invariant Borel set in X. Then, by virtue of Theorem 2.1, for every generalized Poisson exponent M corresponding to an n-times (resp. completely) self-decomposable probability measure on X, the restriction of M to U, denoted by $M|_U$, is also a generalized Poisson exponent corresponding to an n-times (resp. completely) self-decomposable probability measure on X. Hence and by Lemma 3.3 we get the following lemma:

3.4. LEMMA. *Let M be a generalized Poisson exponent corresponding to a n-times (resp. completely) self-decomposable probability measure on X. Then there exists a decomposition $M = \sum_{k=1}^{\infty} M_k$, where every M_k $(k = 1, 2, \ldots)$ is a generalized Poisson exponent corresponding to a n-times (resp. completely) self-decomposable probability measure on X, M_k are concentrated on disjoint sets $\tau(E_k)$, $0 \notin E_k$ and E_k are compact.*

This lemma reduces our problem of examining measures $M \in M(X)$ corresponding to n-times (resp. completely) self-decomposable probability measures on X to the case of measures concentrated on $\tau(E)$, where E is compact and $0 \notin E$. We denote this class by $G_n(E)$ $(n = 1, 2, \ldots, \infty)$. Following Urbanik [8] we shall find a suitable compactification of $\tau(E)$ and determine the extreme points of a certain convex set formed by probability measures on this compactification.

Accordingly, let $[-\infty, \infty]$ be the usual compactification of the real line $(-\infty, \infty)$, E a compact subset of X such that $0 \notin E$ and let S be the unit sphere in X. It is evident that the set $\bar{\tau}(E) := (\tau(E) \cap S) \times [-\infty, \infty]$ endowed with the product topology is compact. Further, each element of $\tau(E)$ can be represented in a unique form $e^{-t}x$, where $x \in \tau(E) \cap S$ and t is a real number. Thus the mapping $e^{-t}x \to (x, t)$ is an embedding of $\tau(E)$ into a dense subset of $\bar{\tau}(E)$. In other words, $\bar{\tau}(E)$ is a compactification of $\tau(E)$. In what follows we shall identify elements $e^{-t}x$ of $\tau(E)$, where $\|x\| = 1$, and elements (x, t) of $\bar{\tau}(E)$. Further, for every $c > 0$ and (x, t) in $\bar{\tau}(E)$ we put $c(x, t) = (x, t - \log c)$. The norm $\|\cdot\|$ can be extended from $\tau(E)$ onto $\bar{\tau}(E)$ by continuity, i.e. we put $\|(x, \infty)\| = 0$ and $\|(x, -\infty)\| = \infty$ for all $x \in \tau(E) \cap S$.

Let Φ be a weight function on X. Since for any $r_1 > r_2 > 0$ the set $\{x \in \bar{\tau}(E): r_1 \geq \|x\| \geq r_2\}$ is compact and by condition (b) it follows that Φ is bounded from below on every set $\{x \in \tau(E): \|x\| \geq r\}$ with $r > 0$. Further, Φ can be extended to $\bar{\tau}(E)$ by assuming $\Phi((x, \infty)) = 0$ and $\Phi((x, -\infty)) = \lim_{\|z\| \to \infty} \Phi(z)$. Let N be a finite Borel measure on $\bar{\tau}(E)$. Put

$$M_N(U) = \int_U \frac{N(du)}{\Phi(du)} \tag{3.1}$$

for every subset U of $\bar{\tau}(E)$ with the property $\inf\{\|u\|: u \in U\} > 0$. This formula defines a σ-finite measure M_N on $\{u \in \bar{\tau}(E): \|u\| > 0\}$.

Let $H_n(E)$ $(n = 1, 2, \ldots)$ denote the class of all finite measures N on $\bar{\tau}(E)$ for which the corresponding measures M_N fulfil the condition

$$M_N(F) + \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1, i_2, \ldots, i_k = 1 \\ \text{distinct}}}^{n} M_N(c_{i_1}^{-1} c_{i_2}^{-1} \ldots c_{i_k}^{-1} F) \geq 0 \tag{3.2}$$

