wird $\varrho = \varrho_1 = \nu = m|_{[0,1]}$,

$$k(z) = \begin{cases} 2 & \text{für} \quad z \in \mathfrak{a}, \\ 1 & \text{für} \quad z \in [0,1]-\mathfrak{a}, \end{cases} \qquad \# \Psi^{-1}\{z\} = 2 \text{ f.a. } z \in [0,1]$$

(also gleich der Hellinger–Hahnschen Vielfachheitsfunktion von M_Ψ).

4. Zum Abschluß wird noch kurz das Beispiel D aus [1] diskutiert: Mit Hilfe eines stetigen singulären Wahrscheinlichkeitsmaßes σ mit Träger $[0,1]$ und der dadurch definierten Funktion $F(x) = \sigma([0,x))$ wird dort eine stetige Funktion $\varphi\colon [0,1] \to \boldsymbol{R}$ eingeführt, so daß für den Operator M_φ auf $L^2([0,1], m)$ gilt: $k(z) = 1$ für ν-f.a. $z \in [0,1]$, $\# \varphi^{-1}\{z\} = 2$ f.a. $z \in [0,1]$; $\nu = \frac{1}{2}(m+\sigma)$. Die Überlegungen von Abschnitt 3–5 kann man hier mit $E_1 = [0, \frac{1}{2}]$, $E_2 = (\frac{1}{2}, 1]$ und $E_3 = \ldots = E_\infty =$ leere Menge durchführen. Man findet $\varrho_1 = \frac{1}{2}\cdot m$, $\varrho_2 = \frac{1}{2}\sigma$. Sei $S \subset [0,1]$ eine Lebesgue-Nullmenge, auf der σ konzentiert ist. Dann ist $\frac{d\varrho_1}{d\varrho} = \chi_{[0,1]-S}$, $\frac{d\varrho_2}{d\varrho} = \chi_S$; $\mathfrak{d}_1 = [0,1]-S$, $\mathfrak{d}_2 = S$. Satz 1 bestätigt, daß $k(z) = 1$ für ν-f.a. $z \in [0,1]$. Die im Beweis von Satz 4 auftretende Ausnahmenullmenge ist hier

$$N_0 = [0, \tfrac{1}{2}] \cap \varphi^{-1}(S) \cup (\tfrac{1}{2}, 1] \cap \varphi^{-1}([0,1]-S) = \tfrac{1}{2}\cdot S \cup \big(1-\tfrac{1}{2}F([0,1]-S)\big);$$

es ist $\varphi(N_0) = S \cup ([0,1]-S) = [0,1]$.

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On the Poisson integral for Lipschitz and C^1-domains

by

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Abstract. Let D be a Lipschitz domain and let Hf denote the solution of the Dirichlet problem with boundary values f. In this note we prove that if $f \in L^p(\sigma)$, where σ is the surface measure of ∂D and $2 \leqslant p \leqslant \infty$, then the nontangential maximal function associated to Hf is also in L^p. Here the exponent 2 is sharp. However, if D is assumed to be a C^1-domain, then the results hold for $1 < p \leqslant \infty$. The method we use is to characterize the corresponding Carleson measures.

1. Introduction. Let D be a bounded Lipschitz domain and denote by δ the surface measure of ∂D. If $E \subset \partial D$ and $P \in D$, we denote by $\omega(P, E)$ the harmonic measure of E evaluated at P, and if f is integrable with respect to the harmonic measure, we denote by

$$Hf(P) = \int_{\partial D} f(Q)\,\omega(P, dQ)$$

the Poisson integral of f. For the basic properties of ω we refer to Helms [5], Chapter 8. We recall that if D is the unit ball and $1 < p < \infty$, then

$$\int_{\partial D} \sup_{0<r<1} |Hf(rQ)|^p\, d\sigma(Q) \leqslant C_p \int_{\partial D} |f|^p\, d\sigma.$$

In this article we shall study the analogues of this result for Lipschitz domains. It turns out that the analogue holds if $2 \leqslant p < \infty$ but not always if $1 < p < 2$. However, if we assume that D is a C^1-domain, then we can extend the result to $1 < p < \infty$. We shall formulate these results in Section 3, but we can give the following characterization of the corresponding Carleson measures.

THEOREM 1. *Let $D \subset \boldsymbol{R}^n$, $n \geqslant 3$, be a Lipschitz domain and let $2 \leqslant p < \infty$. If μ is a positive measure on D, then the following conditions are equivalent*:

(i) *There is a constant M such that for all $P \in \partial D$ and all $r > 0$ we have*

$$\mu\{Q \in D\colon |Q-P| < r\} \leqslant M r^{n-1}. \tag{1.1}$$

(ii) *There is a constant K such that for all $f \in L^p(\sigma)$ we have*

$$\int_D |Hf|^p\, d\mu \leqslant K \int_{\partial D} |f|^p\, d\sigma. \tag{1.2}$$

If condition (1.1) *holds, then the constant K may be chosen to depend only on p, M and D.*

If D is assumed to be a C^1-domain, then the above result holds when $1 < p < \infty$.

For domains with smooth boundaries the result of Theorem 1 is contained in Hörmander [8]. We would like to point out that it has been proved in Dahlberg [3] that $L^2(\sigma) \subset L^1(\omega(P, \cdot))$ and also observed that if $1 < p < 2$ is given, then there is a Lipschitz domain such that $L^p(\sigma) \not\subset L^1(\omega(P, \cdot))$. This explains the restriction $p \geqslant 2$ in the first part of the theorem.

We remark that our results continue to hold in the case when $n = 2$, but to avoid the minor complications due to the logarithmic singularity of the Green functions we have restricted ourselves to the case $n \geqslant 3$.

