[14] I. Labuda, *Sur le théorème de Bartle–Dunford–Schwartz*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), pp. 545–553.
[15] — *Exhaustive measures in arbitrary topological vector spaces*, Studia Math. 58 (1976), pp. 239–248.
[16] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. 33 (1970), pp. 157–165.
[17] — *On integrability and summability in vector spaces*, Illinois J. Math. 16 (1972), pp. 294–307.
[18] W. Matuszewska and W. Orlicz, *A note on modular spaces, IX*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), pp. 801–808.
[19] B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. 44 (1938), pp. 277–304.
[20] R. S. Philips, *Integration in a convex linear topological space*, ibid. 47 (1940), pp. 114–145.
[21] C. E. Rickart, *Integration in a convex linear topological space*, ibid. 52 (1942), pp. 498–521.
[22] S. Rolewicz, *Metric linear spaces*, Monografie Matematyczne, tom 56, PWN, Warszawa 1972.
[23] S. Rolewicz and Cz. Ryll-Nardzewski, *On unconditional convergence in linear metric spaces*, Colloq. Math. 17 (1967), pp. 327–331.
[24] H. Schaefer, *Topological vector spaces*, MacMillan, New-York 1966.
[25] A. H. Schuchat, *Integral representation theorems in topological vector spaces*, Trans. Amer. Math. Soc. 172 (1972), pp. 373–396.
[26] M. Sion, *A theory of semi-group-valued measures*, Lecture Notes in Mathematics 355, Springer-Verlag, Berlin–Heidelberg–New-York 1973.
[27] E. Thomas, *L'intégration par rapport à une mesure de Radon vectorielle*, Ann. Inst. Fourier 20 (1970), pp. 55–191.
[28] — *On Radon maps with values in arbitrary topological vector spaces, and their integral extensions*, preprint.
[29] P. Turpin, *Convexité dans les espaces vectoriels topologiques généraux*, Thèse, Orsay 1974.
[30] — *Intégration par rapport à une mesure à valeurs dans un espace vectoriel topologique non supposé localment convexe*, Colloque: Intégration vectorielle et multivoque, Caen 1975.
[31] L. Waelbroeck, *Topological vector spaces and algebras*, Lecture Notes in Mathematics 230, Springer-Verlag, Berlin–Heidelberg–New-York 1971.
[32] K. Yosida, *Functional analysis*, 2nd ed., Springer-Verlag, New-York 1968.

Received June 26, 1976
Revised version May 14, 1977 (1172)

STUDIA MATHEMATICA, T. LXVI. (1979)

A characterization of some weak semi-continuity of integral functionals

by

TRAN CAO NGUYEN (Warsaw)

Abstract. In this paper we give a characterization of lower semicontinuity of the integral functional $I_f(u) = \int_T f(t, u(t))\mu(dt)$ on the space L_1 of integrable functions from T into a Euclidean space. The semicontinuity is considered with respect to the weak topology $w(L_1, S)$, where S is a subspace of L_∞.

1. Introduction. Consider the integral functional of the form

$$I_f(u) = \int_T f(t, u(t))\mu(dt), \qquad u \in L_1(T, \boldsymbol{R}^n),$$

where T is an abstract space, and μ is a fixed, finite, nonnegative, nonatomic and complete measure on T. We denote by $\mathscr{L}$ the σ-field of μ-measurable subsets of T. We assume that the space $L_1(T, \boldsymbol{R}^n)$ is separable.

Concerning the integrand we assume as little regularity as possible: namely, throughout the paper we assume the following

ASSUMPTION A. *$f\colon T\times \boldsymbol{R}^n \to \boldsymbol{R}\cup\{+\infty\}$ is such that for each integrable u, $f(t, u(t))$ is measurable and there is a u_0 such that $f(t, u_0(t))$ is integrable and the integral is finite.*

Notice that Assumption A does not imply that the domain of I_f is the whole space $L_1(T, \boldsymbol{R}^n)$. A priori, it may happen that both the negative part and the positive part of $f(t, u(t))$ have the integral divergent to infinity and $I_f(u)$ cannot be uniquely defined. If only one of them is divergent, then $I_f(u) = -\infty$ or $+\infty$.

In this paper we shall characterize the property that the epigraph of I_f, that is, the set

$$\operatorname{epi} I_f = \{(u, a) \mid a \geqslant I_f(u)\} \subset L_1 \times \boldsymbol{R},$$

is closed if we consider in L_1 the weak topology $w(L_1, S)$, S being a subspace of L_∞. We say in that case that I_f is $w(L_1, S)$-*lower semicontinuous* ($w(L_1, S)$-l.s.c).

In fact, if the domain of I_f is the whole L_1 or if we put $I_f(u) = +\infty$ whenever the integral cannot be defined uniquely, then the closedness of the epigraph is equivalent to the l.s.c. of $I_f(u)$ for each $u \in L_1$; that is, for each u we have $\liminf_{v \to u} I_f(v) \geqslant I_f(u)$.