for all $c_1, c_2, \ldots, c_n \in (0, 1)$, and for every subset F of the set $\{u \in \bar{\tau}E: \|u\| > 0\}$. Define $H_\infty(E) = \bigcap_{n=1}^{\infty} H_n(E)$. Let $I_n(E)$ $(n = 1, 2, \ldots, \infty)$ denote the subset of $H_n(E)$ $(n = 1, 2, \ldots, \infty)$ consisting of probability measures. It is evident that the sets $I_n(E)$ $(n = 1, 2, \ldots, \infty)$ are convex and compact. Let us consider a measure M from $G_n(E)$ $(n = 1, 2, \ldots, \infty)$ as a measure on $\bar{\tau}(E)$. Set

$$N^M(U) = \int_U \Phi(u) M(du) \tag{3.3}$$

for all Borel subset U of $\bar{\tau}(E)$. It is evident that $M \in G_n(E)$ $(n = 1, 2, \ldots, \infty)$ if and only if $N^M \in H_n(E)$ respectively. Further, for any Borel subset E_1 of $\tau(E) \cap S$ the sets $\tau(E_1)$, $\{(x, -\infty): x \in E_1\}$ and $\{(x, \infty): x \in E_1\}$ are invariant. Consequently, if $N \in H_n(E)$ $(n = 1, 2, \ldots, \infty)$ its restriction to any of these sets is again in $H_n(E)$ respectively. This implies that every extreme point of $I_n(E)$ $(n = 1, 2, \ldots, \infty)$ must be concentrated on orbits of elements of $\bar{\tau}(E)$, i.e. on one of the following sets: $\tau(\{x\})$, $\{(x, -\infty)\}$, $\{(x, \infty)\}$, where $x \in \tau(E) \cap S$. Obviously, all measures δ_z $(z \in \bar{\tau}(E) \setminus \tau(E))$ are extreme points of $I_n(E)$ $(n = 1, 2, \ldots, \infty)$. Then the problem of examining measures $M \in G_n(E)$ is reduced to finding extreme points of sets $I_n(E)$ $(n = 1, 2, \ldots, \infty)$ concentrated on $\tau(\{x\})$ $(x \in \tau(E) \cap S)$.

4. Multiply monotone functions on the real line. Let g be a left-continuous function on the real line. Then it is called *n-times monotone*

($n = 1, 2, \ldots$) if the following conditions are satisfied:

(α) $$\lim_{t\to-\infty} g(x) = 0$$

and

(β) for any $t_1, t_2, \ldots, t_n > 0$ and $a, b \in (-\infty, \infty)$ with $a < b$ we have the inequality

$$\Delta_{t_1,t_2,\ldots,t_n} g(b) \geqslant \Delta_{t_1,t_2,\ldots,t_n} g(a),$$

where $\Delta_{t_1,t_2,\ldots,t_n}$ is the difference operator defined inductively as follows:

$$\Delta_{t_1,t_2,\ldots,t_k} g(s) = \begin{cases} \Delta_{t_1,t_2,\ldots,t_{k-1}} \big(g(s)-g(s-t_k)\big), & k \geqslant 2, \\ g(s)-g(s-t_1), & k = 1 \end{cases}$$

($t_1, t_2, \ldots, t_k > 0$ and $-\infty < s < \infty$).

Further, if for every $n = 1, 2, \ldots$ a function g is n-times monotone, then it is called *completely monotone*.

It is evident that every n-times ($n = 1, 2, \ldots$) monotone function is convex, non-negative and monotone non-decreasing. Hence and by an easy induction one can prove the following propositions:

4.1. Proposition. *Let g be a n-times monotone function ($n = 1, 2, \ldots$) on the real line. Then there is a unique non-negative left-continuous monotone non-decreasing function q such that for every $t \in (-\infty, \infty)$*

$$g(t) = \int_{-\infty}^{t} \int_{-\infty}^{u_{n-1}} \int_{-\infty}^{u_{n-2}} \cdots \int_{-\infty}^{u_1} q(u)\,du\,du_1\,du_2 \ldots du_{n-1}.$$

4.2. Proposition. *Let g be a completely monotone function on the real line. Then there is a unique completely monotone function q such that for every $t \in (-\infty, \infty)$*

$$g(t) = \int_{-\infty}^{t} q(u)du.$$

5. The Urbanik representation for n-times self-decomposable probability measures on X. Consider a compact subset E of X such that $0 \notin E$ and an arbitrary probability measure N concentrated on $\tau(\{x\})$, where x is a fixed point of $\tau(E) \cap S$. Let us fix $n = 1, 2, \ldots$ It is clear that $N \in I_n(E)$ if and only if