2. Technical preliminaries. We start by recalling that a function φ is called a *Lipschitz function* if

$$|\varphi(x) - \varphi(x_1)| \leqslant M|x - x_1|. \tag{2.1}$$

The smallest possible M such that (2.1) holds is called the *Lipschitz constant of* φ, which we denote by $\Lambda(\varphi)$. We say that a bounded domain D is a *Lipschitz domain with Lipschitz constant less than M* if ∂D can be covered by right circular cylinders whose bases have positive distance from ∂D and corresponding to each cylinder L there is a coordinate system (x, y) with $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, with the y-axis parallell to the axis of L and a Lipschitz function φ with $\Lambda(\varphi) < M$ such that

$$L \cap D = \{(x, y)\colon \varphi(x) < y\} \cap L \text{ and } L \cap \partial D = \{(x, y)\colon y = \varphi(x)\} \cap L.$$

The greatest lower bound of all possible M is called the *Lipschitz constant of D*. If in addition the function φ can be taken to be C^1-functions, we say that D is a *C^1-domain*. With this terminology it is easily verified that a C^1-domain has Lipschitz constant 0. If D is a Lipschitz domain, we shall denote by σ the surface measure of ∂D. If $P \in \partial D$ and $r > 0$, we put $\Delta(P, r) = \partial D \cap B(P, r)$, where $B(P, r) = \{Q\colon |Q - P| < r\}$. Since Lipschitz functions are differentiable almost everywhere with respect to Lebesgue measure (Stein [11], p. 250), it follows that for all points Q on ∂D outside a set of vanishing σ-measure there is an inward unit normal, which we denote by n_Q.

THEOREM A (Dahlberg [3]). *Let $D \subset \mathbf{R}^n$, $n \geqslant 3$, be a Lipschitz domain and denote by G the Green function of D. Let $P \in D$ and put $g = G(P, \cdot)$. Then there exists a set $E \subset \partial D$ such that $\sigma(E) = 0$ and for all $Q \in \partial D - E$ the limit $\lim_{t \downarrow 0} (\partial/\partial n_Q) g(Q + t n_Q)$ exists. If we denote this limit by $(\partial/\partial n) g(Q)$, then the following holds:*

(a) *If $Q \in \partial D - E$, then* $0 < (\partial/\partial n) g(Q) < \infty$.

(b) *Let σ_n denote the surface measure of $\{P \in \mathbf{R}^n\colon |P| = 1\}$ and define γ_n by $\gamma_n^{-1} = \sigma_n(n-2)$. If $F \subset \partial D$, then*

$$\omega(P, F) = \gamma_n \int_F (\partial/\partial n) g(Q)\, d\sigma(Q).$$

(c) *There is a number $C > 0$ such that for all $P' \in \partial D$ and all $r \in (0, 1)$ we have*

$$\sigma(\Delta(P', r)) \int_{\Delta(P', r)} [(\partial/\partial n) g(Q)]^2 d\sigma(Q) \leqslant C \int_{\Delta(P', r)} [(\partial/\partial n) g(Q)]^2 d\sigma(Q).$$

We will need to compare positive harmonic functions which vanish on a part of the boundary. The following result was stated in Kemper [9]. For a proof see Dahlberg [3].

THEOREM B. *Let D be a Lipschitz domain. Suppose V is an open set such that $V \cap \partial D \neq \emptyset$. Suppose W is a domain such that $W \subset D$ and $\overline{W} \subset V$. Let $P_0 \in W$. Then there is a constant $C > 0$ such that if u and v are non-negative harmonic functions in D which vanish on $V \cap \partial D$ and satisfy $u(P_0) \leqslant v(P_0)$, then $u(P) \leqslant Cv(P)$ for all $P \in W$.*

We shall be working with a class of domains which we shall now describe. For $m > 0$ let $L(m)$ be the class of Lipschitz functions φ such that $\varphi(0) = 0$, $\Lambda(\varphi) < m$ and the support of φ is contained in $\{x \in \mathbf{R}^{n-1}\colon |x| < 1\}$. If $\varphi \in L(m)$, we put

$$D(\varphi, m) = \{(x, y)\colon \varphi(x) < y < Am,\ |x| < 10\},$$

$$S(\varphi) = \{(x, \varphi(x))\colon |x| \leqslant 1\}.$$

We suppose that A is chosen so large that if $m \geqslant 1$, then $D(\varphi, m)$ is star-shaped with respect to $P_m(0, \frac{1}{2}mA)$ for all $\varphi \in L(m)$ and $P_m \in \Gamma + P$ for all $P \in \{(x, \varphi(x))\colon |x| \leqslant 10\}$, when $\Gamma = \{(x, y)\colon |x| < 2my\}$. Also if A is chosen large enough, we have the following result from Dahlberg [3], Lemma 1:

LEMMA 2.1. *Let $m \geqslant 1$ and $\varphi \in L(m)$. Let G be the Green function of $D(\varphi, m)$. Then there is a number C such that for all $Q \in S(\varphi)$ and all $r \in (0, 1)$ we have*

$$C^{-1} G(Q + (0, r), P_m) \leqslant \omega(P_m, \Delta(Q, r)) \leqslant CG(Q + (0, r), P_m).$$

We will make use of the following consequence from Hunt and Wheeden [7], p. 512:

LEMMA 2.2. *Let $m \geqslant 1$ and $\varphi \in L(m)$. Then there is a number C with the following property: If u is positive and harmonic in $D(\varphi, m)$ and vanishes on $\partial D(\varphi, m) - \Delta(Q, r)$ for some $Q \in S(\varphi)$ and $r \in (0, 1)$, then*

$$u(P_m) \leqslant Cu(Q + (0, r))\, \omega(P_m, \Delta(Q, r)).$$

We have the following estimate of the harmonic measure for sets of the form $\Delta(P,r)$ (see Hunt and Wheeden [7], Lemma 2.1):