If $S = L_\infty$, then $w(L_1, S)$ is the L_1-weak topology, and in this case the characterization of the l.s.c. of I_f is comparatively simple (see [9]). In the case where T is a bounded domain of $\boldsymbol{R}^k$ and μ is the Lebesgue measure Olech [6] gave such a characterization also when S is equal to C (the space of continuous functions) or C^∞. In [7] he proved it for the case $S = C$, using his characterization of the w^*-closure of subsets of L_1 (see [8]).

In this paper we present an extension of Olech's result to a more abstract setting. This allowed us to simplify the proof. In particular, in Theorem A we did not need to use any analogue of the above-mentioned characterization of w^*-closure. Clearly, we need to specify the subspace S. In fact, our first result applies to the case where S is approximatively decomposable. The definition of this property and some other auxiliary facts are given in the next section. In Section 3 we state the main result while the proof of it is given in Section 4. Section 5 contains an alteration of a similar result but under different assumptions concerning S. The approximative decomposability assumption is replaced in Section 5 by the existence of a certain subspace $S^I \subset L_\infty(T, \boldsymbol{R})$ of "multipliers" of S.

2. Approximatively decomposable subspace. A space S or a set of integrable functions is called *decomposable* if and only if for each measurable $A \subset T$ and any s_1, s_2 from S the function $\chi_A s_1 + \chi_{T \setminus A} s_2$ belongs to S (see [10], [11]).

In analogy to this we shall call S *approximatively decomposable* if for each measurable $A \subset T$, any $s_1, s_2 \in S$ and each $\delta > 0$ there is an $s_\delta \in S$ such that $s_\delta(t) = \lambda(t)s_1(t) + (1-\lambda(t))s_2(t)$, $0 \leqslant \lambda(t) \leqslant 1$ is measurable and such that

$$\|s_\delta - (\chi_A s_1 + \chi_{T \setminus A} s_2)\|_{L_1} \leqslant \delta.$$

Clearly, if S has this property, $s_1, \ldots, s_k \in S$ and $T_1, \ldots, T_k$ form a disjoined measurable decomposition of T, then for each $\delta > 0$ there is an $s_\delta \in S$ such that

$$s_\delta(t) \in \operatorname{co}\{s_i(t)\} \quad \text{for each } t$$

and

$$\Big\|s_\delta - \sum \chi_{T_i} s_i\Big\|_{L_1} \leqslant \delta.$$

We notice that if T is a compact topological Hausdorff space, then the space of continuous functions is an approximatively decomposable subspace of L_∞. Similarly, if T is a compact smooth manifold, then the space $C^{(k)}$ or C^∞ is also approximatively decomposable.

One more example. The subspace of simple functions, that is, functions assuming a finite number of values, is decomposable while the space of piece-wise constant functions is approximatively decomposable.

The decomposability of the space was used in [10], [11], [2] to prove the representation $I_f^* = I_{f^*}$ or $I_{f^*}^* = I_{f^{**}}$. In Theorem A we proved $I_{f^*}^\# = I_{\tilde f}$ by using approximative decomposability. We believe that the notion of approximative decomposability may be useful also in other situations and that our result is closely related to those in [10], [11], [2].

3. First characterization.

THEOREM A. *Suppose that f satisfies Assumption A and assume S is an approximatively decomposable linear subspace of $L_\infty(T, \boldsymbol{R}^n)$. Then the following three conditions are equivalent:*

(i) *I_f is $w(L_1, S)$-l.s.c.*

(ii) *The epigraph of I_f is closed and convex.*

(iii) *There is a function $\tilde f$: $T \times \boldsymbol{R}^n \to \boldsymbol{R} \cup \{+\infty\}$ defined by the formula*

$$\tilde f(t, u) = \sup\big(-\Psi_i(t) + \langle s_i(t), u\rangle\big), \tag{1}$$

where the denumerable $\{s_i\}$ belong to S and Ψ_i are integrable on T, so that, for each $u \in L_1(T, \boldsymbol{R}^n)$,

$$f\big(t, u(t)\big) = \tilde f\big(t, u(t)\big) \ \mu\text{-a.e. in } T. \tag{2}$$

The proof of this theorem will be given in the next section. Before that we shall make some remarks and comments.

Condition (ii) differs from (i) only in convexity. We notice also that (ii) and Assumption A $\big(I_f(u_0)$ finite$\big)$ imply that there is an $s_0 \in S$ such that

$$I_f(u) \geqslant a + \langle s_0, u\rangle. \tag{3}$$

Indeed, denote by D the epigraph of I_f. Then from (ii) it follows that

$$D = \bigcap_{\substack{a \in \boldsymbol{R} \\ s \in S}} \{(u, a) \mid\ aa + \langle s, u\rangle \leqslant c(a, s)\},$$

where

$$c(a, s) = \sup_{(u,a) \in D} (aa + \langle s, u\rangle) \tag{4}$$

is a support function. Since D is an epigraph, then $c(a, s) = +\infty$ if $a > 0$. Inequality (3) means that $c(-1, s_0) < \infty$. If $c(a, s) = +\infty$ for each $a < 0$, then D would be of the form $A \times \boldsymbol{R}$, where A is $w(L_1, S)$-closed convex; hence $I_f(u)$ would be equal $-\infty$ on A and $+\infty$ elsewhere, contrary to the assumption that $I_f(u_0)$ is finite.