(5.1) $$M_N(U) + \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,i_2,\ldots,i_k=1 \\ \text{distinct}}}^{n} M_N(c_{i_1}^{-1} c_{i_2}^{-1} \ldots c_{i_k}^{-1} U) \geqslant 0$$

for all $c_1, c_2, \ldots, c_n \in (0, 1)$ and all sets U of the form $U = \{(x, t)\colon a \leqslant t < b\}$ ($-\infty < a < b < \infty$). Setting, for $N \in I_n(E)$,

(5.2) $$g_N(b) = M_N\big(\{(x, t)\colon t < b\}\big)$$

we infer that for any $t_1, t_2, \ldots, t_n > 0$

(5.3) $$\Delta_{t_1,t_2,\ldots,t_n} g_N(b) = M_N(V) + \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,i_2,\ldots,i_k=1 \\ \text{distinct}}}^{n} M_N(c_{i_1}^{-1} c_{i_2}^{-1} \ldots c_{i_k}^{-1} V),$$

where $c_k = e^{-t_k}$ ($k = 1, 2, \ldots, n$) and V is a set of the form $V = \{(x, t)\colon t < b\}$. Hence and by (5.1) for any sets U of the form $U = \{(x, t)\colon a \leqslant t < b\}$ and for $c_k = e^{-t_k}$ ($k = 1, 2, \ldots, n$) we have

$$\Delta_{t_1,t_2,\ldots,t_n} g_N(b) - \Delta_{t_1,t_2,\ldots,t_n} g_N(a) = M_N(U) + \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,i_2,\ldots,i_k=1 \\ \text{distinct}}}^{n} M_N(c_{i_1}^{-1} c_{i_2}^{-1} \ldots c_{i_k}^{-1} U) \geqslant 0$$

which implies that the function g_N is n-times monotone. By Proposition 4.1 there exists a unique non-negative left-continuous monotone non-decreasing function q_N such that for every $t \in (-\infty, \infty)$

(5.4) $$g_N(t) = \int_{-\infty}^{t} \int_{-\infty}^{u_{n-1}} \int_{-\infty}^{u_{n-2}} \cdots \int_{-\infty}^{u_1} q_N(u)\,du\,du_1\,du_2 \ldots du_{n-1}.$$

Moreover, by (3.3), (5.2) and (5.4), we get the formula

(5.5) $$N\big(\{(x, t)\colon a \leqslant t < b\}\big) = \int_a^b \Phi\big((x, t)\big) g_N^*(t)\,dt,$$

where the function g_N^* is defined by

(5.6) $$g_N^*(t) = \int_{-\infty}^{t} \int_{-\infty}^{u_{n-2}} \cdots \int_{-\infty}^{u_1} q_N(u)\,du\,du_1 \ldots du_{n-2}.$$

Consequently, we have

(5.7) $$\int_{-\infty}^{\infty} \Phi\big((x, t)\big) g_N^*(t)\,dt = 1.$$

Conversely, every non-negative monotone non-decreasing left-continuous function q_N with property (5.7) determines, by formulas (5.5) and (5.6), a probability measure N concentrated on $\tau(\{x\})$. Moreover, the corresponding function g_N is n-times monotone which shows that $N \in I_n(E)$. Hence we conclude that a measure $N \in I_n(E)$ is an extreme point of $I_n(E)$ if and only if the corresponding function q_N cannot be decomposed in to a non-trivial convex combination of two functions q_{N_1} and q_{N_2} ($N_1, N_2 \in I_n(E)$). But this is possible only in the case $q_N(t) = 0$ if $t \leqslant t_0$ and $q_N(t) = c$ if $t > t_0$ for some constants t_0 and c. By (5.5) and (5.6) we get the formula

$$N(\{(x, t):\ a \leqslant t < b\}) = c \int_a^b \Phi((x, t))(t-t_0)_+^{n-1} dt, \tag{5.8}$$

where for a real number λ we write $\lambda_+ = \max(\lambda, 0)$. The constant c is determined by (5.7) and (5.8). Namely,

$$c^{-1} = \int_{-\infty}^{\infty} \Phi((x, t))(t-t_0)_+^{n-1} dt. \tag{5.9}$$

We note that by condition (c) the last integral is finite for every $n = 1, 2, \ldots$ Thus we have proved that the extreme points of $I_n(E)$ concentrated on sets $\tau(\{x\})$ are of the form (5.8).