LEMMA 2.3. *Let D be a Lipschitz domain. Then there is a number $\delta > 0$ such that if $P \in \partial D$ and $r > 0$, then $\omega(Q, \Delta(P,r)) \geqslant \delta$ whenever $Q \in D \cap B(P, \delta r)$.*

3. The main result. We start with the following preliminary version of Theorem 1.

LEMMA 3.1. *Let φ be a Lipschitz function such that $\varphi(0) = 0$. Suppose the positive numbers A and B have been chosen such that $A > \sup\{|\varphi(x)|: |x| \leqslant 10B\}$ and the domain $D = \{(x,y): \varphi(x) < y < 10A,\ |x| < 10B\}$ is starshaped with respect to a point $P_0 = (0, A_1)$. Let μ be a positive measure on D such that $\mu(D \cap B(P,r)) \leqslant Mr^{n-1}$ whenever $P \in \{(x, \varphi(x)): |x| \leqslant 10B\}$. Let k be the density of $\omega(P_0, \cdot)$ with respect to σ. Suppose there is a $q \in (1, \infty)$ and an $\varepsilon > 0$ such that if $P \in \{(x, \varphi(x)): |x| \leqslant 4B\}$ and $0 < r < \varepsilon$, then*

$$\Big(r^{1-n} \int_{\Delta(P,r)} k^q\, d\sigma\Big)^{1/q} \leqslant L r^{1-n} \int_{\Delta(P,r)} k\, d\sigma.$$

Then there is a number K, which can be taken to depend only on D, P_0, L and q such that if $f \in L^p(S^*, \sigma)$, where $S^* = \{(x, \varphi(x)): |x| \leqslant B\}$ and $p = q(q-1)^{-1}$, then

$$\mu\{P \in D^*: |Hf(P)| > s\} \leqslant K s^{-p} \int_{S^*} |f|^p\, d\sigma.$$

Here $D^* = \{(x,y): |x| < 2B,\ \varphi(x) < y < 2A\}$.

Proof. Let $f^+ = \max(f, 0)$, $f^- = f^+ - f$. Then $f = f^+ - f^-$. Since $\mu\{P \in D^*: |Hf(P)| > s\} \leqslant \mu\{P \in D^*: Hf^+ > s/2\} + \mu\{P \in D^*: Hf^- > s/2\}$, it follows that it is sufficient to prove the lemma in the case $f \geqslant 0$. Put $S_m = \{(x, \varphi(x)): |x| \leqslant mB\}$ and $D_m = \{(x,y): |x| \leqslant mB,\ \varphi(x) < y < mA\}$. Let $\gamma = \{(x,y): a|x| < y,\ 0 < y < h\}$, where a and h have been chosen so small that $\gamma(P) = \gamma + P \in D_3$ whenever $P \in S_2$. We assume from now on that $f \geqslant 0$. Then we have from Hunt and Wheeden [6], Lemma 4, that if $P \in S_3$, then

$$\sup\{Hf(Q): Q \in \gamma(P)\} \leqslant Cf^*(P), \tag{3.1}$$

where $f^*(P) = \sup_{r>0} \big(\int_{\Delta(P,r)} fk\, d\sigma\big)\big(\omega(P_0, \Delta(P,r))\big)^{-1}$. Let $V = \{P \in D^*: P \in \gamma(Q)$ for some $Q \in S_2\}$ and put $U = D^* - V$. Since $\overline{U} \subset D$, it follows from Harnack's inequality that there is a constant C such that if $P \in U$, then $Hf(P) \leqslant CHf(P_0) \leqslant C\|f\|_p$, where $\|f\|_p = (\int_{\partial D} |f|^p\, d\sigma)^{1/p}$. Hence

$$\mu\{P \in U: Hf(P) > s\} \leqslant C_1 s^{-p} \|f\|_p^p \tag{3.2}$$

for a suitable choice of C_1. If $P = (x,y) \in D$, we put $P^* = (x, \varphi(x))$.

From (3.1) follows the existence of a positive number $\delta > 0$ such that if $P = (x,y) \in V$ and $Hf(P) > s$, then $f^*(Q) > \delta s$ whenever $|Q - P^*| < \delta(y - \varphi(x))$. We now define $R(P) = \{(x', y'): |x - x'| < t(y - \varphi(x)),\ |y - y'| < 2(y - \varphi(x))\}$, where t has been chosen so small that if $Q \in R(P) \cap S_3$, then $|Q - P^*| < \delta(y - \varphi(x))$. Therefore we have from (3.1) that if $P \in V$ and $Hf(P) > s$, then $f^*(Q) > \delta s$ whenever $Q \in R(P) \cap S_3$. Suppose $F \subset \{P \in V: Hf(P) > s\}$ is compact. Then there exists finitely many $P_j \in F$ such that $F \subset \bigcup R(P_j)$. Moreover, we may assume that no point P belongs to more than 2^n of the sets $R(P_j)$ (see Stein and Weiss [12], p. 54). Hence $\mu(F) \leqslant \sum \mu(R(P_j)) \leqslant C\sigma\{Q \in S_3: f^*(Q) > \delta s\}$. Since F was arbitrary, it follows that

$$\mu(V) \leqslant C\sigma\{Q \in S_3: f^*(Q) > \delta s\}. \tag{3.3}$$

We now see from (3.2) and (3.3) that the lemma follows if we can show that $\sigma\{Q \in S_3: f^*(Q) > s\} \leqslant Cs^{-p}\|f\|_p^p$. To this end we note that if $0 < r < \varepsilon$ and $Q \in S_3$, then

$$\int_{\Delta(Q,r)} fk\, d\sigma \leqslant r^{n-1}\Big(r^{1-n} \int_{\Delta(Q,r)} f^p\, d\sigma\Big)^{1/p}\Big(r^{1-n} \int_{\Delta(Q,r)} [k^q\, d\sigma\Big)^{1/q}.$$