It is a general fact that a convex and lower semicontinuous function on a linear topological space either is never equal to $-\infty$ or is equal to $-\infty$ on a convex and closed set and to $+\infty$ elsewhere.

Another general property of convex sets which will be useful for us is the following statement. If (ii) holds, then

$$D = \bigcap_{s \in S} \{(u, a) \mid\ -a + \langle s, u\rangle \leqslant c(-1, s)\}. \tag{5}$$

Indeed, if $(\bar{u}, \bar{a}) \notin D$, then there are α_1, s_1 such that $\alpha_1\bar{a} + \langle s_1, \bar{u}\rangle > c(\alpha_1, s_1)$. Since the latter has to be finite, we have $\alpha_1 \leqslant 0$. If $\alpha_1 < 0$, then we can put $\alpha_1 = -1$, because c is homogeneous, and we conclude that $(\bar{u}, \bar{a})$ does not belong to the right-hand side of (5). If $\alpha_1 = 0$, then there is a $\delta > 0$ such that

$$c\big(-\delta,\ \delta s_0 + (1-\delta)s_1\big) < -\delta\bar{a} + \langle \delta s_0 + (1-\delta)s_1, \bar{u}\rangle,$$

which follows from the continuity of both sides with respect to δ. Thus again $(\bar{u}, \bar{a})$ does not belong to the right-hand side of (5). Hence (5) is proved.

The function $\tilde{f}$ in condition (iii) is $\mathscr{L}\times\mathscr{B}$-measurable, convex and lower semicontinuous in the second variable for any fixed t. This follows clearly from (1). It is interesting to note that f may not have any of these properties and still (2) holds.

However, condition (iii) means more. One can prove that the $\mathscr{L}\times\mathscr{B}$-measurability condition of $\tilde{f}(t, u)$, convexity and l.s.c. in u imply (1), but in general we cannot say more about s_i than that they are measurable and bounded.

To get a better understanding of condition (iii) we will give an equivalent version of it. For this purpose consider the function $\tilde{f}^*(t, \cdot)$ conjugate to $\tilde{f}$; that is,

(6) $$\tilde{f}^*(t, p) = \sup_{u\in \mathbf{R}^n}\big(-\tilde{f}(t, u) + \langle u, p\rangle\big).$$

It is well known that $\tilde{f}^*(t, \cdot)$ is convex and l.s.c. Moreover, we have $\tilde{f}^{**} = \tilde{f}$.

Notice that from (1) and (6) it follows that

$$\tilde{f}^*\big(t, s_i(t)\big) \leqslant \Psi_i(t)$$

and (2) implies

$$\tilde{f}^*\big(t, s_i(t)\big) = \operatorname*{ess\,sup}_{u\in L_1}\Big(-f\big(t, u(t)\big) + \langle u(t), s_i(t)\rangle\Big),$$

where ess sup stands for the essential supremum. The general formula

(7) $$\tilde{f}^*\big(t, s(t)\big) = \operatorname*{ess\,sup}_{u\in L_1}\Big(-f\big(t, u(t)\big) + \langle u(t), s(t)\rangle\Big)$$

for each $s \in S$ such that the right-hand side of (7) is integrable can be proved by using measurable selection theorems (see for example [1], [5]). Denote the right-hand side of (7) by $\Psi_s(t)$.

Let P be the set of all $s \in L_\infty$ such that $\Psi_s(t)$ is integrable, and denote by $Q(t)$ the domain of $\tilde{f}^*(t, \cdot)$; that is,

$$Q(t) = \{p \mid \tilde{f}^*(t, p) < +\infty\}.$$

Each $s \in P$ is a selector of Q, that is, $s(t) \in Q(t)$ μ-a.e. in T. The opposite

does not hold in general, but because of (1) if $s \in L_\infty$ is a selector of Q, then s belongs to the L_1-closure of P.

The following two conditions are equivalent versions of (iii):

(iii′) *There is an* $\tilde{f}(t, u)$ $\mathscr{L}\times\mathscr{B}$*-measurable, convex and* l.s.c *in* u *for each fixed* t *such that* (2) *holds and there is a sequence* $\{s_i\} \subset S$ *such that* $\operatorname{cl}\operatorname{co}\{s_i(t)\} = \operatorname{cl}Q(t)$ μ-a.e. *in* T *and* $P \neq \emptyset$.

(iii″) *There is a denumerable* L_1*-dense subset of* P *contained in* S *and*

$$f\big(t, u(t)\big) = \operatorname*{ess\,sup}_{s\in S}\big(-\Psi_s(t) + \langle u(t), s(t)\rangle\big) \ \mu\text{-a.e. } in\ T.$$

In particular, if S is the subspace of continuous functions, then (iii′) means that $\operatorname{cl}Q(t)$ is equal to a lower semicontinuous set-valued function μ-a.e. in T.