Conversely, let N be a probability measure on $\bar{\tau}(E)$ defined by formula (5.8), where $x \in \tau(E) \cap S$ and the constant c is given by (5.9). Then the corresponding measure M_N is of the form

$$M_N(\{(x, t):\ a \leqslant t < b\}) = c \int_a^b (t-t_0)_+^{n-1} dt$$

and consequently, the measure M_N satisfies condition (5.1). This implies that every measure N defined by formula (5.8) is in $I_n(E)$ and then it is an extreme point of $I_n(E)$

Let z be an arbitrary element of $\tau(E)$. Substituting $x = \frac{z}{\|z\|}$ and $t_0 = \log \frac{1}{\|z\|}$ into formulas (5.8) and (5.9) we get all extreme points $N_z^{(n)}$ of $I_n(E)$ concentrated on $\tau(E)$ as follows:

$$N_z^{(n)}(U) = c_n(z) \int_0^{\infty} 1_U(e^{-t}z)\Phi(e^{-t}z)t^{n-1} dt, \tag{5.10}$$

where 1_U denotes the indicator of a subset U of $\bar{\tau}(E)$ and

$$c_n^{-1}(z) = \int_0^{\infty} \Phi(e^{-t}z)t^{n-1} dt. \tag{5.11}$$

Now putting $N_z^{(n)} = \delta_z$ for $z \in \bar{\tau}(E) \setminus \tau(E)$ we get the following lemma:

5.1. Lemma. *The set $\{N_z^{(n)}:\ z \in \bar{\tau}(E)\}$ is identical with the set of all extreme points of $I_n(E)$ and the mapping $z \to N_z^{(n)}$ is a homeomorphism between $\bar{\tau}(E)$ and the set of extreme points of $I_n(E)$.*

Once the extreme points of $I_n(E)$ are found we can apply a well-known Krein–Milman–Choquet Theorem ([6], Chapter 3). Since each element of $H_n(E)$ is of the form cN_1, where $N_1 \in I_n(E)$ and $c > 0$, we then get the following proposition:

5.2. Proposition. *A measure N belongs to $H_n(E)$ if and only if there exists a finite Borel measure m on $\bar{\tau}(E)$ such that*

$$\int_{\bar{\tau}(E)} f(x) N(dx) = \int_{\bar{\tau}(E)} \int_{\bar{\tau}(E)} f(u) N_z(du) m(dz)$$

for every continuous function f on $\bar{\tau}(E)$. If N is concentrated on $\tau(E)$, then m does the same.

From this proposition and by (3.1) and (5.10) we get, after some computation, the following corollary:

5.3. Corollary. *Let M be a measure from $M(X)$ concentrated on $\tau(E)$. Then $M \in G_n(E)$ ($n = 1, 2, \ldots$), if and only if there exists a finite measure m on $\tau(E)$ such that*

$$\int_{\tau(E)} f(x) M(dx) = \int_{\tau(E)} c_n(z) \int_0^{\infty} f(e^{-t}z) t^{n-1} dt\, m(dz)$$

for every M-integrable function f on $\tau(E)$. The function $c_n(z)$ is given by formula (5.11).

We now turn to the consideration of arbitrary measures $M \in M(X)$ corresponding to a n-times self-decomposable probability measure μ on X. By Lemma 3.4 there exists a decomposition $M = \sum_{k=1}^{\infty} M_k$, where $M_k \in M(X)$ are restrictions of M to disjoint sets $\tau(E_k)$, $0 \notin E_k$ and E_k are compact. Then we have $M_k \in G_n(E_k)$ ($k = 1, 2, \ldots$). Let m_k denote a finite measure on $\tau(E_k)$ corresponding to M_k in the representation given by Corollary 5.3. Then

$$\int_X f(x) M(dx) = \sum_{k=1}^{\infty} \int_{\tau(E_k)} c_n(z) \int_0^{\infty} f(e^{-t}z) t^{n-1} dt\, m_k(dz)$$

for every M-integrable function f. Substituting $f = \Phi$ into this formula, we get the equation

$$\int_X \Phi(x) M(dx) = \sum_{k=1}^{\infty} m_k(\tau(E_k)).$$

Consequently, setting $m = \sum_{k=1}^{\infty} m_k$, we get a finite measure on X satisfying the equation

$$(5.12)\qquad \int_X f(x)M(dx) = \int_X c_n(z)\int_0^\infty f(e^{-t}z)t^{n-1}dt\, m(dz)$$

for every M-integrable function f on X. Moreover, $m(\{0\}) = 0$.