From our condition on k follows now that

$$\sup_{0<r<\varepsilon} \big(\omega(P_0, \Delta(Q,r))\big)^{-1} \int_{\Delta(Q,r)} fk\, d\sigma \leqslant C(Mf^p)^{1/p}(Q), \tag{3.4}$$

where $Mf(Q) = \sup_{0<r} r^{1-n} \int_{\Delta(Q,r)} f\, d\sigma$. From Lemma 2.3 follows the existence of a number $c = c(\varepsilon, D)$ such that $\omega(P_0, \Delta(Q, \varepsilon)) \geqslant c$. If $r > \varepsilon$ and $Q \in S_3$, then $\big(\omega(P_0, \Delta(Q,r))\big)^{-1} \int_{\Delta(Q,r)} fk\, d\sigma \leqslant C\|f\|_p^p$, which together with (3.4) implies that $f^*(Q) \leqslant C(Mf^p)^{1/p} + C\|f\|_p$. From the ordinary maximal inequality (see Stein [11], Chapter 1) follows now that $\sigma\{Q \in S_3: f^*(Q) > s\} \leqslant Cs^{-p}\|f\|_p^p$ which proves the lemma.

Let D be a Lipschitz domain and let $1 < q < \infty$. We say that a function $f \in L^q(\sigma)$ *satisfies condition* B_q if there is a constant B_q such that for all $P \in \partial D$ and all $r \in (0,1)$

$$\Big[\big(\sigma(\Delta(P,r))\big)^{-1} \int_{\Delta(P,r)} f^q\, d\sigma\Big]^{1/q} \leqslant B_q\big(\sigma(\Delta(P,r))\big)^{-1} \int_{\Delta(P,r)} f\, d\sigma.$$

LEMMA 3.2. *Let $D \subset \mathbf{R}^n$, $n \geqslant 3$, be a Lipschitz domain and let $q \in (1, \infty)$. Suppose there is a point $P_0 \in D$ such that k satisfies condition B_q, where k is the density of $\omega(P_0, \cdot)$ with respect to σ. Suppose μ is a positive measure on D such that $\mu(B(P,r) \cap D) \leqslant Mr^{n-1}$ for all $P \in \partial D$ and all $r > 0$. Suppose $p < s < \infty$, where $p = q(q-1)^{-1}$. Then there is a constant K, which can be taken to depend only on D, k, s and M such that*

$$\int_D |Hf|^s\, d\mu \leqslant K \int_{\partial D} |f|^s\, d\sigma.$$

Proof. As in the proof of Lemma 3.1 it is sufficient to treat the case $f \geq 0$. From the definition of a Lipschitz domain follows the existence of finitely many domains $D_0, D_1, \ldots, D_N$ with the following properties. First, $D_i \subset D$ for all i and $\bar{D}_0 \subset D$. Secondly, each of the domains $D_1, \ldots, D_N$ is congruent to a domain of the type considered in Lemma 3.1 and to each $i \geq 1$ there is a right circular cylinder L_i whose bases are on positive distance from ∂D and $L_i \cap D = D_i$, $L_i \cap \partial D = L_i \cap \partial D_i$. At last, the domains can be chosen so that $\bigcup_1^N S_i^* = \partial D$ and $D_0 \cup \left(\bigcup_1^N D_i^*\right) = D$.

Suppose now that f has its support on S_i^* and $f \geq 0$. From Theorem B follows the existence of a constant C such that $Hf(P) \leq CHf(P_0) \leq C\|f\|_p$ for all $P \in D - D_1^*$. Hence

$$\int_D (Hf)^s d\mu \leq \int_{D_i^*} (Hf)^s d\mu + C\|f\|_p^p. \tag{3.5}$$

Let G and G_i be the Green functions of D and D_i, respectively. Let k_i denote the density of the harmonic measure of D_i evaluated at P_i. If V is an open set such that $\bar{V} \subset L_i$ and $P_i \notin \bar{V}$, then it follows from Theorem B that

$$C^{-1}G(P_0, Q) \leq G_i(P_i, Q) \leq CG(P_0, Q) \quad \text{for all } Q \in V. \tag{3.6}$$

From Theorem A and (3.6) follows now that to each compact set $K \subset L_i \cap \partial D$ there is a number C such that if $Q \in K$, then $C^{-1}k(Q) \leq k_i(Q) \leq Ck(Q)$. Hence we have from Lemma 3.1 that $\mu\{P \in D_i^*: H_i f > s\} \leq Cs^{-p}\|f\|_p^p$ when H_i denotes the Poisson integral with respect to D_i. The argument leading to (3.5) gives that $Hf|D_i \leq H_i f + C\|f\|_p$. Hence $\mu\{P \in D_i^*: Hf > s\} \leq Cs^{-p}\|f\|_p^p$ which taken together with the Marcinkiewicz interpolation theorem (Stein [11], p. 272) and (3.5) gives that $\int_D (Hf)^s d\mu \leq K_i\|f\|_p^p$. Since any $f \in L^p(\sigma)$ can be written as $\sum_{i=1}^N h_i f$, where h_i is the characteristic function of S_i^*, the lemma is proved.