The implication (iii) $\Rightarrow$ (iii′) follows immediately from (1), (7). Let (iii′) hold. Without loss of generality we may assume that

(8) $$\operatorname{cl}\{s_i(t)\} = \operatorname{cl}Q(t).$$

Since L_1 is separable, there is a denumerable subset $\{u_k\}$ L_1-dense in P. Thus $u_k(t) \in \operatorname{cl}Q(t)$ μ-a.e. in T for each k. By (8) for each j there is a sequence $\{s_{i_1}, \ldots, s_{i_m}\} \subset \{s_i\}$ such that

$$u_k(t) = \sum_{n=1}^{m} s_{i_n}\chi_{T_n}(t) + \eta(t),$$

where $\|\eta(\cdot)\|_{L_1} < 1/2j$. Therefore from the approximative decomposability of S there is an $s_k^j \in S\cap P$ satisfying $\|s_k^j - u_k\|_{L_1} < 1/j$. Then $\{s_k^j\} \subset S$ is a L_1-dense subset of P. Since we may include $\{s_i\}$ in the sequence $\{s_k^j\}$,

$$\operatorname{cl}\{s_k^j(t)\} = \operatorname{cl}Q(t) \ \mu\text{-a.e. in } T$$

and

$$\tilde{f}(t, u) = \sup_{j,k}\Big(-\tilde{f}^*\big(t, s_k^j(t)\big) + \langle s_k^j(t), u\rangle\Big).$$

We have

$$\begin{aligned}\tilde{f}^{**}\big(t, u(t)\big) &\geqslant \operatorname*{ess\,sup}_{s\in S}\Big(-\tilde{f}^*\big(t, s(t)\big) + \langle s(t), u(t)\rangle\Big)\\ &\geqslant \operatorname*{ess\,sup}_{s_k^j}\Big(-\tilde{f}^*\big(t, s_k^j(t)\big) + \langle s_k^j(t), u(t)\rangle\Big) = \tilde{f}\big(t, u(t)\big).\end{aligned}$$

This together with (2) completes the proof of the implication (iii′) $\to$ (iii″).

Let (iii″) hold and suppose $\{s_i\}$ is the L_1-dense subset of P. Defining

$$\tilde{f}(t, u) = \sup_i\big(-\Psi_{s_i}(t) + \langle s_i(t), u\rangle\big)$$

and using the definition of Ψ_s, one can check that (iii) holds.

4. Proof of Theorem A. First we state two results which will be needed in the proof.

PROPOSITION 1. *Let $K \subset L_1(T, \mathbf{R}^n)$ be decomposable; then for any $\varphi \in L_\infty$ we have*

$$\sup_{x\in K}\langle u, \varphi\rangle = \sup_{u\in K}\int_T \langle u(t), \varphi(t)\rangle \mu(dt) = \int_T \operatorname{ess\,sup}_{u\in K}\langle u(t), \varphi(t)\rangle \mu(dt). \tag{9}$$

PROPOSITION 2 (an extension of the Liapunov theorem). *Under the same assumption as above, for any $\varphi_1, \ldots, \varphi_k \in L$ the set*

$$B = \Big\{(x_1, \ldots, x_k) \mid x_i = \int_T \langle u(t), \varphi_i(t)\rangle \mu(dt), \quad u \in K, \quad i = 1, \ldots, k\Big\}$$

is convex.

The proofs of both propositions can be found in [9].

To prove that (i) implies (ii) we need only to check that D is convex. We will prove this if we show that the set

$$D_1 = \bigcap_{\substack{a\leqslant 0\\ s\in S}} \{(u, a) \mid \ aa + \langle s, u\rangle \leqslant c(a, s)\},$$

where $c(a, s)$ is defined by (4), is contained in D.

Let $(\bar u, \bar a)$ be any fixed point of D_1 and consider a neighbourhood of it in the $w(L_1, S)$ topology, that is, a set

$$N = \{(u, a) \mid \ |a - \bar a| < \varepsilon, \ |\langle u - \bar u, s_i\rangle| < \varepsilon, \ i = 1, \ldots, k\},$$

where $s_i \in S$ is arbitrary but fixed. To prove that $N \cap D$ is not empty we need to show that $\bar x = (\bar a, \langle \bar u, s_1\rangle, \ldots, \langle \bar u, s_k\rangle)$ belongs to the closure of the set

$$B = \{x \in \mathbf{R}^{k+1} \mid \ x_0 = a, \ x_i = \langle u, s_i\rangle, (u, a) \in D, \ i = 1, \ldots, k\}.$$

But we have

$$B = \Big\{x \in \mathbf{R}^{k+1} \mid \ x_0 = \int_T v(t)\mu(dt), x_i = \int_T \langle u(t), s_i(t)\rangle \mu(dt),$$
$$v(t) \geqslant f\big(t, u(t)\big) \ \mu\text{-a.e. in } T, \ u \in L_1\Big\}.$$