Putting,

$$(5.13)\qquad K_n(x, y) = c_n(x)\int_0^\infty K(e^{-t}x, y)t^{n-1}dt,$$

where the function c_n is given by (5.11) and the kernel K is given by (2.3), we get the formula

$$\int_X K(x, y)M(dx) = \int_X K_n(x, y)m(dx)$$

which, by (2.2) and (2.3), yields the following theorem:

5.4. THEOREM. *Let Φ be an arbitrary weight function on X. A probability measure μ on X is n-times self-decomposable ($n = 1, 2, \ldots$) if and only if there exist a finite measure m on X vanishing at 0 and a covariance operator R and an element $x_0 \in X$ such that*

$$(5.14)\qquad \hat{\mu}(y) = \exp\Big(i\langle y, x_0\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \int_X K_n(x, y)m(dx)\Big)$$

for all $y \in X^*$. The kernel K_n is defined by formula (5.13).

6. The Urbanik representation for completely self-decomposable probability measures on X. Consider a compact subset E of X such that $0 \notin E$ and an arbitrary probability measure N concentrated on $\tau(\{x\})$, where x is a fixed point of $\tau(E)\cap S$. It is clear that $N \in I_\infty(E)$ if and only if for every $n = 1, 2, \ldots$ equation (5.1) holds. Hence and by (5.3) the function g_N defined by means of formula (5.2) is completely monotone. By Proposition 4.2 it follows that there exists a unique completely monotone function p_N on the real line such that

$$(6.1)\qquad g_N(t) = \int_{-\infty}^{t} p_N(u)\,du \qquad (-\infty < t < \infty).$$

which together with (3.3) and (5.2) implies the formula

$$(6.2)\qquad N\big(\{(x, t)\colon a \leqslant t < b\}\big) = \int_a^b \Phi\big((x, t)\big)p_N(t)\,dt.$$

Consequently, we have

$$(6.3)\qquad \int_{-\infty}^{\infty} \Phi\big((x, t)\big)p_N(t)\,dt = 1.$$

Conversely, every completely monotone function p_N on the real line with property (6.3) determines, according to formula (6.2), a probability measure N concentrated on $\tau(\{x\})$. It is evident that $N \in I_\infty(E)$. Hence we conclude that a measure $N \in I_\infty(E)$ concentrated on $\tau(\{x\})$ is an extreme point of $I_\infty(E)$ if and only if the corresponding function p_N cannot be decomposed into a non-trivial convex combination of two functions p_{N_1} and p_{N_2} ($N_1, N_2 \in I_n(E)$). Given $t > 0$ and a function p with such a property define two auxiliary functions p_1 and p_2 as follows:

$$p_1(u) = \frac{p(u)+p(u-t)}{1+c}$$

and

$$p_2(u) = \frac{p(u)-p(u-t)}{1-c} \qquad (-\infty < u < \infty),$$

where $c = \int_{-\infty}^{\infty} \Phi\big((x, t)\big)p(u-t)\,dt$. It is evident that for sufficiently large t we have $0 < c < 1$ and then the functions p_1 and p_2 are both completely monotone. Moreover, they are normalized by condition (6.3). Now, for every $u \in (-\infty, \infty)$ we have the equation

$$p(u) = \tfrac{1}{2}(1+c)p_1(u) + \tfrac{1}{2}(1-c)p_2(u).$$

Consequently, for all $u \in (-\infty, \infty)$ and sufficiently large $t > 0$

$$p(u-t) = p(u)\int_{-\infty}^{\infty} \Phi\big((x, t)\big)p(u-t)\,dt$$

which, by a simple reason, implies that the function p is of the form

$$p(u) = ae^{su} \qquad (a, s > 0;\ -\infty < u < \infty).$$

Suppose that Φ is a weight function on X satisfying condition (*). Then, by Proposition 3.2, the integral

$$\int_{-\infty}^{\infty} \Phi\big((x, t)\big)p(t)\,dt = a\int_{-\infty}^{\infty} \Phi(e^{-t}x)e^{st}\,dt$$

is finite if and only if $0 < s < 2$. In this case, by (6.3), the constant a is given by