We shall next show that if D is a C^1-domain, then the density of the harmonic measure satisfies condition B_q for all $q \in (1, \infty)$. We start with the following fact. Suppose u is harmonic in a domain $D \subset \mathbf{R}^n$ and $2 < q < \infty$. Then the function

$$F_q(\nabla u) = \left(|\nabla_x u|^2 + t\left((\partial/\partial y)u\right)^2\right)^{q/2} - q|\nabla_x u|^2\left(|\nabla_x u|^2 + t\left((\partial/\partial y)u\right)^2\right)^{(q-2)/2} \tag{3.7}$$

is superharmonic in D (see Küran [10]), where $t = (q-1)^{-1}(n-1)^{-1}$ and $|\nabla_x u|^2 = \sum_{i=1}^{n-1}\left((\partial/\partial x_i)u\right)^2$.

Let $\delta > 0$ be chosen so that for all $\varphi \in L(1)$ we have $B(P_1, 5\delta) \subset D(\varphi, 1)$ and put $D^*(\varphi) = D(\varphi, 1) - \overline{B(P_1, \delta)}$.

Lemma 3.3. *Let $n \geq 3$ and suppose $0 < m < (n-1)^{-1/2}$. For $\varphi \in L(m)$ let γ denote the harmonic measure of $\partial D^*(\varphi) - \{(x, \varphi(x)): |x| \leq 10\}$. Suppose $2 < q < 1 + m^{-1}(n-1)^{-1/2}$. Then there are positive numbers A and B, which can be taken to depend only on m and q, such that*

$$F_q(\nabla g) + A\gamma \geq 2B\left((\partial/\partial y)g\right)^q \quad \text{in} \quad D^*(\varphi),$$

where g is the Green function of $D(\varphi, 1)$ with pole at P_1.

Proof. Put $L = \partial D^*(\varphi) - \{(x, \varphi(x)): |x| \leq 10\}$. Since $0 \leq g(P) \leq |P - P_1|^{2-n}$, it follows from the standard Schauder estimates that there are numbers $A_1 > 0$ and $B_1 > 0$, which can be taken to depend only on q, such that $F_q(\nabla g) \geq -A_1$ and $|\nabla g| \leq B_1$ on L, where ∇g denotes the gradient of g. We now assume $\varphi \in C^\infty(\mathbf{R}^{n-1}) \cap L(m)$. Then g is smooth up to the boundary of $D^*(\varphi)$ and, in particular, the tangential part of the gradient vanishes on $L_1 = \{(x, \varphi(x)): |x| < 10\}$. Hence, if $P \in L_1$, then $|\nabla_x g(P)| \leq m|(\partial/\partial y)g(P)|$. From (3.7) follows now that there is a number $B = B(q, m)$ such that $F_q(\nabla g) \geq 2B|(\partial/\partial y)g(P)|^q$ on L_1. Hence we can now choose a number $A = A(q, m) > 0$ such that $F_q(\nabla g) - 2B|(\partial/\partial y)g|^q + A \geq 0$ on L. From (3.7) we have that $v = F_q(\nabla u) - 2B|(\partial/\partial y)g|^q + A\gamma$ is superharmonic in $D^*(\varphi)$ and has boundary values ≥ 0 on $\partial D^*(\varphi)$ which gives the lemma in the case when $\varphi \in C^\infty(\mathbf{R}^{n-1}) \cap L(m)$.

If $\varphi \in L(m)$ and not assumed to be C^∞, we can find $\varphi_i \in C^\infty \cap L(m)$ such that $\varphi_i \geq \varphi$ and $\varphi_i \to \varphi$ uniformly. Let g_i denote the Green function of $D(\varphi_i, 1)$ with pole at P_1 and let γ_i denote the harmonic measure of $D^*(\varphi_i) - \{(x, \varphi_i(x)): |x| \leq 10\}$. Then it follows from the arguments given in Helms [5], p. 89, that $g_i \to g$ uniformly on compact subsets of $D(\varphi, 1) - \{P_1\}$ and $\gamma_i \to \gamma$ uniformly on compact subsets of $D^*(\varphi)$. Hence the lemma follows from the previous case.

We shall need the following simple consequence from Theorems A and B.

Lemma 3.4. *Let D_1 and D_2 be two Lipschitz domains in $\mathbf{R}^n$, $n \geq 3$. Let $P_i \in D_i$, $i = 1, 2$, and denote by k_i the density of the harmonic measure of D_i evaluated at P_i. Suppose there is an open set V such that $V \cap D_1 = V \cap D_2$ and $V \cap \partial D_1 = U \cap \partial D_2 \neq \emptyset$. Then to each compact set $F \subset V \cap \partial D_1$ there is a number $C > 0$ such that $C^{-1}k_1(P) \leq k_2(P) \leq Ck_1(P)$ a.e. on F.*

Proof. Let G_i denote the Green function of D_i, $i = 1, 2$. From Theorem B follows the existence of a neighbourhood W of F such that $W \subset V$ and $C^{-1}G_1(P_1, Q) \leq G_2(P_2, Q) \leq CG_1(P_1, Q)$ for all $Q \in W$. From Theorem A follows that k_i is a.e. given as the normal derivative of $G_i(P_i, \cdot)$ from which the lemma follows.

For $m > 0$ let $A(m) = \{(x, y): |x| < my, \ |x|^2 + y^2 = 1\}$ and put $e = (0, 1) \in A(m)$. Let δ denote the Beltrami operator of $S^{n-1} = \{P \in \mathbf{R}^n: |P| = 1\}$ and let $\lambda(m)$ be the first eigenvalue of $\delta f + \lambda f = 0$, $f = 0$ on the boundary of $A(m)$. Let φ_m be the corresponding eigenfunction, normalized by $\varphi_m(e) = 1$. Let $\alpha(m)$ denote the positive root of the equation $t(t+n-2) = \lambda(m)$. It follows from Courant–Hilbert [2], p. 321, that $\alpha(m)$ is a continuous increasing function of m and

$$\lim_{m\to 0} \alpha(m) = \alpha(0) = 1. \tag{3.8}$$

The fact which is important for us is that the function

$$h_m(P) = |P|^{\alpha(m)}\varphi_m(P|P|^{-1}) \tag{3.9}$$

is non-negative and harmonic in $\Gamma(m) = \{(x, y)\colon |x| < my\}$ and vanishes on the boundary of $\Gamma(m)$.