Hence, by Proposition 2, B is convex and the closure $\bar B$ of B is given by

$$\bar B = \bigcap_{b\in\mathbf{R}^{k+1}} \Big\{x \in \mathbf{R}^{k+1} | b_0 x_0 + \sum_{i=1}^{k} b_i x_i \leqslant c\Big(b_0, \sum b_i s_i\Big)\Big\}.$$

Indeed,

$$\sup_{(x_0, x)\in B}\Big(b_0 x_0 + \sum_{i=1}^{k} b_i x_i\Big) = \sup_{(u,a)\in D}\Big(b_0 a + \langle u, \sum_{i=1}^{k} b_i s_i\rangle\Big).$$

Since $a\bar a + \langle \bar u, s\rangle \leqslant c(a, s)$ for each (a, s), we have, in particular, $b_0 a + \sum b_i \langle \bar u, s_i\rangle \leqslant c\big(b_0, \sum b_i s_i\big)$ for each b_0. Therefore $\bar x \in \bar B$, which was to be proved.

This part of proof is the same as in the case of $w(L_1, C)$ considered by Olech [8], but we have included it here for the convenience of the reader.

We shall now prove the implication (ii) → (iii). Since the space $L_1(T, \mathbf{R}^n) \times \mathbf{R}$ is separable, by the Lindelöf theorem [3] (see Theorem 4 on p. 12) it follows from (5) that there exists a denumerable sequence s_i such that

$$D = \bigcap_i \{(u, a) \mid -a + \langle u, s_i\rangle \leqslant c(s_i)\}, \tag{10}$$

where $c(s_i) = \int_T \Psi_{s_i}(t)\mu(dt) = c(-1, s_i)$. Define $\tilde f$ by relation (1) with s_i the same as in (10) and with $\Psi_i(t) = \Psi_{s_i}(t)$. Notice that (3) implies that

$$I_f(u) = \int_T f\big(t, u(t)\big)\mu(dt) > -\infty$$

for each u such that I_f is defined on it. Then I_f is defined on the whole L_1. Indeed, let $u_1 \in L_1$ be such that both the negative part and the positive part of $f(t, u_1(t))$ have the integral divergent to infinity. Denote

$$B = \{t \mid \ f\big(t, u_1(t)\big) < 0\},$$

and define

$$\tilde u(t) = \begin{cases} u_1(t), & t \in B,\\ u_0(t), & t \notin B,\end{cases}$$

where u_0 is the function in Assumption A. Then $\int_T f\big(t, \tilde u(t)\big)\mu(dt) = -\infty$ which contradicts the above statement. Now we shall prove that

$$\tilde f\big(t, u(t)\big) \leqslant f\big(t, u(t)\big) \quad \mu\text{-a.e. in } T \tag{11}$$

for every $u \in L_1$. If $I_f(\bar u) < +\infty$, then by the definition of Ψ_i we have

$$f\big(t, \bar u(t)\big) \geqslant -\Psi_i(t) + \langle \bar u(t), s_i(t)\rangle \quad \mu\text{-a.e. in } T;$$

thus $\tilde f\big(t, u(t)\big) = \sup_i \big(-\Psi_i(t) + \langle \bar u(t), s_i(t)\rangle\big) \leqslant f\big(t, u(t)\big)$ μ-a.e. in T. Let $I_f(\bar u) = +\infty$ and suppose that

$$f\big(t, \bar u(t)\big) < \tilde f\big(t, \bar u(t)\big)$$

on a set B of positive measure. Because $I_f(\bar u) > -\infty$, we may assume without loss of generality that $\int_B f\big(t, \bar u(t)\big)\mu(dt) < +\infty$. Define

$$\tilde u(t) = \begin{cases} \bar u(t), & t \in B,\\ u_0(t), & t \notin B.\end{cases}$$

Then $I_f(\tilde{u}) < +\infty$, and thus

$$\tilde{f}(t, \overline{u}(t)) \leqslant f(t, \overline{u}(t)) \ \mu\text{-a.e. in } B,$$

which contradicts the above supposition. Thus (11) holds for each u.

Now let $I_{\tilde{f}}(u) < +\infty$. Then

$$I_{\tilde{f}}(\overline{u}) = \int_T \sup_i (-\Psi_i(t) + \langle s_i(t), \overline{u}(t)\rangle)\mu(dt)$$
$$\geqslant \int_T (-\Psi_i(t) + \langle s_i(t), \overline{u}(t)\rangle)\mu(dt)$$
$$= -c(s_i) + \langle s_i, \overline{u}\rangle;$$

hence, by (10), $(\overline{u}, I_{\tilde{f}}(\overline{u})) \in D$ and therefore $I_{\tilde{f}}(\overline{u}) \geqslant I_f(\overline{u})$. This together with (11) implies (2). Let $I_{\tilde{f}}(\overline{u}) = +\infty$. Then, supposing that $\tilde{f}(t, \overline{u}(t)) < f(t, \overline{u}(t))$ on a set B of positive measure and introducing a function u as above, we can reduce this case to the previous one. Thus (2) holds and the implication (ii) $\Rightarrow$ (iii) has been proved.