$$(6.4)\qquad a^{-1} = \int_{-\infty}^{\infty} \Phi(e^{-t}x)e^{st}\,dt.$$

Putting, for $z \in \tau(E)$ with $0 < \|z\| < 1$,

$$(6.5)\qquad N_Z(U) = c(z)\int_{-\infty}^{\infty} 1_U\Big(\frac{z}{\|z\|}e^{-t}\Big)\Phi\Big(\frac{z}{\|z\|}e^{-t}\Big)e^{2\|z\|t}\,dt,$$

where I_U denotes the indicator of the subset U of $\bar{\tau}(E)$ and

$$c(z)^{-1} = \int_{-\infty}^{\infty} \Phi\left(\frac{z}{\|z\|}e^{-t}\right) e^{2\|z\|t}dt \tag{6.6}$$

we infer, by above arguments, that the set $\{N_z:\ z \in \tau(E) \text{ and } 0 < \|z\| < 1\}$ contains all extreme points of the set $I_\infty(E)$ which are concentrated on $\tau(E)$. Our further aim is to prove that every measure N_z ($z \in \tau(E)$ and $0 < \|z\| < 1$) defined by formula (6.5) is an extreme point of the set $I_\infty(E)$.

Accordingly, by (6.5) it follows that the measure N_z is concentrated on $\tau(\{z\})$. Further, putting $x = \frac{z}{\|z\|}$ we get the formula for the corresponding measure M_{N_z} of N_z:

$$M_{N_z}(\{(x, t):\ a \leqslant t < b\}) = c(z)\int_a^b e^{2\|z\|t}dt \qquad (-\infty < a < b < \infty).$$

Consequently, $M_{N_z} \in G_\infty(E)$ and then the measure N_z defined by formula (6.5) is an extreme point of the set $I_\infty(E)$. Thus we have proved the following lemma:

6.1. Lemma. *The set $\{N_z:\ z \in \tau(E)$ and $\|z\| < 1\}$ is identical with the set of extreme points of the set $I_\infty(E)$ concentrated on $\tau(E)$ and the mapping $z \to N_z$ is a homeomorphism between $\{z \in \tau(E):\ \|z\| < 1\}$ and the set of extreme points of $I_\infty(E)$ concentrated on $\tau(E)$.*

Denoting by $e(I_\infty(E))$ the set of all extreme points of $I_\infty(E)$ and taking into account the fact that each element of $H_\infty(E)$ is of the form cN_1, where $N_1 \in I_\infty(E)$, we then get the following proposition:

6.2. Proposition. *A measure N belongs to $H_\infty(E)$ if and only if there exists a finite Borel measure m on $e(I_\infty(E))$ such that*

$$\int_{\bar{\tau}(E)} f(x)N(dx) = \int_{e(I_\infty(E))}\Big(\int_{\bar{\tau}(E)} f(u)\tau(du)\Big)m(d\tau)$$

for every continuous function f on $\bar{\tau}(E)$. If N is concentrated on $\tau(E)$, then m is concentrated on the subset of $e(I_\infty(E))$ consisting of probability measures concentrated on the set $\tau(E)$.

Combining (3.1), (6.5), Lemma 6.1 and the last proposition we get the following corollary:

6.3. Corollary. *Let M be a measure from $M(X)$ concentrated on $\tau(E)$ and Φ be a weight function on X satisfying condition $(*)$. Then $M \in G_\infty(E)$ if and only if there exists a finite measure m on the set $\{z \in \tau(E):\ \|z\| < 1\}$ such that*

$$\int_{\tau(E)} f(x)M(dx) = \int_{\{z\in\tau(E):\,\|z\|<1\}} c(z) \int_{-\infty}^{\infty} f\left(\frac{z}{\|z\|}e^{-t}\right) e^{2\|z\|t}dt\, m(dz)$$

for every M-integrable function f on $\tau(E)$. The function $c(z)$ is given by formula (6.6).

Consider an arbitrary measure $M \in M(X)$ corresponding to a completely self-decomposable probability measure on X. By Lemma 3.4 there exists a decomposition $M = \sum_{k=1}^{\infty} M_k$, where $M_k \in M(X)$ are restrictions of M to disjoint sets $\tau(E_k)$, $0 \notin E_k$ and E_k are compact. Then we have $M_k \in G_\infty(E_k)$ ($k = 1, 2, \ldots$). Let m_k denote a finite measure on $\{z \in \tau(E_k):\ \|z\| < 1\}$ corresponding to M_k in the representation given by Corollary 6.3. Then, for every M-integrable function f

