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The algebra of compact operators does not have any finite-codimensional ideal

by

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Abstract. The Lie algebra of compact operators on an infinite dimensional Hilbert space has no non-trivial finite-dimensional quotient. It follows that, as a Banach–Lie algebra, all its extensions are trivial.

Let H be an infinite-dimensional real or complex Hilbert space and let $C(H)$ be the algebra of all compact operators on H. It is well known that any non-trivial two-sided ideal of $C(H)$ contains the ideal of all finite rank operators (Theorem 1.7 in [2]); the standard examples are the von Neumann–Schatten ideals [5]. The purpose of this note is to show that any such ideal has infinite codimension. We shall indeed prove the following stonger statement, which is phrased in the Lie algebra setting; as a corollary we obtain that any extension (in a suitable sense) of the Lie algebra considered is trivial.

THEOREM. *Let $\mathbf{gl}(H, C)$ be the Lie algebra defined by the commutator product on $C(H)$ and let $\mathbf{a}$ be a non-trivial ideal in $\mathbf{gl}(H, C)$. Then $\mathbf{a}$ contains the space $\mathbf{sl}(H, C_0)$ of all finite rank operators with zero trace and $\mathbf{a}$ has infinite codimension.*

That $\mathbf{a}$ contains $\mathbf{sl}(H, C_0)$ is an easy corollary of Schur's lemma and of the simplicity of the Lie algebra $\mathbf{sl}(H, C_0)$; see for example [3], page I.2. For the second statement, consider a finite-dimensional Lie algebra $\mathbf{g}$, say, of dimension $d > 0$, and a morphism $\pi\colon \mathbf{gl}(H, C) \to \mathbf{g}$. We have to show that the kernel of π is $\mathbf{gl}(H, C)$ itself. We do this below in the complex case; the real case will follow from a standard complexification argument.

We denote by $\mathbf{N}$ the set of natural integers including zero and by c_0 the space of sequences $(x_n)_{n\in \mathbf{N}}$ of complex numbers converging towards zero.

LEMMA 1. *Let H_0 be a closed subspace of H of infinite dimension and infinite codimension. Let $(e_n)_{n\in \mathbf{N}}$ be an orthonormal basis in H_0 and let $(x_n)_{n\in \mathbf{N}}$*

be in c_0. Then the compact operator X defined on H by

$$Xe_n = \begin{cases} x_k e_{2k+1} & \text{if} \quad n = 2k, \\ 0 & \text{if} \quad n = 2k+1 \end{cases}$$

and by

$$Xv = 0 \quad \text{if} \quad v \perp H_0$$

is in the kernel of π.

Proof. For each $n \in N$, let y_n be a cube root of x_n; the sequence $(y_n)_{n\in N}$ is in c_0. Let $D = \{0, 1, \ldots, d+2\}$ and let $(f_{\beta,n})_{\beta\in D, n\in N}$ be an orthonormal basis of H with $f_{0,n} = e_{2n}$ and $f_{d+2,n} = e_{2n+1}$ for each $n \in N$. For each $\alpha \in D$, let Y_α be the compact operator defined by

$$Y_\alpha f_{\beta,n} = \begin{cases} y_n f_{\alpha,n} & \text{if} \quad \beta = \alpha, \\ 0 & \text{if} \quad \beta \neq \alpha. \end{cases}$$

We may suppose the y_n's are not all zero, so that the linear span V of $\{Y_1, Y_2, \ldots, Y_{d+1}\}$ is of dimension $d+1 > \dim(g)$; it follows that there exists Y in $(V-\{0\}) \cap \mathrm{Ker}(\pi)$. Let us chose the notations such that

$$Y = Y_1 + \sum \mu_\alpha Y_\alpha,$$

where the summation runs on the integers between 2 and $d+1$.

Let S be the compact operator defined on H by

$$Sf_{\alpha,n} = \begin{cases} y_n f_{d+2,n} & \text{if} \quad \alpha = 1, \\ 0 & \text{if} \quad \alpha \neq 1; \end{cases}$$

it follows from the definitions of S and of the Y_α's that $Z = [S, Y]$ is described by

$$Zf_{\alpha,n} = \begin{cases} (y_n)^2 f_{d+2,n} & \text{if} \quad \alpha = 1, \\ 0 & \text{if} \quad \alpha \neq 1. \end{cases}$$

Let T be the compact operator defined on H by

$$Tf_{\alpha,n} = \begin{cases} y_n f_{1,n} & \text{if} \quad \alpha = 0, \\ 0 & \text{if} \quad \alpha \neq 0; \end{cases}$$

it follows from the definitions that $X = [Z, T]$ is described by

$$Xf_{\alpha,n} = \begin{cases} (y_n)^3 f_{d+2,n} & \text{if} \quad \alpha = 0, \\ 0 & \text{if} \quad \alpha \neq 0. \end{cases}$$

Hence X has the desired properties. ■

Let H_0 be as in Lemma 1. Its orthogonal complement will be identified with countably many copies of H_0, so that $H = \oplus H_j$ where the summation runs over $j \in N$. If $(x_n)_{n\in N}$ is in c_0, we shall denote, as Pearcy and Topping [4] do, by $\mathrm{UDiag}\{(x_n)\}$ the operator on H whose matrix, with respect to $H = \oplus H_j$, is

$$\begin{bmatrix} 0 & x_0 & 0 & 0 & \ldots \\ 0 & 0 & x_1 & 0 & \ldots \\ 0 & 0 & 0 & x_2 & \ldots \\ . & . & . & . & . \end{bmatrix}$$

with non-zero (operator) entries on the diagonal just above the main diagonal, where x_n is written for the corresponding scalar multiple of the identity operator on H_0. We shall write $(x_n^e)_{n\in N}$ for the sequence with $x_n^e = x_n$ for n even and $x_n^e = 0$ for n odd; we shall write $(x_n^0)_{n\in N}$ for the sequence with $x_n^0 = x_n - x_n^e$. Then

$$\mathrm{UDiag}\{(x_n)\} = \mathrm{UDiag}\{(x_n^e)\} + \mathrm{UDiag}\{(x_n^0)\}$$

is in the kernel of π, because the two summands on the right-hand side are in this kernel by Lemma 1.