To prove that (iii) implies (i) notice that by (7) Ψ_i in (1) can be assumed to be equal to $\tilde{f}^*(t, s_i(t))$ μ-a.e. in T. Thus

$$\tilde{f}(t, u(t)) = \sup\left(-\tilde{f}^*(t, s_i(t)) + \langle s_i(t), u(t)\rangle\right).$$

This implies that for any fixed $u \in L_1$ and each $\varepsilon > 0$ there is a sequence $s_1, \ldots, s_k$ and a disjoined measurable decomposition $T_1, \ldots, T_k$ of T such that

$$\tilde{f}(t, u(t)) = \sum_{i=1}^{k}\left(-\tilde{f}^*(t, s_i(t)) + \langle s_i(t), u(t)\rangle\right)\chi_{T_i}(t) + \eta(t), \tag{12}$$

where the L_1-norm of $\eta(\cdot)$ is less than ε. But S is approximatively decomposable, and hence for each $\delta > 0$ there is an $s_\delta \in S$, $s_\delta(t) \in \mathrm{co}\{s_i(t)\}_{i\leqslant k}$ such that the L_1-norm of $s_\delta - \sum_{i=1}^{k}\chi_{T_i}s_i$ is less than δ. Let $s_\delta(t) = \sum_{i=1}^{k}\lambda_i(t)s_i(t)$, where $\lambda_i(t) \geqslant 0$ and $\sum_{i=1}^{k}\lambda_i(t) = 1$. By (12) we have

$$\begin{aligned}\tilde{f}(t, u(t)) = &\left(-\tilde{f}^*(t, s_\delta(t)) + \langle s_\delta(t), u(t)\rangle\right) + \\ &+\left(\tilde{f}^*(t, s_\delta(t)) - \sum_{i=1}^{k}\lambda_i(t)\tilde{f}^*(t, s_i(t))\right) + \\ &+\sum_{i=1}^{k}(\lambda_i(t) - \chi_{T_i}(t))\tilde{f}^*(t, s_i(t)) + \\ &+\left\langle\sum_{i=1}^{k}s_i(t)\chi_{T_i}(t) - s_\delta(t), u(t)\right\rangle + \eta(t).\end{aligned} \tag{13}$$

Because of the convexity of $\tilde{f}^*(t, \cdot)$ the second term of the right-hand side of (13) is nonpositive; thus, integrating (13), we get

$$I_{\tilde{f}}(u) \leqslant -I_{\tilde{f}^*}(s_\delta) + \langle s_\delta, u\rangle + r + \int_T \eta(t)\mu(dt), \tag{14}$$

where

$$r = \sum\int_T (\lambda_i(t) - \chi_{T_i}(t))\tilde{f}^*(t, s_i(t))\mu(dt) + \left\langle\sum s_i\chi_{T_i} - s_\delta, u\right\rangle.$$

Since u, s_i are fixed and both $\lambda_i - \chi_{T_i}$ and $\sum s_i\chi_{T_i} - s_\delta$ are in L_∞ and small in the L_1-norm, we can choose a δ so small that $|r| < \varepsilon$, which implies that

$$I_f(u) \leqslant \sup_S\left(-I_{\tilde{f}^*}(s) + \langle s, u\rangle\right).$$

The opposite inequality is obvious; thus

$$I_f(u) = \sup_S\left(-I_{\tilde{f}^*}(s) + \langle s, u\rangle\right).$$

Hence the epigraph of I_f is closed, and thus (i) holds and the proof of the theorem is completed.

Remark. In Theorem A we do not suppose that S separates points of L_1. In L_1 we can introduce the equivalence relation π: $(u_1, u_2) \in \pi$ iff $\langle u_1, s\rangle = \langle u_2, s\rangle$ for every $s \in S$. Then on the quotient space L_1/π the functional I_f can be defined uniquely. Indeed, if there is a $(u_1, u_2) \in \pi$ such that $I_f(u_1) < I_f(u_2)$, then the sequence $\{u_1\}$ converges to u_2 in $w(L_1, S)$ but $\liminf_{\{u_1\}} I_f(u_1) < I_f(u_2)$, which contradicts the lower semicontinuity of I_f.