$$\int_X f(x)M(dx) = \sum_{k=1}^{\infty} \int_{\tau(E_k)\cap B} c(z) \int_{-\infty}^{\infty} f\left(\frac{z}{\|z\|}e^{-t}\right) e^{2\|z\|t}dt\, m_k(dz),$$

where B denotes the open unit ball in X. Substituting $f = \Phi$ into the last formula, we get the equation

$$\int_X f(x)M(dx) = \sum_{k=1}^{\infty} m_k(\tau(E_k)\cap B).$$

Consequently, setting $m = \sum_{k=1}^{\infty} m_k$ we get a finite measure on B satisfying the equation

$$\int_X f(x)M(dx) = \int_B c(z) \int_{-\infty}^{\infty} f\left(\frac{z}{\|z\|}e^{-t}\right) e^{2\|z\|t}dt\, m(dz) \tag{6.7}$$

which, by (6.6), can be written in the form

$$\int_X f(x)M(dx) = \int_B \int_0^\infty f(sx, y)\frac{ds}{s^{2\|x\|+1}}\left[\int_0^\infty \Phi(tx)\frac{dt}{t^{2\|x\|+1}}\right]^{-1} m(dx). \tag{6.8}$$

Moreover, the measure m fulfils the condition $m(\{0\}) = 0$. Hence and by (2.2) and (2.3) we get the following theorem:

6.4. Theorem. *Let Φ be a weight function on X satisfying condition $(*)$. A probability measure μ on X is completely self-decomposable if and only if there exist a finite measure m on the open unit ball B in X vanishing at 0 and a covariance operator R and on element $x_0 \in X$ such that*

$$\hat{\mu}(y) = \exp\Big\{i\langle y, x_0\rangle - \tfrac{1}{2}\langle y, Ry\rangle + \int_B \int_0^\infty K(sx, y)\frac{ds}{s^{2\|x\|+1}}\Big(\int_0^\infty \Phi(tx)\frac{dt}{t^{2\|x\|+1}}\Big)^{-1} m(dx)\Big\} \tag{6.9}$$

for all $y \in X^*$. The kernel K is given by (2.3).

7. The Hilbert space case. Let H be a real separable Hilbert space. Recall that a complex-valued function φ on H is an infinitely divisible characteristic functional if and only if it can be represented in the form

$$\varphi(y) = \exp\big(i\langle y, x_0\rangle - \tfrac{1}{2}\langle Dy, y\rangle + \int_H K(x, y)M(dx)\big) \tag{7.1}$$

($y \in H$), where $x_0 \in H$, $K(x, y) = e^{i\langle y,x\rangle} - 1 - \frac{i\langle y, x\rangle}{1+\|x\|^2}$, D is an S-operator, M is a generalized Poisson exponent. Taking $\Phi(x) = \frac{\|x\|^2}{1+\|x\|^2}$ as a weight function on H satisfying condition (*) we get, by Theorems 5.4 and 6.4 and after some computation, the following theorems:

7.1. THEOREM. *The class of all n-times self-decomposable ($n = 1, 2, \ldots$) probability measures on H coincides with the class of probability measures μ on H for which characteristic functionals are of the form*

$$\hat{\mu}(y) = \exp\big(i\langle y, x_0\rangle - \tfrac{1}{2}\langle Dy, y\rangle + \int_H Q_n(x, y)w(dx)\big)$$

($y \in H$), where x_0 and D are the same as in (7.1), w is a finite measure on H vanishing at 0 and

$$Q_n(x, y) = \Big[\int_0^{\|x\|}\Big(\log\frac{\|x\|}{t}\Big)^{n-1}\frac{t\,dt}{1+t^2}\Big]^{-1}\int_0^\infty K(e^{-s}x, y)s^{n-1}\,ds.$$

7.2. THEOREM. *The class of all completely self-decomposable probability measures on H coincides with the class of probability measure μ on H for which characteristic functionals are of the form*

$$\hat{\mu}(y) = \exp\Big\{i\langle y, x_0\rangle - \tfrac{1}{2}\langle Dy, y\rangle + \int_B\Big(\int_0^\infty K(sx, y)\frac{ds}{s^{2\|x\|+1}}\Big)\frac{\sin\pi\|x\|}{\|x\|^{2\|x\|}}m(dx)\Big\}$$

($y \in H$), where K, x_0 and D are the same as in (7.1), m is a finite measure on the open unit ball B in H vanishing at 0.

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