LEMMA 3.5. *Let m, φ and q be as in Lemma 3.3. Suppose in addition that $(q-1)\alpha(m) \leqslant q$. Denote by k the density of the harmonic measure of $D(\varphi, 1)$ evaluated at P_1. Then there is a number $C > 0$ such that for all $P \in S(\varphi)$ and all $r \in (0, 1)$ we have*

$$\Big(r^{1-n} \int_{A(P',r)} k^{q+1}\, d\sigma\Big)^{1/(q+1)} \leqslant Cr^{1-n} \int_{A(P',r)} k\, d\sigma.$$

Proof. Define $v(P) = F_q(\nabla g) + (A+1)\gamma - B|(\partial/\partial y)g|^q$, where A and B are as in Lemma 3.3. Then v is non-negative and superharmonic in $D^*(\varphi)$ and $v \geqslant 1$ on L, where L is as in the proof of Lemma 3.3. Moreover, $v \geqslant B|(\partial/\partial y)g|^q$ in $D^*(\varphi)$. For $0 < t < 1$ let $v_t(P) = v(tP + (1-t)P_1)$. If t is sufficiently near 1, then v_t is non-negative and superharmonic in $D_1(\varphi) = D(\varphi, 1) - \overline{B(P_1, 2\delta)}$. Fix a point $P_2 \in D_1(\varphi)$ and denote by k_1 the density of the harmonic measure of $D_1(\varphi)$ evaluated at P_2. Since v_t is superharmonic and continuous in $D_1(\varphi)$, we have

$$v_t(P_2) \geqslant \int_{\partial D_1(\varphi)} v_t(P) k_1(P)\, d\sigma(P). \tag{3.10}$$

We now observe that from Lemma 2.1 follows that ∇g is non-tangentially bounded a.e. on $S(\varphi)$, which means that ∇g has non-tangential limits a.e. on $S(\varphi)$. Also it follows that $\lim_{t\to 1} |\nabla g(P_t)| \geqslant C^{-1}k(P)$ a.e. on $S(\varphi)$. From Lemma 3.4 follows that there is a constant $C > 0$ such that $k_1(P) \geqslant C^{-1}k(P)$ a.e. on $S(\varphi)$. Hence we find from (3.8) and Fatou's lemma that $Cv(P_2) \geqslant \int_{S(\varphi)} (k(P))^{q+1} d\sigma(P)$. This means $k \in L^{q+1}(S(\varphi), \sigma)$. However, since the support of φ is contained $m\{x \in \mathbf{R}^{n-1}\colon |x| < 1\}$, it follows that

$$\int_{L_1} k^{q+1}\, d\sigma < \infty, \tag{3.11}$$

where $L_1 = \{(x, \varphi(x))\colon |x| < 10\}$. Let $P' \in S(\varphi)$ and $0 < r < 1$ and put $u(P) = \int_{A(P',r)} |\partial/\partial y(g(x))|^q \omega(P, dQ)$. From (3.11) follows that there is a constant C such that $u(P) \leqslant C$ on L. Hence $u \leqslant Cv$ in $D^*(\varphi)$ for a suitable constant C. From Lemma 2.2 follows the existence of a constant C such that

$$u(P_1) \leqslant Cu(P' + (0, r))\, \omega(P_1, A(P', r)). \tag{3.12}$$

We now observe

$$\begin{aligned} u(P' + (0, r)) &\leqslant C(F_q(\nabla g) + A\gamma)(P' + (0, r)) \\ &\leqslant C(|\nabla g|^q + \gamma)(P' + (0, r)). \end{aligned} \tag{3.13}$$

Let $P_r = P' + (0, r)$. Then there is a number $\varepsilon > 0$ such that $B(P_r, \varepsilon r) \subset D(\varphi) - (P_1)$ for all $r \in (0, 1)$. Since g is positive and harmonic in $B(P_r, \varepsilon r)$, we have that there is a number C, depending only on ε and n, such that $|\nabla g(P_r)| \leqslant Cr^{-1}g(P_r)$. Let $\Gamma = \Gamma(m) = \{(x, y)\colon |x| < my\}$ and for $P \in S(\varphi)$ define $\Gamma(P) = \Gamma + P$. Fix an $r_0 > 0$ such that $B(P, r_0) \cap \Gamma(P) \subset D(\varphi, 1)$ for all $P \in S(\varphi)$. Then there is a number $C > 0$ such that $g(Q) \geqslant C$, $Q \in \bigcup_{P\in S(\varphi)} \{Q\colon Q \in \Gamma(P), |Q-P| = r_0\}$. If s denotes the harmonic measure $\{P \in \Gamma\colon |P| = r_0\}$ with respect to $\Gamma \cap B(0, r_0)$, then $g(Q) \geqslant Cs(Q-P)$ for all $Q \in B(P, r_0) \cap \Gamma(P)$. From Theorem B follows that there is a number $C > 0$ such that $s(Q) \geqslant ch_m(Q)$ for all $Q \in B(0, \frac{1}{2}r_0) \cap \Gamma$, where h_m is defined by (3.9). Hence $g(P_r) \geqslant Cr^{\alpha(m)} > 0$ for all r, $0 < r < 1$. From Theorem B and the assumption on q follows the existence of a number $C > 0$ such that $\gamma(P_r) \leqslant Cg(P_r) \leqslant C(r^{-1}g(P_r))^q$.