5. Another characterization. Here we introduce other assumptions on the space S to get the lower semicontinuity of the functional I_f. Consider the following assumptions:

There is a linear subspace S^I of $L_\infty(T, \mathbf{R})$ such that

(α) *If $\gamma \in L_\infty^*$ and $\langle\gamma, s^I\rangle = 0$ for each $s^I \in S^I$, then γ is a singular functional.*

(β) *$\sigma s \in S$ if $\sigma \in S^I$, $s \in S$; $1 \in S^I$ and either S^I is an algebra or $\frac{1}{|\sigma|}(\cdot) \in S^I$ for each $\sigma \in S^I$ such that $\sigma(t) \leqslant \varepsilon < 0$ for some ε.*

Denote

$$K = \{(v, u) \in L_1 |\ v(t) \geqslant f(t, u(t))\ \mu\text{-a.e. in } T\}.$$

Let $S' = S^I \times S$ be the Cartesian product. If K is $w(L_1, S)$ closed, then there is a denumerable $\{(s_i^I, s_i)\} \subset S'$, $s_i^I(t) \leqslant 0$, and $\{\Psi_i\} \subset L_1$ such that it can be represented in the form

$$(15)\qquad K = \bigcap_i \{(v, u) \in L_1 |\ \langle s_i^I(t), v(t)\rangle + \langle s_i(t), u(t)\rangle - \Psi_i(t) \leqslant 0\}$$

μ-a.e. in T.

This fact follows from the decomposability of K and the separability of L_1 and the proof can be found in [8].

THEOREM B. *Assume that f satisfies Assumption* A *and there exists a linear subspace S^I of $L_\infty(T, \mathbf{R})$ satisfying* (α) *and* (β). *Then the conditions* (i)-(iii) *are equivalent to the following condition:*

(iv) *K is $w(L_1, S')$-closed and there is an $s \in S$ and a $\Psi(t)$ integrable and such that*

$$(16)\qquad -f(t, u(t)) + \langle s(t), u(t)\rangle - \Psi(t) \leqslant 0 \ \mu\text{-a.e. in } T.$$

for every $u \in L_1$.

Before proving Theorem B we recall some information concerning the space L_∞^*.

The linear functional $\lambda \in L_\infty^*$ is said to be *singular with respect to μ* if there is a sequence of measurable sets E_n such that $E_{n+1} \subset E_n$, $\mu(E_n) \to 0$ as $n \to \infty$ and $\langle \lambda, \chi_{T \setminus E_n} \varphi\rangle = 0$ for every $\varphi \in L_\infty$ and any E_n.

The linear functional $\lambda \in L_\infty^*$ is said to be *absolutely continuous with respect to μ* if it has the form

$$\langle \lambda, \varphi\rangle = \int_T \langle \lambda(t), \varphi(t)\rangle \mu(dt) \quad \text{for every } \varphi \in L_\infty,$$

where $\lambda(\cdot) \in L_1$.

It is known (see for example [4], [12]) that every linear functional $\lambda \in L_\infty^*$ can be represented uniquely as the sum $\lambda_a + \lambda_s$, where λ_a, λ_s are absolutely continuous and singular parts of λ with respect to μ. $\lambda \in L_\infty^*$ is nonpositive iff for each $\varphi \in L_\infty$, $\varphi(t) \geqslant 0$ μ-a.e. in T we have the inequality $\langle \lambda, \varphi\rangle \leqslant 0$. For a nonpositive $\lambda \in L_\infty^*$ both λ_a and λ_s are also nonpositive.

Proof of Theorem B. In Theorem A the approximative decomposability was used only in the implication (iii) $\Rightarrow$ (i); hence to prove Theorem B it is enough to verify the implications (iv) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv).

Assume that (iv) holds, and let $\{u_a\} \subset L_1$ converge $w(L_1, S)$-weakly to $u_0 \in L_1$. From (16) we have

$$\gamma = \liminf_a \int_T f(t, u_a(t)) \mu(dt) > -\infty.$$

If $\gamma = +\infty$, then there is nothing to prove. Then let $\gamma < +\infty$. There is a subnet still denoted by $\{a\}$ and such that $\lim_a I_f(u_a) = \gamma$. From that and from (16) it follows that

$$\int_T |-f(t, u_a(t)) + \langle s(t), u_a(t)\rangle - \Psi(t)| \mu(dt) < M.$$

Hence, by virtue of the Alaoglu theorem ([3], Theorem 2, p. 424), there exists another subnet $-f(t, u_a(t)) + \langle s(t), u_a(t)\rangle - \Psi(t)$ converging in the $w(L_\infty^*, L_\infty)$ topology to $\lambda \in L_\infty^*$. Denote

$$\nu = -\lambda + \langle s, u_0\rangle(\cdot) - \Psi(\cdot) \in L_\infty^*.$$

Then by (β)

$$(17)\qquad \lim \int_T \big\langle f(t, u_a(t)), s^I(t)\big\rangle \mu(dt) = \langle \nu, s^I\rangle$$

for every $s^I \in S^I$. We wish to show that

$$(18)\qquad (v_0, u_0) \in K, \text{ where } v_0(t) = d\nu_a/d\mu(t).$$

Since K is assumed to be closed, $\bar{K} = K$ and representation (15) holds for K. Therefore to show (18) it is enough to prove that for each fixed i

$$(19)\qquad \langle s_i^I(t), v_0(t)\rangle + \langle s_i(t), u_0(t)\rangle - \Psi_i(t) \leqslant 0 \ \mu\text{-a.e. in } T.$$