LEMMA 2. *Any compact operator on H can be written as a finite sum $\sum [A_j, B_j]$, where the A_j's are compact and where each B_j is of the form $\mathrm{UDiag}\{(x_n)\}$.*

Proof. See Section 3 of [4]. (On line 12, page 250 of that paper read $a_{2n} = 0$ instead of $a_{2n} = a_{2n-1}$.) ■

The theorem follows. It suggests two questions.

(1) Has $\boldsymbol{gl}(H, C)$ any non-trivial sub Lie algebra of finite codimension? (We guess no.)

(2) Let $\mathrm{GL}(H, C)$ be the group of those invertible operators on H which are congruent to the identity modulo $C(H)$; is any homomorphism $\mathrm{GL}(H, C) \to \mathrm{GL}(n, \boldsymbol{R})$ trivial? (We guess yes; if the homomorphism is assumed to be continuous, for the topology inherited from the norm on $\mathrm{GL}(H, C)$, then the question becomes an easy exercise.)

Consider now $\boldsymbol{gl}(H, C)$ as a Banach-Lie algebra, with the usual norm on operators. Extending classical concepts, we define an *abelian extension* of $\boldsymbol{gl}(H, C)$ by a finite-dimensional commutative algebra $\boldsymbol{a}$ to be a short exact sequence

$$\{0\} \to \boldsymbol{a} \xrightarrow{\lambda} \boldsymbol{g} \xrightarrow{\mu} \boldsymbol{gl}(H, C) \to \{0\},$$

where $\boldsymbol{g}$ is a Banach-Lie algebra and where λ, μ are continuous morphisms; $\boldsymbol{a}$ is the *kernel* of the extension; the *trivial* extension is that for which $\boldsymbol{g}$ is the direct product $\boldsymbol{a} \times \boldsymbol{gl}(H, C)$ and λ, μ the canonical inclusion and projection, respectively.

COROLLARY. *Any abelian extension of $\boldsymbol{gl}(H, C)$ with finite-dimensional kernel is trivial.*

Proof. Let $\boldsymbol{a}$, $\boldsymbol{g}$, λ and μ be as above. By the theorem, the $\boldsymbol{gl}(H, C)$-module structure defined by the extension on $\boldsymbol{a}$ is trivial. As the image of λ is of finite dimension, it is a complemented subspace of the Banach space $\boldsymbol{g}$ and μ has a continuous linear section. Now the cohomology of $\boldsymbol{gl}(H, C)$ with continuous cochains and trivial scalar coefficients reduces to zero in degree two; see [3], page IV.8. The usual argument (sketched in [1], § 3, exercise 12i) shows that te extension is inessential. Hence $\boldsymbol{g}$ is a semi-direct product of $\boldsymbol{gl}(H, C)$ and $\boldsymbol{a}$, relative to some morphism from $\boldsymbol{gl}(H, C)$ to the algebra of derivations of $\boldsymbol{a}$ (see [1], § 1 no 8). This morphism is trivial, again by the theorem above, and the product is direct. ■

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Lipschitz classes and Poisson integrals on stratified groups*

by

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Abstract. It is shown that the Lipschitz classes Γ_a on stratified groups can be characterized in terms of Poisson integrals, and some interpolation and approximation theorems are proved.

Introduction. It is well known that the classical Lipschitz classes Λ_a $(a > 0)$ on $\boldsymbol{R}^n$ can be characterized in terms of Poisson integrals; see [7]. In this paper we generalize this result to the Lipschitz classes Γ_a $(a > 0)$ on stratified groups studied in [3]. To some extent our arguments are adaptations of those in [7], but the non-commutativity and non-ellipticity in the general situation present a number of difficulties which do not occur in the classical case. From the Poisson integral characterization we obtain a simple proof that the classes Γ_a form a scale of interpolation spaces, a result which has been proved with different techniques by Krantz [6]. Actually, the logical order of the paper is somewhat different; we prove the interpolation theorems for the spaces defined by Poisson integrals and then use them in showing that these spaces coincide with the spaces Γ_a.

The plan of the paper is as follows. In Section 1 we recall the basic facts about stratified groups and the spaces Γ_a. (For proofs and further details the reader is referred to [3].) In Section 2 we construct the Poisson kernel and derive its fundamental properties. In Section 3 we define spaces Γ_a^* in terms of the Poisson integral and prove the interpolation and approximation theorems. Sections 4 and 5 are devoted to the proof that $\Gamma_a = \Gamma_a^*$.

1. Let $\mathfrak{g}$ be a stratified Lie algebra in the sense of [3]; that is, $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over $\boldsymbol{R}$ together with a vector space decomposition $\mathfrak{g} = \bigoplus_{1}^{m} V_j$ such that $[V_1, V_j] = V_{j+1}$ for $j < m$ and $[V_1, V_m] = \{0\}$. We define a one-parameter family $\{\gamma_r\colon r > 0\}$ of

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