From (3.13) follows now that $u(P_r) \leqslant C(r^{-1}g(P_r))^q$ and using Lemma 2.1 we find $u(P_r) \leqslant C(r^{1-n}\omega(P_1, A(P', r)))^q$. Considering (3.10) we now have

$$r^{1-n}u(P_1) \leqslant C(r^{1-n}\omega(P_1, A(P', r)))^{q+1}.$$

From Lemma (2.1) follows now that $\lim_{r\to 1} (\partial/\partial y)g(P + (0, r)) \geqslant C^{-1}k(P)$ a.e. on $\{(x, \varphi(x))\colon |x| < 10\}$. Hence $u(P_1) \geqslant C^{-q} \int_{A(P',r)} k^{q+1}\, d\sigma$ from which the lemma follows.

Proof of Theorem 1. Let $D \subset \mathbf{R}^n$ be a Lipschitz domain; then it follows from Lemma 2.3 that (ii) implies (i). If k denotes the density of the harmonic measure evaluated at some point P, then k satisfies a B_2-condition (Theorem A). Hence it follows from Coifman–Fefferman [1] that k satisfies a B_q-condition for some $q > 2$. Lemma 3.2 shows now that (i) implies (ii) in this case.

If D is a C^1-domain, then it follows from (3.8), Lemma 3.4 and Lemma 3.5 that k satisfies a B_q-condition for all $q \in (1, \infty)$. As above, considering Lemma 3.2 completes the proof.

We shall now turn to some consequences of Theorem 1. Let D be a Lipschitz domain and let Γ be a given open bounded circular cone with vertex at 0. We say that a compact set $F \subset \partial D$ is *Γ-regular* if there is an open right circular cylinder L with axis parallell to the axis of Γ and a coordinate system (x, y) with $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, with the y-axis parallell to the axis of Γ such that $L\cap D = \{(x, y)\colon \varphi(x) < y\}\cap L$ and $L\cap\partial D = \{(x, y)\colon y = \varphi(x)\}\cap L$ for some Lipschitz function φ. Moreover, we require that $L\cap\partial D \supset F$ and there exists an open circular cone Γ' with vertex at 0 such that $\bar{\Gamma}-\{0\} \subset \Gamma'$ and $\Gamma'+P \subset D$ for all $P \in \partial D\cap L$.

THEOREM 2. *Let $D \subset \mathbf{R}^n$, $n \geqslant 3$, be a Lipschitz domain and assume that $F \subset \partial D$ is a compact set which is Γ-regular for some open circular cone Γ with axis along $e \in s^{n-1}$. Let $2 \leqslant p < \infty$ and put for $f \in L^p(\sigma)$ $f^*(P) = \sup\{|Hf(Q)|\colon Q \in \Gamma+P\}$. Then*

$$\int_F (f^*)^p\, d\sigma \leqslant C \int_{\partial D} |f|^p\, d\sigma \tag{3.14}$$

and $\lim_{\delta\to 0} \int_F |Hf(P+\delta e) - Hf(P)|\, d\sigma(P) = 0$.

If, in addition, D is a C^1-domain, then the above results hold for all $p \in (1, \infty)$.

Proof. To prove the theorem it is sufficient to prove (3.14) for the appropriate range of p. It is sufficient to prove (3.14) for the case when $f \geqslant 0$. From our assumptions of Γ and Harnack's inequality it follows $f^*(P) \leqslant Cf^{**}(P)$, where $f^{**}(P) = \sup_{0<t<h} Hf(P+te)$, where h is the height of Γ. If s is a non-negative function with $0 \leqslant s \leqslant h$, we define $F^* = \{P+ +s(P)e\colon P \in F\}$ and let $T\colon F \to F$ be defined by $\varphi(P) = P+s(P)$. Define now the positive measure μ on F^* by $\mu(M) = \sigma(T^{-1}(M))$ for $M \subset F^*$. Then μ is a measure such that $\mu(B(P, r)\cap F) \leqslant Cr^{n-1}$ for all $P \in F^*$, where C is independent of s. From Theorem 1 follows now that $\int_{F^*} (Hf)^p\, d\mu \leqslant K \int_{\partial D} f^p\, d\sigma$, where K is independent of s. By a suitable choice of s we can arrange that $Hf(T(P)) \geqslant \frac{1}{2} f^{**}(P)$ for all $P \in F$, which proves the theorem.

We shall now show a converse of Theorem 2.

THEOREM 3. *Let $D \subset \mathbf{R}^n$, $n \geqslant 3$, be a Lipschitz domain and suppose $V \subset \partial D$ is a relatively open set such that $\bar{V}$ is Γ-regular for some open circular cone Γ. Let $2 \leqslant p \leqslant \infty$ and put $V_\varepsilon = \{P+\varepsilon e\colon P \in V\}$, where e is the axis of Γ. Let u be harmonic in D and suppose that $\limsup_{\varepsilon\to 0} \left(\int_{V_\varepsilon} |u|^p\, d\sigma\right)^{1/p} < \infty$. Then there is a function $f \in L^p(\sigma)$ such that $\lim_{P\to Q} (u(P) - Hf(P)) = 0$ for all $Q \in V$.*

If, in addition, D is assumed to be a C^1-domain, then the above results also hold when $1 < p < 2$.

Proof. Define f_ε by $f_\varepsilon(Q) = u(Q+\varepsilon e)$ if $Q \in V$ and zero otherwise. Then there is a number $M > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then $\|f_\varepsilon\|_p \leqslant M$. Pick a sequence $\varepsilon_j \to 0$ such that f_{ε_j} converges weakly to $f \in L^p(\sigma)$ as $j \to \infty$.