Suppose that S^I is an algebra. From (17), (15) we deduce that $\big\langle s_i^I(t), f(t, u_a(t))\big\rangle + \langle s_i(t), u_a(t)\rangle - \Psi_i(t)$ converges to a non-positive $\lambda^i \in L_\infty^*$ in the $w(L_\infty^*, L_\infty)$ topology. On the other hand, by (17) for each $s^I \in S^I$ we find that

$$\big\langle s^I(\cdot), s_i^I(\cdot) f(\cdot, u_a(\cdot)) - \langle s_i, u_a\rangle(\cdot) - \Psi_i(\cdot)\big\rangle$$

converges to

$$\langle s^I(\cdot), (s_i^I v_0)(\cdot) + \langle s_i, u_0\rangle(\cdot) - \Psi_i(\cdot)\rangle + \langle s^I s_i^I, \nu_s\rangle.$$

The latter functional is singular, therefore, using assumption (α), we have

$$s_i^I(t) v_0(t) + \langle s_i(t), u_0(t)\rangle - \Psi_i(t) - \lambda_a^i(t) = 0 \ \mu\text{-a.e. in } T.$$

Since $\lambda_a(t) \leqslant 0$, we have (19) and hence also (18) follows. Notice that ν_s is nonnegative and therefore, applying (17) for $s^I(t) \equiv 1$, we obtain by (18)

$$\gamma = \liminf_a \int_T f(t, u_a(t)) \mu(dt) = \langle \nu_a, 1\rangle + \langle \nu_s, 1\rangle \geqslant \int_T f(t, u_0(t)) \mu(dt).$$

Thus (i) holds if S^I is an algebra.

Suppose now that the other alternative of (β) holds.

Notice that, as in the proof of (5) in Section 3, by (16) one can prove that (15) holds also with $s_i^I(t) \leqslant \varepsilon_i < 0$. Thus, if the second part of (β) holds, then we can put in (15) $s_i^I(t) \equiv -1$ for each i. Therefore we may now repeat the proof word for word except that we need not use the assumption that S^I is an algebra. Hence (i) holds also in this case. Thus the proof of the implication (iv) $\Rightarrow$ (i) is completed.

Assume now that (iii) holds. Then by (1) and (2) we clearly have

$$(20)\qquad K = \bigcap_i \{(v, u) \in L_1 |\ -v(t) + \langle s_i(t), u(t)\rangle \leqslant \Psi_i(t) \ \mu\text{-a.e. in } T\},$$

where $s_i \in S$ and Ψ_i are integrable. Thus (16) holds. By the same argument as above one can see that if (v_a, u_a) converges to (v_0, u_0) in the $w(L_1, S)$ topology, and $-v_a(t)+\langle s_i(t), u_a(t)\rangle \leqslant \Psi_i(t)$ μ-a.e. in T for each a, then the same inequality holds for the limit function. This and (20) prove that K is $w(L_1, S')$-closed. Hence (iv) follows and this completes the proof of Theorem B.

Acknowledgement. The author wishes to thank Professor C. Olech for suggesting the problem and for helpful discussions. In particular, it was he who suggested to the author the notion of approximative decomposability. Thanks are also due to the referee, Professor Ioffe, whose opinion helped the author to avoid errors and to present an improved version of the paper.

References

[1] Ch. Castaing, *Sur les multi-applications mesurables*, Rev. Inf. Rech. Op. 1 (1967), pp. 91–126.
[2] — *Intégrales convexes duales*, C. R. Acad. Sci. Paris 275 (1972), pp. 1331–1334.
[3] N. Dunford and J. T. Schwartz, *Linear operators. Part I. General theory*, Interscience Publications, INC, New York 1967.
[4] A. D. Ioffe and V. L. Levin, *Subdifferential of convex functions*, Trudy Moskov. Mat. Obšč. 26 (1972), pp. 3–73.
[5] A. D. Ioffe and V. M. Tikhomirov, *Theory of extremum problems*, Moscow 1974.
[6] C. Olech, *Existence theory in optimal control problem—the underlying ideas*, International Conference of differential equations, Academic Press, New York 1975, pp. 612–635.
[7] — *A necessary and sufficient condition for lower-semi-continuity of certain integral functionals*, Mathematical Structures — Computational Mathematics–Mathematical Modelling, Sofia 1975, pp. 378–379.
[8] — *The characterization of the weak* closure of certain sets of integrable functions*, SIAM J. Control 12 (1974), pp. 311–318.
[9] — *Existence theory in optimal control*, in: Control Theory and Topics in Functional Analysis, Vol. I, I.A.E.A. VIENNA 1976, pp. 291–328.
[10] R. T. Rockafellar, *Integrals which are convex functionals*, Pacific J. Math. 24 (1968), pp. 525–539.
[11] M. Valadier, *Convex integrands on Souslin locally convex spaces*, ibid. 59 (1975), pp. 267–276.
[12] K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. 72 (1952), pp. 46–66.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
and
INSTITUTE OF MATHEMATICS
ACADEMY OF SCIENCES OF VIETNAM

Received February 26, 1976
Revised version June 6, 1977 (1202)