Fix a point $Q \in V$ and let L be a right circular cylinder with its basis at a positive distance from ∂D and its axis parallell to e such that $Q \in L\cap\partial D$ $\subset \overline{L\cap\partial D} \subset V$ and $\overline{L\cap D} - \partial D \subset D$. Define $u_\varepsilon(P) = |u(P+\varepsilon e) - Hf_\varepsilon(P)|$ if $P \in L\cap D$ and zero otherwise. Then there is a number $\varepsilon' > 0$ such that u_ε is subharmonic in L whenever $0 < \varepsilon < \varepsilon'$. Also, it follows from Theorem 2 that if ε' is chosen sufficiently small, then there is a number δ_0 such that

$$\left(\int_V \left(u_\varepsilon(P+\delta e)^p\, d\sigma\right)\right)^{1/p} \leqslant C, \quad 0 < \delta < \delta_0,\ 0 < \varepsilon < \varepsilon'. \tag{3.15}$$

Let R be an open set such that $\overline{L\cap D} - \partial D \subset R \subset \bar{R} - \partial D \subset \{P+se\colon P \in V,\ 0 < s < \delta_0/2\}$. Hence we have from (3.15) that $\int_R u_\varepsilon\, dP \leqslant \text{Const} \times \left(\int_R u_\varepsilon^p\, d\sigma\right)^{1/p} \leqslant \text{Const}$ if $0 < \varepsilon < \varepsilon'$. Also, there is a number $t > 0$ such that if $P \in L\cap D$, then $B(P, td(P)) \subset R$, where $d(P)$ denotes the distance from P to ∂D. Since u_ε is subharmonic, we find

$$u_\varepsilon(P) \leqslant Cd(P)^{-n} \int_{B(P,td(P))} u_\varepsilon\, dP \leqslant Cd(P)^{-n} \int_R u_\varepsilon\, dP \leqslant Kd(P)^{-n}.$$

Define $F(P) = Kd(P)^{-n}$ if $P \in L\cap D$, $F(P) = +\infty$ if $P \in L\cap\partial D$ and zero otherwise. Then F is upper semicontinous in L, $\int_L (\log^+ F)^q\, dP < \infty$ for all $q > 0$ and $u_\varepsilon \leqslant F$ whenever $0 < \varepsilon < \varepsilon'$. Now it is known that if a non-negative function F is upper semicontinuous in a domain $\Omega \subset \mathbf{R}^n$ and $(\log^+ F)^{n-1+\varepsilon}$ is locally integrable in Ω for some $\varepsilon > 0$, then $S(F)$ contains a largest element v, where $S(F) = \{u\colon u$ is subharmonic in Ω and $u \leqslant F\}$ (see Domar [4], Theorem 2). Hence there is a function v subharmonic in L, vanishing outside $L\cap\bar{D}$ such that if $0 < \varepsilon < \varepsilon'$, then $u_\varepsilon \leqslant v$ in L. Hence $|u(P) - Hf(P)| = \lim_{j\to\infty} u_{\varepsilon_j} \leqslant v(P)$. It follows from the Wiener criterion (Helms [5], p. 220) that $L - \bar{D}$ is not thin (Helms [5], p. 209) at Q. Hence $v(Q) = \limsup_{P\to Q, P\in L-\bar{D}} v(P) = 0$, which implies that $\limsup_{P\to Q} |u(P) - Hf(P)| \leqslant \limsup_{P\to Q} v(P) = 0$. Since Q was arbitrary, Theorem 3 follows.

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On weakly* conditionally compact dynamical systems

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1. Let (X, φ) be a topological dynamical system, i.e. X is a compact metric space and $\varphi\colon X \to X$ is a continuous mapping. Denote by $C(X)$ the space of all continuous, real (or complex) valued functions on X, and let U_φ be an operator defined as follows: $U_\varphi f = f \circ \varphi$, $f \in C(X)$.

A sequence (f_n) of elements of a Banach space E is said to be *weakly* conditionally compact* if for every sequence of positive integers (n_k) there is a subsequence (n_{k_i}) such that for every linear continuous functional $\Phi \in E^*$ the sequence of scalars $(\Phi(f_{n_{k_i}}))$ is convergent. In the case of $E = C(X)$ it means that the sequence $(f_{n_{k_i}}(x))$ is pointwise convergent (not necessarily to a continuous function).

If for every sequence (n_k) there exists a subsequence (n_{k_i}) and an element $f \in E$ such that $(f_{n_{k_i}})$ is weakly convergent to f, then the sequence (f_n) is said to be weakly conditionally compact.

DEFINITION. A system (X, φ) is said to be *weakly* [weakly] conditionally compact* if for every $f \in C(X)$ the sequence $(U^n f)$ is weakly* [weakly] conditionally compact. For brevity, we shall call these systems w*cc [wcc] systems.

The aim of the paper is to study some spectral properties, the strict ergodicity (under some additional assumptions) and the sequence entropy of w*cc systems.

In view of Rosenthal's theorem [8] for every $f \in C(X)$ there are two possibilities:

(1) The sequence $(U^n f)$ contains a subsequence $(U^{n_k} f)$ such that for some $c > 0$ and for every sequence of numbers (real or complex) $a_0, \ldots, a_{m-1}$ the following inequality holds:

$$\sup_{x\in X} \Big|\sum_{k=0}^{m-1} a_k U^{n_k} f(x)\Big| \geqslant c \sum_{k=0}^{m-1} |a_k|. \tag{1}$$

(2) The sequence $(U^n f)$ is w*cc.